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Growth and representation of analytic and harmonic functions in the unit disc

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(*Presented by M. M. Sheremeta*)

Abstract. Let $u(z)$ be harmonic in $\{|z| < 1\}$, $\alpha \ge 0$, $0 < \gamma \le 1$. Let $B(r, u) = \max\{u(z) : |z| \le r\}$, $\omega(\delta, \psi)$ be the modulus of continuity of a function ψ defined on $[0, 2\pi]$. We prove that $u(z)$ has the form

$$u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} P_\alpha(r, \varphi - t)\, d\psi(t)$$

where $\psi \in BV[0, 2\pi]$, and $\omega(\delta, \psi) = O(\delta^\gamma)$ $(\delta \downarrow 0)$, if and only if $B(r, u) = O((1 - r)^{\gamma-\alpha-1})$, $r \uparrow 1$ and $\sup_{0<r<1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})|\, d\varphi < +\infty$. Here u_α is α-fractional integral of $u(re^{i\varphi})$, $P_\alpha(r, t) = \Gamma(1 + \alpha)\Re(\frac{2}{(1-re^{it})^{\alpha+1}} - 1)$.

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1. Introduction and main results

1.1. Analytic functions in the unit disc

Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $A(D)$ the class of analytic functions in D. For $f \in A(D)$ let $M(r, f) = \max\{|f(z)| : |z| = r\}$ be the maximum modulus, $T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$, $0 < r < 1$, the Nevanlinna characteristic, $x^+ = \max\{x, 0\}$.

Usually, the orders of the growth of analytic functions in D are defined as

$$\rho_M[f] = \limsup_{r\uparrow 1} \frac{\log^+ \log^+ M(r, f)}{-\log(1 - r)}, \qquad \rho_T[f] = \limsup_{r\uparrow 1} \frac{\log^+ T(r, f)}{-\log(1 - r)}.$$

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It is well known that

$$\rho_T[f] \le \rho_M[f] \le \rho_T[f] + 1, \tag{1.1}$$

and all cases are possible. This is in contrast to entire functions where the corresponding orders are equal. We cite a couple of results concerning (1.1).

Given $\alpha > 1$, ρ satisfying $\rho \le \alpha \le \rho + 1$, C. N. Linden [1] constructed an analytic function in $\overline{D} \setminus \{1\}$ of the form of so-called Naftalevich–Tsuji product

$$g(z) = \prod_{n=1}^{\infty} E\Big(\frac{1-|a_n|^2}{1-\bar{a}_n z}, p\Big), \qquad \sum_n (1-|a_n|)^{p+1} < \infty,$$

with the property $\rho_T[g] = \rho$, $\rho_M[g] = \alpha$. Here

$$E(w,p) = (1-w)\exp\{w + w^2/2 + \dots + w^p/p\}, \quad p \in \mathbb{Z}_+,$$

is the Weierstrass primary factor, a_n are the zeros of $g(z)$.

Another approach is used in a paper by M. Sheremeta [2], where, in particular, given $\alpha > 0$ a class of analytic functions f represented by gap series (with Hadamard's gaps) is extracted such that

$$\int_0^1 (1-r)^{1+\alpha} T(r,f)\, dr < +\infty \Leftrightarrow \int_0^1 (1-r)^{1+\alpha} \log M(r,f)\, dr < +\infty.$$

Prof. O. Skaskiv posed the following problem

Problem 1.1. *Given $0 \le \rho \le \alpha \le \rho + 1$, describe the class of analytic function in D such that $\rho_T[f] = \rho$, $\rho_M[f] = \alpha$.*

In order to solve Problem 1.1 one needs a parametric representation of functions analytic in D and of finite order of the growth. Such representation was obtained [3] in 1960th by M. M. Djrbashian using the Riemann–Liouville fractional integral.

For $\alpha > 0$ consider two subclasses of $A(D)$

$$A_\alpha : \sup_{0<r<1} \int_0^{2\pi} \bigg(\int_0^r (r-t)^{\alpha-1} \log|f(te^{i\varphi})|\, dt \bigg)^+ d\varphi < +\infty,$$

$$A_\alpha^* : \sup_{0<r<1} \int_0^{2\pi} \bigg(\int_0^r (r-t)^{\alpha-1} \log^+|f(te^{i\varphi})|\, dt \bigg) d\varphi < +\infty.$$

Obviously, $A^*_\alpha \subset A_\alpha$. Note that $f \in A^*_\alpha$ means $\int_0^1 T(t,f)(1-t)^{\alpha-1}dt < +\infty$, i.e. f belongs to the convergence class of order α.

