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A class of periodic integral equations with numerical solving by a fully discrete projection method

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Abstract. For a class of integral periodic equations of the first kind the problem of stable approximate solving is considered. The error estimates in the metric of Sobolev spaces for a fully discrete projection method with two discretization parameters are established. For choosing the level of discretization a balancing principle is used.

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Introduction

As is known elliptic pseudodifferential equations are frequently found in various problems of natural sciences that can be described by a boundary value problems such as Laplace, Neumann or Helmholz equations. Moreover by solving such equations the special class of periodic integral equations arises that contains the elliptic pseudodifferential equations. Such problems are well-known and were investigated, for example, in [2,4,7]. It is known that the mentioned problems are unstable by direct solving in the case of perturbed data but can be selfregularized by discretization on the appropriate pair of spaces. Therefore to obtain a good approximation it is necessary to find a fitting discretization parameter that is considered as regularization parameter. A too large discretization parameter leads to instability in approximation and a too small one gives high error bound. Thus it is necessary to use special techniques for choosing an appropriate discretization level. In this paper a balancing principle is proposed for this purpose. Due to this principle, our method does not need knowledge of the smoothness level for the exact solution.

For an approximative solving the mentioned periodic integral equations fully discrete collocation and projection methods are widely used (for a detailed review on this topic see [7]). In the framework of the paper only fully discrete projection methods will be considered. Moreover here we apply some modification of it that was firstly proposed for solving the integral Symm equation (see Example 1.1) in [4]. Further, such a way for discretization was investigated in [8] by solving the same equation but on the scale of Sobolev spaces. In the paper this approach will be applied on wide class of pseudodifferential equation with saving the order of accuracy.

1. Statement of the problems

In the space $L_2(0, 1)$ we consider the following integral equation

$$\mathcal{A}u(t) = f(t), \quad t \in [0, 1], \tag{1.1}$$

where f is a 1- periodic function and the operator $\mathcal{A}$ has the form

$$\mathcal{A} = \sum_{p=0}^{q} A_p, \quad A_p u(t) = \int_0^1 k_p(t - s) a_p(t, s) u(s) \, ds. \tag{1.2}$$

Let's denote by $C^\infty = C^\infty([0, 1]^2)$ the space C^∞ of infinite smooth 1-biperiodic functions of two variables. Suppose that $a_p \in C^\infty([0, 1]^2)$, $p = 0, \ldots, q$, and

$$a_0(t, t) \neq 0, \quad \forall t \in [0, 1]. \tag{1.3}$$

Moreover, assume that $k_p(t)$ is a 1-periodic function with known Fourier coefficients $\hat{k}_p(n)$ with respect to the trigonometric basis. Additionally, we suppose that for some $\alpha \in \mathbb{R}$ and $\beta > 0$ the following inequalities

$$c_{00}|n|^\alpha \leq |\hat{k}_0(n)| \leq c_0|n|^\alpha, \quad n \in \mathbb{Z}/0, \tag{1.4}$$

$$|\hat{k}_0(n) - \hat{k}_0(n - 1)| \leq c\underline{n}^{\alpha-\beta}, \quad n \in \mathbb{Z}, \tag{1.5}$$

$$|\hat{k}_p(n)| \leq c\underline{n}^{\alpha-\beta}, \quad n \in \mathbb{Z}, \; p = 1, \ldots, q, \tag{1.6}$$

hold true, where c, c_0, $c_{00} > 0$ and

$$\underline{n} = \begin{cases} |n|, & n \in \mathbb{Z}/0 \\ 1, & n = 0. \end{cases}$$