Throughout this paper by $(1-w)^\alpha$, $w \in D$, $\alpha \in \mathbb{R}$, we mean the branch of the power function such that $(1-w)^\alpha\big|_{w=0} = 1$.

Theorem A. *The class A_α coincides with the class of functions represented in the form*

$$f(z) = C_\lambda z^\lambda B_\alpha(z) \exp\left\{\int_0^{2\pi} \frac{d\psi(\theta)}{(1-e^{-i\theta}z)^{\alpha+1}}\right\} \equiv C_\lambda z^\lambda B_\alpha(z) \exp\{g_\alpha(z)\}, \quad (1.2)$$

where $\psi \in BV[0,2\pi]$, (z_k) is the zero sequence of $f(z)$ such that $\sum_k (1-|z_k|)^{\alpha+1} < +\infty$; $B_\alpha(z) = \prod_k \big(1-\frac{z}{z_k}\big)\exp\{-W_\alpha(z,z_k)\}$ is a Djrbashian product

$$W_\alpha(z,\zeta) = \sum_k \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)\Gamma(1+k)} \times \left((\bar\zeta z)^k \int_{|\zeta|}^1 \frac{(1-x)^\alpha}{x^{k+1}}\,dx - \left(\frac{z}{\zeta}\right)^k \int_0^{|\zeta|} (1-x)^\alpha x^{k-1}\,dx\right).$$

In this paper we confine by the case when $f(z)$ has no zeros and of finite order of the growth. Then $f(z) = C_\alpha \exp\{g_\alpha(z)\}$, for some $\alpha \ge 0$.

Radial and non-tangential limits of $g_\alpha(z)$ were investigated in many papers, e.g. D. J. Hallenbeck, T. H. MacGregor [4, 5], and M. M. Sheremeta [6], even for complex-valued functions ψ of bounded variation. It turns out that $g_\alpha(z)$ admits above estimates in terms of the modulus of continuity for ψ. We cite a typical result [6].

Let $S(\theta,\gamma)$ be the closed Stolz angle having vertex $e^{i\theta}$ and opening γ, i.e. $S(\theta,\gamma) = \{z \in D : |\arg(e^{i\theta}-z)| < \gamma/2\}$. A function g defined in D is said to have a nontangential limit at $e^{i\theta}$ provided that $\lim_{z\to e^{i\theta}, z\in S(\theta,\gamma)} g(z)$ exists for every $\gamma \in [0,\pi)$.

Theorem B. *Let $\alpha > -1$, $\theta \in [0,2\pi]$, $\psi \in BV[0,2\pi]$, and ω be a nonnegative, increasing continuous, semi-additive function on $[0,+\infty)$, and $\omega(0) = 0$. If*

$$\int_0^1 t^{-\alpha-2}\omega(t)\,dt = \infty, \quad |\psi(t)-\psi(\theta)| = o\big(\omega(|t-\theta|)\big), \qquad t \to \theta,$$

and g_α *is given by* (1.2) *then*

$$|g_\alpha(z)| \Big/ \int\limits_{|1-ze^{-i\theta}|}^{1} t^{-\alpha-2}\omega(t)\,dt$$

has the nontangential limit zero at $e^{i\theta}$.

Since lower estimates for $|g_\alpha(z)|$ are known only in particular cases (see [7], Theorem D and Remark 1.3 below), it is interesting to obtain results which give lower estimates for $|g_\alpha(z)|$ in the general situation.

The main purpose of this paper is to describe the growth of $|g_\alpha(z)|$ in terms of the modulus of continuity for ψ and find counterparts for harmonic functions in D.

Problem 1.1 is not solved, but Theorem 3.3 and the corollary describe large classes of analytic functions f with the property $\rho_T[f] = \rho$, $\rho_M[f] = \alpha$, $0 \le \rho \le \alpha \le \rho + 1$. Theorem 3.2 yields asymptotic formulas for g_α in Stolz angles when ψ is not continuous.

1.2. Representation and the growth of harmonic functions

We need some definitions. Let $U_\theta(\delta) = \{x \in [0, 2\pi] : |x - \theta| < \delta\}$, $\delta > 0$. For $\psi\colon [0, 2\pi] \to \mathbb{R}$ define the moduli of continuity $\omega(\delta, \theta; \psi) = \sup\{|\psi(x) - \psi(y)| : x, y \in U_\theta(\delta)\}$, $\omega(\delta; \psi) = \sup_{\theta\in[0,2\pi]} \omega(\delta, \theta; \psi)$.

Following [8] we say that $\psi \in \Lambda_\gamma$ if $\omega(\delta; f) = O(\delta^\gamma)$ $(\delta \downarrow 0)$.

The fractional integral of order $\alpha > 0$ for $h\colon (0, 1) \to \mathbb{R}$ is defined by the formulas [3]

$$D^{-\alpha}h(r) = \frac{1}{\Gamma(\alpha)} \int\limits_0^r (r - x)^{\alpha-1} h(x)\,dx,$$

$$D^0h(r) \equiv h(r), \quad D^\alpha h(r) = \frac{d^p}{dr^p}\{D^{-(p-\alpha)}h(r)\}, \quad \alpha \in (p - 1; p],\ p \in \mathbb{N}.$$

Let $H(D)$ be the class of harmonic functions in D. We put $u_\alpha(re^{i\varphi}) = r^{-\alpha}D^{-\alpha}u(re^{i\varphi})$, where the fractional integral is taken on the variable r. We define $B(r, u) = \max\{u(z) : |z| \le r\}$ for a subharmonic function u in D.

Let

$$S_\alpha(z) = \Gamma(1 + \alpha)\Big(\frac{2}{(1 - z)^{\alpha+1}} - 1\Big), \quad P_\alpha(r, t) = \Re S_\alpha(re^{it}).$$

Remark 1.1. Note that $S_0(z)$ is the Schwartz kernel, $P_0(r, t)$ is the Poisson kernel; $P_\alpha(r, t) = D^\alpha(r^\alpha P_0(r, t))$.

Our starting point is the following two theorems

Theorem C (M. Djrbashian). *Let* $u \in H(D)$, $\alpha > -1$. *Then*

$$u(re^{i\varphi}) = \int_0^{2\pi} P_\alpha(r, \varphi - \theta)\, d\psi(\theta), \tag{1.3}$$

where $\psi \in BV[0, 2\pi]$, *if and only if*

$$\sup_{0<r<1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})|\, d\varphi < +\infty.$$

Remark 1.2. Actually, for $\alpha = 0$ it is the classical result of Nevanlinna on representation of $\log|F(z)|$ when F belongs to the Nevanlinna class N.

Theorem D (Hardy–Littlewood). *Let* $u \in H(D)$, $0 < \gamma \leq 1$. *Then*

$$u(re^{i\varphi}) = \int_0^{2\pi} P_0(r, \varphi - t) v(t)\, dt \tag{1.4}$$

for some function $v \in \Lambda_\gamma$, *if and only if*

$$B\Big(r, \frac{\partial u}{\partial \varphi}\Big) = O((1-r)^{\gamma-1}), \quad r \uparrow 1.$$

Remark 1.3. Theorem D was originally proved [9] for analytic function (cf. Theorem 1.2).

Applying methods of proofs of Theorems B and C, we prove the following theorem which describes the growth of functions of form (1.3).

Theorem 1.1. *Let* $u \in H(D)$, $\alpha \geq 0$, $0 < \gamma < 1$. *Then* $u(z)$ *has form* (1.3) *where* ψ *is of bounded variation on* $[0, 2\pi]$, *and* $\psi \in \Lambda_\gamma$, *if and only if*

$$B(r, u) = O((1-r)^{\gamma-\alpha-1}), \quad r \uparrow 1$$

and

$$\sup_{0<r<1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})|\, d\varphi < +\infty.$$

Note that Theorem D corresponds to the case when ψ is absolutely continuous, but in Theorem 1.1 the general situation is considered.

Similar to that as we deduce Theorem 1.1 from the proposition below, one can deduce the following generalization of Theorem D.

Theorem 1.2. *Let $u \in H(D)$, $0 < \gamma < 1$, $\alpha \geq 0$. Then*

$$u(re^{i\varphi}) = \int_0^{2\pi} P_\alpha(r, \varphi - t) v(t)\, dt$$

for some function $v \in \Lambda_\gamma$, if and only if

$$B\Big(r, \frac{\partial u}{\partial \varphi}\Big) = O((1-r)^{\gamma-\alpha-1}), \quad r \uparrow 1.$$

It is not difficult to prove a counterpart of the last theorem for analytic functions.