Denote by H^{λ_1} and H^{λ_1,λ_2}, $-\infty < \lambda_1, \lambda_2 < \infty$, Sobolev spaces of 1-periodic and 1-biperiodic functions with the norms

$$\|u\|_{\lambda_1} := \left(\sum_{n\in\mathbb{Z}} |\underline{n}|^{2\lambda_1} |\hat{u}(n)|^2 \right)^{1/2} < \infty,$$

$$\|a\|_{\lambda_1,\lambda_2} := \left(\sum_{(k,l)\in\mathbb{Z}^2} |\underline{k}|^{2\lambda_1} |\underline{l}|^{2\lambda_2} |\hat{a}(k,l)|^2 \right)^{1/2} < \infty$$

respectively. Here

$$\hat{u}(n) = \int\limits_0^1 e_{-n}(t)u(t)\,dt, \quad \hat{a}(k,l) = \int\limits_0^1 \int\limits_0^1 e_{-k}(t)e_{-l}(s)a(t,s)\,dt\,ds$$

are the Fourier coefficients of the functions $u(t)$ and $a(t,s)$ with respect to the trigonometric basis $\{e_k\}_{k=-\infty}^{+\infty}$, where $e_k(t) = e^{i2\pi kt}$, $t \in [0,1]$. Further we will consider the equation (1.1) in the scale H^{λ_1}.

Note, that in the general case the operator $\mathcal{A} : H_0 \to H_0$ is unstable and the corresponding equation in space H_0 should be regularized. Following [7, Ch. 7], we write the operator $\mathcal{A}$ (1.2) as

$$\mathcal{A} = D + A_0' + \sum_{p=1}^q A_p, \tag{1.7}$$

where $D \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ is performing the isomorphism between the spaces H^λ and $H^{\lambda-\alpha}$ and the operators $A_0', A_p \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha+\beta})$, $p = 1,\ldots,q$, are compact on the pair of spaces H^λ and $H^{\lambda-\alpha}$. This yields the statement that the operator $\mathcal{A}$ performs the isomorphism between H^λ and $H^{\lambda-\alpha}$ and the equation (1.1) is uniquely solvable. Thereafter there are constants $c_\lambda', c_\lambda'' > 0$, such that for any $v \in H^\lambda$ the following relation

$$c_\lambda' \|v\|_\lambda \le \|\mathcal{A}v\|_{\lambda-\alpha} \le c_\lambda'' \|v\|_\lambda \tag{1.8}$$

holds true.

Further we will assume that the exact solution of equation (1.1) belongs to some Sobolev spaces, namely $u \in H^\mu$ for some $\mu > \alpha + 1/2$ and $\|u\|_\mu \le 1$. Then due to (1.8) we have that $f \in H^{\mu-\alpha}$ and $\|f\|_{\mu-\alpha} \le c_\mu''$.

Note that the set of classical elliptic pseudodifferential equations satisfies the conditions (1.3)–(1.6) (see for detail [6]). Below, we cite examples of some equations that satisfy the conditions (1.3)–(1.6).

Example 1.1. The typical example of an equation from the class under consideration is the integral Symm equation

$$\mathcal{A}u(t) := \int_0^1 k_0(t-s)u(s)\,ds + \int_0^1 a_1(t,s)u(s)\,ds = f(t), \qquad (1.9)$$

$$k_0(t-s) = \log|\sin\pi(t-s)|, \qquad (1.10)$$

$$a_1(t,s) = \begin{cases} \log\frac{|x(t)-x(s)|}{|\sin\pi(t-s)|}, & t \neq s \\ \log(|x'(t)/\pi|), & t = s, \end{cases}$$

where $x(t)$ is some function such that $|x'(t)| \neq 0$ for $t \in [0,1]$. It is well-known that the kernel $a_1(t,s)$ of the operator A_1 is a C^∞-smooth and 1-biperiodic function and Fourier coefficients of $k_0(t)$ have the form

$$\hat{k}_0(n) = \begin{cases} \frac{1}{2|n|}, & n \in \mathbb{Z}/0 \\ \log 2, & n = 0. \end{cases}$$

It is evident that the conditions (1.3)-(1.6) are satisfied for $a_0(t,s) = k_1(t,s) \equiv 1$, $\alpha = -1$ and any $\beta > 0$.

Example 1.2. The integral equation

$$\int_0^1 |x(t)-x(s)|^2 \log|x(t)-x(s)|u(s)\,ds = f(t), \quad t \in [0,1],$$

arises in the solution of biharmonic Dirichlet problems in a bounded domain with smooth Jordan boundary (for more detailed information, see for example, [1], [7, Ch. 6]). We rewrite the equation in the form

$$\int_0^1 k_0(t-s)a_0(t,s)u(s)\,ds + \int_0^1 a_1(t,s)u(s)\,ds = f(t),$$

where

$$a_0(t,s) = \frac{|x(t)-x(s)|^2}{\sin^2\pi(t-s)} \quad \text{for} \quad t \neq s, \quad a_0(t,t) = \frac{|x'(t)|^2}{\pi^2},$$

$$a_1(t,s) = |x(t)-x(s)|^2 \log\frac{|x(t)-x(s)|}{|\sin\pi(t-s)|} \quad \text{for} \quad t \neq s, \quad a_1(t,t) \equiv 0,$$

$$k_0(t) = \sin^2\pi t \log|\sin\pi t|.$$

The Fourier coefficients k_0 are known and have the following form

$$\hat{k}_0(0) = -\frac{1}{2}\log 2 + \frac{1}{4}, \quad \hat{k}_0(\pm 1) = \frac{1}{4}\log 2 - \frac{3}{16},$$

$$\hat{k}_0(n) = \frac{1}{4|n|(n^2 - 1)}, \quad |n| \geq 2.$$

It is easy to see that the conditions (1.4)–(1.6) are satisfied for $\alpha = -3$, $\beta = 1$. Thus, the equation under consideration is also included in the investigated class of problems.

To make the smoothness properties of functions a_p, $p = 0, \ldots, q$, more precise we introduce the space of Gevre's functions of Roumieu type (see [3, p. 112]):

$$G_{\eta_1,\eta_2} = \left\{ a \in C^\infty : \|a\|_{\eta_1,\eta_2}^2 \right.$$

$$\left. := \sum_{k,l=-\infty}^{\infty} |\hat{a}(k,l)|^2 e^{2\eta_2(|k|^{1/\eta_1} + |l|^{1/\eta_1})} < \infty \right\}, \quad \eta_1, \eta_2 > 0. \quad (1.11)$$

Note that with $\eta_1 = 1$ by (1.11) it follows that the function $a(t,s)$ has analytic continuations in both variables into the strip $\{z : z = t + is, |s| < \frac{\eta_2}{2\pi}\}$ of the complex plane. Further we suppose that $a_p \in G_{\eta_1,\eta_2}$, $p = 0, \ldots, q$, for some $\eta_1 \geq 1$ and $\eta_2 > 0$.

2. Auxiliary statements

For the further presentation of our results we will use the following notations. We introduce the n-dimensional subspaces of trigonometric polynomials

$$\mathcal{T}_N = \{u_N : u_N(t) = \sum_{k \in Z_N} c_k e_k(t)\},$$

$$Z_N = \left\{ k : -\frac{N}{2} < k \leq \frac{N}{2}, \ k = 0, \pm 1, \pm 2, \ldots \right\}. \quad (2.1)$$

We denote by P_N and $P_{N,N}$ the orthogonal projectors

$$P_N u(t) = \sum_{k \in Z_N} \hat{u}(k) e_k(t) \in \mathcal{T}_N,$$

$$P_{N,N} a(t,s) = \sum_{l,k \in Z_N} \hat{a}(k,l) e_k(t) e_l(s) \in \mathcal{T}_N \times \mathcal{T}_N,$$

and by Q_N and $Q_{N,N}$ the interpolation projectors, such that $Q_N u(t) \in \mathcal{T}_N$, $Q_{N,N} a(t,s) \in \mathcal{T}_N \times \mathcal{T}_N$ and on the uniform grid we have

$$(Q_N u)(jN^{-1}) = u(jN^{-1}), \quad j = 1, 2, \ldots, N,$$

$$(Q_{N,N} a)(jN^{-1}, iN^{-1}) = a(jN^{-1}, iN^{-1}), \quad j, i = 1, 2, \ldots, N.$$