Remark 1.4. Similar to [6], one can prove that if $u(z)$ is represented by (1.3), then

$$u(re^{i\theta}) = O\Bigg(\int_{1-r}^{2\pi} \frac{\omega(\tau, \varphi; \psi)}{\tau^{\alpha+2}}\, d\tau\Bigg), \quad r \uparrow 1,\ re^{i\theta} \in S(\varphi, \tau),\ 0 \leq \tau < \pi.$$

Problem 1.2. *Obtain necessary and sufficient conditions for local growth of $u \in H(D)$.*

2. Proof of Theorem 1.1

2.1. The case $\alpha = 0$

First, it is suitable to prove an important particular case of Theorem 1.1 in spirit of Theorem D.

Proposition 2.1. *Let $u \in H(D)$, $0 < \gamma \leq 1$. Then $u(z)$ has the form*

$$u(re^{i\varphi}) = \int_0^{2\pi} P_0(r, \varphi - t)\, d\psi(t) \tag{2.1}$$

where ψ is of bounded variation on $[0, 2\pi]$, and $\psi \in \Lambda_\gamma$, if and only if

$$B(r, u) = O((1-r)^{\gamma-1}), \quad r \uparrow 1$$

and

$$\sup_{0<r<1} \int_0^{2\pi} |u(re^{i\varphi})|\, d\varphi < +\infty.$$

In the sequel, the symbol C with indices stands for some positive constants.

Proof of Proposition 2.1. First, we consider the case $\gamma = 1$. Note that the class Λ_1 consists of functions that are integrals of bounded functions. Thus it is sufficient to apply Theorem (6.3) [8, Ch.IV], which states that (1.4) holds if and only if $B(r,u)$ is bounded as $r \uparrow 1$.

Consider the case $\gamma \in (0,1)$.

Necessity. The proof of necessity is standard (cf. [11, Ch.8.2], [6]).

The following estimates of $P_0(r,t)$ are well known

$$\Big|\frac{\partial}{\partial t}P_0(r,t)\Big| \le \frac{2}{(1-r)^2}, \quad \Big|\frac{\partial}{\partial t}P_0(r,t)\Big| \le \frac{\pi^2}{t^2}, \quad r \ge \frac{1}{2}, \ |t| \le \pi. \tag{2.2}$$

We extend ψ on $\mathbb{R}$ by the formula $\psi(t+2\pi)-\psi(t) = \psi(2\pi)-\psi(0)$. Since $P_0(r,t)$ is a periodic and even function in t, we have

$$u(re^{i\varphi}) = \int_{-\pi+\varphi}^{\pi+\varphi} P_0(r,\theta-\varphi)d(\psi(\theta)-\psi(\varphi))$$

$$= (\psi(\theta)-\psi(\varphi))P_0(r,\theta-\varphi)\Big|_{-\pi+\varphi}^{\pi+\varphi} - \int_{-\pi+\varphi}^{\pi+\varphi} \frac{\partial}{\partial\theta}(P_0(r,\theta-\varphi))(\psi(\theta)-\psi(\varphi))\,d\theta$$

$$= (\psi(2\pi)-\psi(0))P_0(r,\pi) - \int_{-\pi}^{\pi} \frac{\partial}{\partial\tau}(P_0(r,\tau))(\psi(\tau+\varphi)-\psi(\varphi))\,d\tau.$$

Hence, using (2.2), we obtain

$$\begin{aligned} &|u(re^{i\varphi})| \\ &\le \frac{C_1(\psi)(1-r)}{1+r} + \Bigg(\int_{|\tau|\le 1-r} + \int_{1-r\le|\tau|\le\pi}\Bigg)\Big|\frac{\partial}{\partial\tau}P_0(r,\tau)\Big|\omega(|\tau|,\psi;\varphi)\,d\tau \\ &\le o(1) + 2\int_{|\tau|\le 1-r}\frac{\omega(|\tau|,\psi;\varphi)}{(1-r)^2}\,d\tau + \int_{1-r\le|\tau|\le\pi}\frac{\pi^2}{\tau^2}\omega(|\tau|,\psi;\varphi)\,d\tau \\ &\le o(1) + 4\frac{\omega(1-r,\psi;\varphi)}{1-r} + 2\pi^2\int_{1-r\le\tau\le\pi}\frac{\omega(\tau,\psi;\varphi)}{\tau^2}\,d\tau \\ &\le (2\pi^2+4)\int_{1-r\le\tau\le\pi}\frac{\omega(\tau,\psi;\varphi)}{\tau^2}\,d\tau + O(1), \quad r\uparrow 1. \end{aligned} \tag{2.3}$$