It is well-known (see, for example, [7, Ch.8]), that

$$\|u - P_N u\|_\lambda \leq \left(\frac{N}{2}\right)^{\lambda - \mu} \|u\|_\mu, \quad \lambda \leq \mu, \ u \in H^\mu, \tag{2.2}$$

$$\|u - Q_N u\|_\lambda \leq c_{\lambda,\mu} N^{\lambda - \mu} \|u\|_\mu, \quad 0 \leq \lambda \leq \mu, \ \mu > \frac{1}{2}, \ u \in H^\mu, \tag{2.3}$$

where $c_{\lambda,\mu} = \left(\frac{1}{2}\right)^{\lambda - \mu} \gamma_\mu$ and $\gamma_\mu = \left(1 + 2\sum_{j=1}^{\infty} \frac{1}{j^{2\mu}}\right)^{\frac{1}{2}}$.

Moreover, for any $v_N \in \mathcal{T}_N$ according to the inverse Bernshtein inequality it holds

$$\|v_N\|_\mu \leq \left(\frac{N}{2}\right)^{\mu - \lambda} \|v_N\|_\lambda, \quad \lambda \leq \mu. \tag{2.4}$$

3. Fully discrete projection method

Let's approximate $\mathcal{A}$ in the following way

$$\mathcal{A}_M = \sum_{p=0}^{q} A_{p,M}, \tag{3.1}$$

where the operators $A_{p,M}, p = 0, \ldots, q$, have the form

$$A_{p,M} u(t) = \int_0^1 k_p(t-s) a_{p,M}(t,s) u(s) \, ds, \quad a_{p,M} = Q_{M,M} a_p \tag{3.2}$$

We approximate the right-hand side of equation (1.1) as follows

$$f_N := Q_N f,$$

where $N > M$. The main idea of the fully discrete projection method (FDPM) for equation (1.1) consists in solving the equation

$$P_N \mathcal{A}_M u_{M,N} := \sum_{p=0}^{q} P_N A_{p,M} u_{M,N} = Q_N f, \tag{3.3}$$

where $A_{p,M}$ have the form (3.2) and $u_{M,N} \in \mathcal{T}_N$ is considered as approximate solution of (1.1). Note that in virtue of (1.4) and (1.5) it holds true that $A_{0,M} \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ and $A_{p,M} \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha+\beta})$, $p = 1, \ldots, q$.

Let's introduce the following auxiliary functions

$$z_1 := z_1(\lambda'', \lambda_\nu) = \gamma_{\lambda''+1} + \gamma_{\lambda_\nu+1} + \gamma_{\lambda''+1}\gamma_{\lambda_\nu+1},$$

$$z_2 := z_2(\lambda'', \lambda) = \begin{cases} 2^{|\lambda''|+|\lambda|+2}\gamma_{|\lambda''|}\gamma_{|\lambda|}, & |\lambda''| > \frac{1}{2}, \ |\lambda| > \frac{1}{2} \\ 2^{\nu+|\lambda''|+2}\gamma_\nu\gamma_{|\lambda|}, & |\lambda''| \le \frac{1}{2}, \ |\lambda| > 1/2 \\ 2^{|\lambda''|+\nu+2}\gamma_\nu\gamma_{|\lambda''|}, & |\lambda''| > \frac{1}{2}, \ |\lambda| \le 1/2 \\ 2^{2(\nu+1)}\gamma_\nu^2, & |\lambda''| \le \frac{1}{2}, \ |\lambda| \le 1/2, \end{cases}$$

$$z_3 := z_3(\lambda - \alpha) = c_0 \begin{cases} 2^{\lambda-\alpha+1}\gamma_{\lambda-\alpha}, & \lambda - \alpha > \frac{1}{2} \\ 2^{\lambda-\alpha+1}\gamma_\nu, & 0 \le \lambda - \alpha \le \frac{1}{2} \\ 2^{|\lambda-\alpha|}\gamma_\nu, & \lambda - \alpha \le 0, \end{cases}$$

where

$$\lambda'' = \begin{cases} \max\{\lambda', \nu\}, & \lambda - \alpha > \frac{1}{2} \\ \lambda', & \lambda - \alpha \le \frac{1}{2}, \end{cases} \qquad \lambda' = \begin{cases} \lambda - \alpha, & \lambda - \alpha > \frac{1}{2}, \\ \nu, & 0 \le \lambda - \alpha \le \frac{1}{2}, \\ |\lambda - \alpha| + \nu, & \lambda - \alpha \le 0, \end{cases}$$

$\lambda_\nu = \max\{|\lambda|, \nu\}$ and $\nu > 1/2$ is some arbitrary parameter.

The following result is taken from [9] and describes some approximative property of the operator discretized by scheme (3.1).

Lemma 3.1 ([9]). *Let the conditions (1.4)–(1.6) be satisfied and the operator $\mathcal{A}_M$ has the form (3.1). Then for all $M : M \ge 2\max\{(\frac{\eta_1(\lambda''+1)}{\eta_2})^{\eta_1},$ $(\frac{\eta_1(\lambda_\nu+1)}{\eta_2})^{\eta_1}\}$ and any $\nu > 1/2$ it holds true*

$$\|(\mathcal{A} - \mathcal{A}_\mathcal{M})v\|_{\lambda-\alpha} \le z_4 e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}} \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} \|v\|_\lambda,$$

where $z_4 := z_4(\lambda, \nu, \alpha) = c_2 z_1 z_2 z_3$, $c_2 = (q+1)\max_p\{\|a_p\|_{\eta_1,\eta_2}\}$.

Further, we rewrite the stability inequality for the operator $P_N\mathcal{A}$ that was established in [7].

Lemma 3.2 ([7, Lemma 9.8.2.]). *Let the conditions (1.3)–(1.6) be fulfilled. Then for any $\lambda \in \mathbb{R}$ and $v \in \mathcal{T}_N$ it holds true that*

$$\|v\|_\lambda \le d_\lambda \|P_N\mathcal{A}v\|_{\lambda-\alpha},$$

where $d_\lambda > 0$ is some constant.

The estimate of the accuracy for FDPM on the class of problems (1.1)–(1.6) with unperturbed input data is established in the following assertion (for detail see [9]).

Theorem 3.1 ([9])**.** *Let the conditions* (1.3)–(1.6) *be fulfilled, and the operator $\mathcal{A}_M$ has the form* (3.1)*. Then for any $\nu > 1/2$, $\lambda \le \mu$, $\mu > \alpha + 1/2$ and for all*

$$M : \left[M \ge 2 \max \left\{ \left(\frac{\eta_1(\lambda'' + 1)}{\eta_2} \right)^{\eta_1}, \left(\frac{\eta_1(\lambda_\nu + 1)}{\eta_2} \right)^{\eta_1} \right\}, \right.$$
$$\left. z_4 e^{-2\eta_2 \left(\frac{M}{2} \right)^{1/\eta_1}} \left(\frac{M}{2} \right)^{\lambda'' + \lambda_\nu} \le 1/2 \right]$$

it holds true that

$$\|u - u_{M,N}\|_\lambda \le z_5 \left(\frac{N}{2} \right)^{\lambda - \mu} + 2d_\lambda z_4 e^{-2\eta_2 \left(\frac{M}{2} \right)^{1/\eta_1}} \left(\frac{M}{2} \right)^{\lambda'' + \lambda_\nu}, \quad (3.4)$$

where $z_5 := z_5(\lambda) = 1 + 4d_\lambda c_\lambda''$.

Following [7], we suppose that instead of the functions $a_p(t, s)$, $p = 0, \dots, q$, and $f(t)$ we are given only some their perturbations $a_{p,\varepsilon}(t, s)$, $p = 0, \dots, q$, and $f_\delta(t)$ such that in the points of uniform grids we have

$$M^{-2} \left(\sum_{i,j=1}^M |a_{p,\varepsilon}(iM^{-1}, jM^{-1}) - a_p(iM^{-1}, jM^{-1})|^2 \right)^{\frac{1}{2}} \le \varepsilon, \ p = 0, \dots, q,$$

$$N^{-1} \left(\sum_{j=1}^N |f_\delta(jN^{-1}) - f(jN^{-1})|^2 \right)^{1/2} \le \delta \|f\|_{\mu - \alpha}.$$