Here, we have used increasing of the modulus of continuity. Since $\psi \in \Lambda_\gamma$, $\omega(\tau, \psi; \varphi) = O(\tau^\gamma)$ as $\tau \downarrow 0$. Thus, (2.3) yields

$$B(r, u) \le C_2(\gamma)(1-r)^{\gamma-1}, \quad r \uparrow 1.$$

Sufficiency. Let $u(re^{i\varphi})$ be harmonic for $r < 1$, and $\int_0^{2\pi} |u(re^{i\varphi})|\, d\varphi \le C_3$.

Remark 2.1. By Theorem C, we have (2.1), where $\psi \in BV[0, 2\pi]$, and one can take ψ such that at any point θ of continuity of ψ for some sequence (r_n) ([3], [10, p. 57]).

$$\psi(\theta) = \lim_{r_n \uparrow 1} \int\limits_0^\theta u(r_n e^{i\phi})\, d\phi. \tag{2.4}$$

Let $F(z)$ be an analytic function in D_1 such that $\Re F(z) = u(z)$. By the theorem of Zygmund [8, Th. (2.30), Ch. VII] $B(r, u) = O((1-r)^{\gamma-1})$ implies $M(r, F) = O((1-r)^{\gamma-1})$ as $r \uparrow 1$.

Define the analytic function $\Phi(z) = \int_0^z F(\zeta)\, d\zeta$, $z \in D$. For any fixed $\varphi \in [0, 2\pi]$ and $0 < r' < r'' < 1$, we have

$$|\Phi(r''e^{i\varphi}) - \Phi(r'e^{i\varphi})| = \left| \int\limits_{r'}^{r''} F(\rho e^{i\varphi}) e^{i\varphi}\, d\rho \right|$$
$$\le C_4 \int\limits_{r'}^{r''} (1-\rho)^{\gamma-1}\, d\rho \le \frac{C_4}{\gamma}(1-r')^{1-\gamma}.$$

Therefore, by Cauchy's criterion, there exists $\lim_{r\uparrow 1} \Phi(re^{i\varphi}) \equiv \Phi(e^{i\varphi})$ uniformly in φ. Consequently, $\tilde{\Phi}(\varphi) \stackrel{\text{def}}{=} \Phi(e^{i\varphi})$ is a continuous function on $[0, 2\pi]$.

Let us prove that $\tilde{\Phi} \in \Lambda_\gamma$. Let $h \in (0, 1)$, $z_0 = e^{i\varphi}$, $z_1 = (1-h)e^{i\varphi}$, $z_2 = (1-h)e^{i(\varphi+h)}$, $z_3 = e^{i(\varphi+h)}$.

Then by Cauchy's theorem

$$\Phi(z_3) - \Phi(z_0) = \int\limits_{[0, z_3]} F(z)\, dz + \int\limits_{[z_0, 0]} F(z)\, dz$$
$$= \left(\int\limits_{[z_0, z_1]} + \int\limits_{z_1}^{z_2} + \int\limits_{[z_2, z_3]} \right) F(z)\, dz.$$

For sufficiently small $h > 0$, we have

$$\left| \int_{[z_0,z_1]} F(z)\,dz \right| \le \int_{1-r}^{1} \frac{C_4}{(1-h)^{1-\gamma}}\,dr = \frac{C_4}{\gamma} h^{\gamma}.$$

Similarly, $|\int_{[z_2,z_3]} F(z)\,dz| \le \frac{C_4}{\gamma} h^{\gamma}$. It is obvious that $|\int_{z_1}^{z_2} F(z)\,dz| \le C_4 h^{\gamma}$.

Therefore, $|\Phi(e^{i(\varphi+h)}) - \Phi(e^{i\varphi})| \le C_4(\frac{2}{\gamma}+1)h^{\gamma}$, so $\tilde{\Phi} \in \Lambda_\gamma$.