It is easy to show (see, for example, [7, p. 100]), that the mentioned estimates are equivalent to

$$\|Q_{M,M}(a_p - a_{p,\varepsilon})\|_{0,0} \le \varepsilon, \quad p = 0, \dots, q, \quad (3.5)$$

$$\|Q_N(f_\delta - f)\|_0 \le \delta \|f\|_{\mu - \alpha} \quad (3.6)$$

respectively. Then, taking into account the perturbation of the input data the FDPM has the form

$$P_N \mathcal{A}_{M,\varepsilon} u_{M,N,\varepsilon,\delta} = Q_N f_\delta, \quad (3.7)$$

where $u_{M,N,\varepsilon,\delta} \in \mathcal{T}_N$ and $\mathcal{A}_{M,\varepsilon} = \sum_{p=0}^q A_{p,\varepsilon,M}$, $A_{p,\varepsilon,M} v(t) = \int_0^1 k(t - s) \times Q_{M,M} a_{p,\varepsilon}(t, s) v(s) \, ds$.

The above described variants of FDPM (3.3) and (3.7) for $M = N$ are widely used for solving periodic integral equations (1.1) with conditions (1.4)–(1.6) (for the detail see [7]). The distinction of our approach to solution from classical one is that in (3.3) and (3.7) the idea of separating discretization levels for the right-hand side and the operator will be realized. Now our aim is to find error estimates for FDPM (3.7) with separated discretization levels on a wide class of equations (1.1) with the conditions (1.4)–(1.6) in the metric of the spaces H^λ, $\lambda < \mu$. Moreover we pose the problem to consider both cases, a priori and a posteriori, for choosing the discretization levels. The a posteriori way allows us to choose the discretization levels without precise knowledge of the smoothness of the exact solution and to keep the same order of error bound as in the a priori case. In the a priori case such approach allows to reduce the information expenses (i.e. the volume of discrete information in view of the values of the functions $f(t)$ and $a_p(t, s)$ in the points of the uniform grid) that is necessary for achieving the given accuracy in comparison with similar estimates known earlier in [7]. This is possible due to the separation of the discretization parameters for the right-hand side and the operator of the equation under consideration.

To achieve the goal of our investigation we state at first some auxiliary estimates.

Lemma 3.3. *For any $\nu > 1/2$ and $\lambda \geq \alpha$ it holds true that*

$$\|\mathcal{A}_M - \mathcal{A}_{M,\varepsilon}\|_{H^\lambda \to H^{\lambda-\alpha}} \leq z_2 z_3(q+1)\left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu}\varepsilon.$$

Proof. According to Lemma 3 in [9] we have

$$\|Au\|_{\lambda-\alpha} \leq z_2 z_3 \|a\|_{\lambda'',\lambda_\nu}\|u\|_\lambda,$$

where $(Au)(t) = \int_0^1 k(t-s)a(t,s)u(s)\,ds$.

Now using the above estimate and inequalities (2.4), (3.5) we find

$$\|(\mathcal{A}_M - \mathcal{A}_{M,\varepsilon})v\|_{\lambda-\alpha} \leq \sum_{p=0}^{q}\left\|\int_0^1 k(t-s)Q_{M,M}(a_{p,\varepsilon}-a_p)(t,s)v(s)\,ds\right\|_{\lambda-\alpha}$$

$$\leq z_2 z_3 \sum_{p=0}^{q}\|Q_{M,M}(a_{p,\varepsilon}-a_p)\|_{\lambda'',\lambda_\nu}\|v\|_\lambda$$

$$\leq z_2 z_3(q+1)\left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu}\varepsilon\|v\|_\lambda,$$

which is the required result.   □

Lemma 3.4. *Let the conditions of Theorem* 3.1 *be fulfilled and* $u_{M,N,\delta} \in \mathcal{T}_N$ *be the solution of the equation*

$$P_N \mathcal{A}_M u_{M,N,\delta} = Q_N f_\delta.$$

Then, for any $\lambda \in [\alpha, \mu]$ *and* M *such that*

$$M : \left[M \geq 2 \max \left\{ \left(\frac{\eta_1(\lambda'' + 1)}{\eta_2} \right)^{\eta_1}, \left(\frac{\eta_1(\lambda_\nu + 1)}{\eta_2} \right)^{\eta_1} \right\}, \right.$$
$$\left. \max\{1, d_\lambda\} z_4 e^{-2\eta_2 \left(\frac{M}{2} \right)^{1/\eta_1}} \left(\frac{M}{2} \right)^{\lambda'' + \lambda_\nu} \leq 1/2 \right], \quad (3.8)$$

it holds true that

$$\|u_{M,N,\delta} - u_{M,N}\|_\lambda \leq \overline{c}_\lambda \left(\frac{N}{2} \right)^{\lambda - \alpha} \delta, \quad (3.9)$$

where $\overline{c}_\lambda = 2 d_\lambda c_\mu''$.

Proof. Using Lemmas 3.1 and 3.2, inequalities (2.4) and (3.6) we find

$$\|u_{M,N} - u_{M,N,\delta}\|_\lambda \leq d_\lambda \|P_N \mathcal{A}(u_{M,N} - u_{M,N,\delta})\|_{\lambda - \alpha}$$
$$\leq d_\lambda \|P_N \mathcal{A}_M(u_{M,N} - u_{M,N,\delta})\|_{\lambda - \alpha} + d_\lambda \|P_N (\mathcal{A} - \mathcal{A}_M)(u_{M,N} - u_{M,N,\delta})\|_{\lambda - \alpha}$$
$$\leq d_\lambda \|Q_N f - Q_N f_\delta\|_{\lambda - \alpha} + d_\lambda \|(\mathcal{A} - \mathcal{A}_M)(u_{M,N} - u_{M,N,\delta})\|_{\lambda - \alpha}$$
$$\leq d_\lambda \left(\frac{N}{2} \right)^{\lambda - \alpha} \delta \|f\|_{\mu - \alpha} + d_\lambda z_4 e^{-2\eta_2 \left(\frac{M}{2} \right)^{1/\eta_1}} \left(\frac{M}{2} \right)^{\lambda'' + \lambda_\nu} \|u_{M,N} - u_{M,N,\delta}\|_\lambda.$$

Therefrom, taking into account the condition of the lemma the estimate yields

$$\|u_{M,N,\delta} - u_{M,N}\|_\lambda$$
$$\leq \frac{d_\lambda}{1 - d_\lambda z_4 e^{-2\eta_2 \left(\frac{M}{2} \right)^{1/\eta_1}} \left(\frac{M}{2} \right)^{\lambda'' + \lambda_\nu}} \left(\frac{N}{2} \right)^{\lambda - \alpha} \delta \|f\|_{\mu - \alpha}$$
$$\leq \overline{c}_\lambda \left(\frac{N}{2} \right)^{\lambda - \alpha} \delta.$$
$$\square$$

Lemma 3.5. *Let* $M \in \mathbb{N}$ *satisfy the condition* (3.8) *and moreover*

$$M : 2\varepsilon < z_1 \max_p \{\|a_p\|_{\eta_1, \eta_2}\} e^{-2\eta_2 \left(\frac{M}{2} \right)^{1/\eta_1}}. \quad (3.10)$$

Then, for all $v \in \mathcal{T}_N$ *it holds true that*

$$\|v\|_\lambda \leq d_\lambda' \|P_N \mathcal{A}_{M,\varepsilon} v\|_{\lambda - \alpha},$$

where $d_\lambda' = 4 d_\lambda$.

Proof. Using Lemmas 3.1, 3.2 and 3.3 we find

$$\|v\|_\lambda \le d_\lambda \|P_N \mathcal{A} v\|_{\lambda-\alpha}$$
$$\le d_\lambda (\|P_N(\mathcal{A} - \mathcal{A}_M)v\|_{\lambda-\alpha} + \|P_N(\mathcal{A}_\mathcal{M} - \mathcal{A}_{M,\varepsilon})v\|_{\lambda-\alpha} + \|P_N \mathcal{A}_{M,\varepsilon} v\|_{\lambda-\alpha})$$
$$\le d_\lambda (z_4 e^{-2\eta_2 \left(\frac{M}{2}\right)^{1/\eta_1}} \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} + z_2 z_3 (q+1) \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} \varepsilon) \|v\|_\lambda$$
$$+ d_\lambda \|P_N \mathcal{A}_{M,\varepsilon} v\|_{\lambda-\alpha}.$$