For $R \in (0,1)$ define

$$\lambda_R(\theta) = \int_0^{\theta} F(Re^{i\sigma})\,d\sigma = \int_0^{\theta} \frac{d\Phi(Re^{i\sigma})}{iRe^{i\sigma}}$$
$$= \frac{\Phi(Re^{i\theta})}{iRe^{i\theta}} - \frac{\Phi(R)}{iR} + \frac{1}{R}\int_0^{\theta} \Phi(Re^{i\sigma})e^{-i\sigma}\,d\sigma.$$

Since $\Phi(z)$ is continuous in $\{z : |z| \le 1\}$, $\Phi(Re^{i\sigma}) \underset{\theta}{\rightrightarrows} \Phi(e^{i\sigma})$ as $R \uparrow 1$. Consequently, as $R \uparrow 1$

$$\lambda_R(\theta) \rightrightarrows -\Phi(e^{i\theta})ie^{-i\theta} + i\Phi(1) + \int_0^{\theta} \Phi(e^{i\sigma})e^{-i\sigma}\,d\sigma \equiv \lambda(\theta) \in C[0,2\pi].$$

Since $\tilde{\Phi} \in \Lambda_\gamma$, we have $\lambda \in \Lambda_\gamma$. On the other hand, $u(re^{i\varphi}) = \int_0^{2\pi} P_0(r, \varphi - t)\,d\psi(t)$, where, by (2.4) and the definition of λ

$$\psi(\theta) = \lim_{r\uparrow 1} \int_0^{\theta} \Re F(re^{i\phi})\,d\phi = \lambda(\theta).$$

Thus, $\psi \in \Lambda_\gamma$. $\square$

2.2. The case $\alpha > 0$

Necessity. Let u has form (1.3), where $\psi \in BV[0,2\pi] \cap \Lambda_\gamma$. This implies

$$u_\alpha(re^{i\varphi}) = \int_0^{2\pi} P_0(r, \varphi - \theta)\,d\psi(\theta). \tag{2.5}$$

By the proposition we have $\sup_{r<1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})|\,d\varphi < +\infty$ and $B(r, u_\alpha) = O((1-r)^{\gamma-1})$ as $r \uparrow 1$.

We use the following formula [3, Chap. IX, (2.9)]

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} P_\alpha(r/\rho, \varphi-\theta)u_\alpha(\rho e^{i\theta})\,d\theta, \qquad 0\le r<\rho<1.$$

Taking $\rho=(1+r)/2$ and using the estimate $B(r,u_\alpha)=O((1-r)^{\gamma-1})$ $(r\uparrow 1)$, we obtain

$$|u(re^{i\varphi})| = O\biggl(\int_0^{2\pi} \frac{(1-\rho)^{\gamma-1}\,d\theta}{|1-\frac{r}{\rho}e^{i(\varphi-\theta)}|^{1+\alpha}}\biggr)$$
$$= O\bigl((1-r)^{\gamma-1}(\rho-r)^{-\alpha}\bigr) = O\bigl((1-r)^{\gamma-1-\alpha}\bigr), \quad r\uparrow 1.$$

The necessity is proved.

Sufficiency. Let $\int_0^{2\pi}|u_\alpha(re^{i\varphi})|\,d\varphi<+\infty$ uniformly for all $r\in(0,1)$. Then by Theorem C we have (2.5), where $\psi\in BV[0,2\pi]$.

We need the following elementary lemma.

Lemma 2.1. *Let* $(\forall x\in[0,1))$ $f\in L[0,x)$, *and* $0\le\eta<\beta$, *and* $|f(x)|=O((1-x)^{-\beta})$ *as* $x\uparrow 1$. *Then* $|D^{-\eta}f(x)|=O((1-x)^{\eta-\beta})$ *as* $x\uparrow 1$.

Proof. Using the definition of the fractional integral and standard estimates we obtain

$$|D^{-\eta}f(x)| = \biggl|\frac{1}{\Gamma(\eta)}\int_0^x f(t)(x-t)^{\eta-1}\,dt\biggr|$$
$$= O\biggl(\int_0^x \frac{(x-t)^{\eta-1}}{(1-t)^\beta}\,dt\biggr) = O\biggl(\int_0^{x-2(1-x)} + \int_{x-2(1-x)}^x\biggr)\frac{(x-t)^{\eta-1}}{(1-t)^\beta}\,dt$$
$$= O\biggl(\int_0^{x-2(1-x)} (1-t)^{\eta-1-\beta}\,dt + (1-x)^{\eta-1}\int_{x-2(1-x)}^x \frac{dt}{(1-t)^\beta}\biggr)$$
$$= O((1-x)^{\eta-\beta}), \qquad x\uparrow 1.$$

The lemma is proved. □

Since $B(r,u)=O((1-r)^{\gamma-1-\alpha})$ as $r\uparrow 1$, by the lemma $B(r,u_\alpha)=O((1-r)^{\gamma-1})$ as $r\uparrow 1$. Therefore, by Proposition 2.1, $\psi\in\Lambda_\gamma$.

3. Further results for analytic functions

There is an analogue of Theorem C for analytic functions proved by M. Djrbashian [3]. The following theorem can be proved as the proposition and Theorem 1.1 were proved.

Theorem 3.1. *Let $f(z)$ be an analytic function in D, $\alpha \geq 0$, $0 < \gamma < 1$. Then $f(z)$ has the form*

$$f(re^{i\varphi}) = \int_0^{2\pi} S_\alpha(r, \varphi - t)\, d\psi(t) + i\Im f(0) \tag{3.1}$$

where ψ is of bounded variation on $[0, 2\pi]$, and $\psi \in \Lambda_\gamma$, if and only if

$$B(r, |f|) = O((1-r)^{\gamma-\alpha-1}), \quad r \uparrow 1$$

and

$$\sup_{0<r<1} \int_0^{2\pi} |\Re f_\alpha(re^{i\varphi})|\, d\varphi < +\infty,$$

where $f_\alpha(re^{it}) = r^{-\alpha} D^{-\alpha} f(re^{it})$.

Theorem 1.1 does not cover the case when $\psi \in \Lambda_0$, in particular, when ψ is not continuous. Here, following [6] we are able to prove a more precise result. It looks to be known, but I have not found it in a literature.

Theorem 3.2. *Let $f(z)$ have the form*

$$f(z) = \int_0^{2\pi} (1 - ze^{-it})^{-\alpha}\, d\psi(t), \qquad z \in D$$

where $\alpha > 0$, $\psi \in BV[0, 2\pi]$. If $\{t_k\}$ is the set of the discontinuity points of ψ with jumps $\{h_k\}$, then

$$f(z) = \frac{h_k + o(1)}{(1 - ze^{-it_k})^\alpha}, \qquad z \to e^{it_k},\ z \in S(t_k, \tau),\ \tau \in [0, \pi), \tag{3.2}$$

and

$$f(z) = \frac{o(1)}{(1 - ze^{-it})^\alpha}, \quad z \to e^{it},\ z \in S(t, \tau),\ t \notin \{t_k\},\ \tau \in [0, \pi). \tag{3.3}$$

Proof. Since $\psi \in BV[0,2\pi]$, the set $\{t_k\}$ is at most countable. Without loss of generality we can assume that ψ is continuous from the right. Then $h_k = \psi(t_k) - \psi(t_k - 0)$. It is sufficient to prove (3.2) for $k = 1$.

We may assume that $t_1 \in (0, 2\pi)$. Let

$$H_1(t) = \begin{cases} 0, & 0 \le t < t_1, \\ h_1, & t_1 \le t \le 2\pi. \end{cases}$$

We extend ψ on $\mathbb{R}$ by the formula $\psi(t+2\pi) - \psi(t) = \psi(2\pi) - \psi(0)$ as well as H_1. The function $g(t) \stackrel{\text{def}}{=} \psi(t) - H_1(t)$ is continuous at the points $t = t_1 + 2\pi k$, $k \in \mathbb{Z}$, so $\omega(\delta, t_1, g) = o(1)$ as $\delta \downarrow 0$. We have

$$f(z) = \int_0^{2\pi} (1 - ze^{-it})^{-\alpha}\, dg(t) + \frac{h_1}{(1 - ze^{-it_1})^{\alpha}}. \tag{3.4}$$

Let $\omega_1(\delta) = \max\{\sqrt{\omega(\delta, t_1, g)}, \delta^{\alpha/2}\}$. It is easy to see that $\omega_1(\delta)$ satisfies the hypotheses of Theorem B on $\omega(\delta)$. Applying Theorem B to the integral from (3.4) we obtain (3.2) from (3.4).

Relationship (3.3) follows directly from Theorem B if we choose $\omega_2(\delta) = \max\{\sqrt{\omega(\delta, t, \psi)}, \delta^{\alpha/2}\}$. □

For $\psi \in BV[0,2\pi]$ we define $\tau[\psi]$ to be $\sup \gamma$ satisfying $\psi \in \Lambda_\gamma$. In particular, $\omega(\delta, \psi) \in \Lambda_{\tau[\psi]-\varepsilon} \setminus \Lambda_{\tau[\psi]+\varepsilon}$.

Theorem 3.3. *Let $F(z)$ be analytic in D,*

$$\log|F(re^{i\varphi})| = \int_0^{2\pi} P_\alpha(r, \varphi - t)\, d\psi(t),$$

where $\psi \in BV[0,2\pi]$, $\tau[\psi] = \tau \in [0,1)$. Then $\rho_M[F] = \alpha + 1 - \tau$, $\rho_T[F] \le \alpha$. If, in addition, ψ is not absolutely continuous, then $\rho_T[F] = \alpha$.

Corollary 3.1. *Suppose that the conditions of Theorem* 3.3 *hold, and $\tau = 0$. Then $\rho_M[F] = \rho_T[F] + 1 = \alpha + 1$.*

Proof of Theorem 3.3. First, let $\tau \in (0,1)$. By Theorem 1.1

$$\sup_{r<1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})|\, d\varphi < +\infty.$$

Since $\omega(\delta, \psi) \in \Lambda_{\tau-\varepsilon} \setminus \Lambda_{\tau+\varepsilon}$, $0 < \varepsilon \le \min\{\tau, 1-\tau\}$, applying Theorem 1.1 again, we have

$$\log M(r, F) = B(r, \log|F|) = O((1-r)^{\tau-\alpha-1-\varepsilon}),$$

$$\log M(r,F) \neq O((1-r)^{\tau-\alpha-1+\varepsilon}), \quad r \uparrow 1,$$

i.e. $\rho_M[F] = \alpha + 1 - \tau$.

Further,

$$\begin{aligned} T(r,f) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi} P_\alpha(r,\varphi-t)\,d\psi(t)\right)^+ d\varphi \\ &\leq \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} P_\alpha^+(r,\varphi-t)\,d\psi(t)\,d\varphi \\ &\leq \frac{1}{4\pi^2}\int_0^{2\pi} d\psi(t)\int_0^{2\pi}\frac{2}{|1-re^{i\theta}|^{\alpha+1}}\,d\theta \\ &\qquad\qquad = \begin{cases} O((1-r)^{-\alpha}), & \alpha > 0, \\ O\left(\log\frac{1}{1-r}\right), & \alpha = 0, \end{cases} \quad r \uparrow 1, \end{aligned}$$

i.e. $\rho_T[f] \leq \alpha$.

In order to complete the proof of Theorem 3.3 we need the following result of F. A. Shamoian [12], which compares the classes A_α and A_α^*.

Theorem E ([12, Theorem 3]). *The function*

$$F(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} S_\alpha(ze^{-i\theta})\,d\psi(\theta)\right\} \in A_\alpha^*$$

if and only if: 1) ψ *is absolutely continuous;*

$$2) \int_0^{2\pi}\int_0^{2\pi}\frac{|\psi(\theta+t)-2\psi(\theta)+\psi(\theta-t)|}{t^2}\,dt\,d\theta < +\infty.$$

As we noted above $F \in A_\alpha^*$ if and only if $T(r,F)$ belongs to the convergence class of order α. Therefore, if ψ is not absolutely continuous, F has the growth at least of the divergence class of order α, i.e. $\rho_T[F] \geq \alpha$.

If $\tau = 0$, then similarly one can deduce that $\rho_T[f] \leq \alpha$. Since $\omega(\delta;\psi) \notin \Lambda_\varepsilon$, $\varepsilon > 0$,

$$\log M(r,f) \neq O((1-r)^{-\alpha-1+\varepsilon}), \quad r \uparrow 1,$$

i.e. $\rho_M[F] \geq \alpha + 1$. Using the inequality $\rho_M[F] \leq \rho_T[F] + 1$, we obtain $\rho_M[F] = \rho_T[F] + 1 = \alpha + 1$, i.e. the statement of the corollary. □

Remark 3.1. The condition $\tau < 1$ in Theorem 3.3 is essential. In fact, by the Cauchy theorem on residues

$$\int_0^{2\pi} \frac{d\theta}{(1 - e^{-i\theta} z)^n} = 2\pi, \qquad n \in \mathbb{N},\ z \in D.$$

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