Hence, due to the conditions of lemma, we obtain

$$\|v\|_\lambda$$
$$\le \frac{d_\lambda}{1 - d_\lambda (z_4 e^{-2\eta_2 \left(\frac{M}{2}\right)^{1/\eta_1}} \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} + (q+1) z_2 z_3 \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} \varepsilon)}$$
$$\times \|P_N \mathcal{A}_{M,\varepsilon} v\|_{\lambda-\alpha} \le 4 d_\lambda \|P_N \mathcal{A}_{M,\varepsilon} v\|_{\lambda-\alpha},$$

which proves the statement.   □

Lemma 3.6. *Let the conditions of Lemma* 3.5 *be fulfilled, then for any* $2 \le N \le \delta^{-\frac{1}{\lambda-\alpha}}$ *it holds*

$$\|u_{M,N,\delta} - u_{M,N,\delta,\varepsilon}\|_\lambda \le z_6 \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} \varepsilon,$$

where $z_6 := z_6(\lambda'', \lambda, \alpha) = c_3 d'_\lambda (q+1) z_2 z_3$, *and* $c_3 = 2 + z_5 + 2^{\alpha-\lambda} \bar{c}_\lambda$.

Proof. Using Lemmas 3.3 and 3.5 and the fact that the elements $u_{M,N,\delta} \in \mathcal{T}_N$ and $u_{M,N,\delta,\varepsilon} \in \mathcal{T}_N$ are solutions of the equations $P_N \mathcal{A}_M u_{M,N,\delta} = Q_N f_\delta$ and $P_N \mathcal{A}_{M,\varepsilon} u_{M,N,\delta,\varepsilon} = Q_N f_\delta$ respectively, we find

$$\|u_{M,N,\delta} - u_{M,N,\delta,\varepsilon}\|_\lambda \le d'_\lambda \|P_N \mathcal{A}_{M,\varepsilon}(u_{M,N,\delta} - u_{M,N,\delta,\varepsilon})\|_{\lambda-\alpha}$$
$$= d'_\lambda \|P_N(\mathcal{A}_{M,\varepsilon} - \mathcal{A}_M) u_{M,N,\delta}\|_{\lambda-\alpha}$$
$$\le d'_\lambda (q+1) z_2 z_3 \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} \varepsilon \|u_{M,N,\delta}\|_\lambda. \quad (3.11)$$

It remains to estimate the norm of the element $u_{M,N,\delta}$. So, from Theorem 3.1 and Lemma 3.4 for all $\lambda : \alpha < \lambda \le \mu$, it follows

$$\|u_{M,N,\delta}\|_\lambda \le \|u\|_\lambda + \|u - u_{M,N}\|_\lambda + \|u_{M,N} - u_{M,N,\delta}\|_\lambda$$
$$\le \|u\|_\mu + \left(z_5 \left(\frac{N}{2}\right)^{\lambda-\mu} + 2 d_\lambda z_4 (\lambda - \alpha) e^{-2\eta_2 \left(\frac{M}{2}\right)^{1/\eta_1}} \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu}\right)$$
$$+ \bar{c}_\lambda \left(\frac{N}{2}\right)^{\lambda-\alpha} \delta. \quad (3.12)$$

Therefrom due to (3.8) for all $2 \leq N \leq \delta^{-1/(\lambda-\alpha)}$ it holds true that

$$\|u_{M,N,\delta}\|_\lambda \leq c_3.$$

Then from (3.11) the statement of the lemma follows.   □

4.   Selection of the discretization levels

Generalizing the results of the previous section we rewrite the general error estimate of FDPM. By virtue of Theorem 3.1, Lemmas 3.4 and 3.6, the accuracy of the method (3.7) is bounded by

$$\|u - u_{N,M,\delta,\varepsilon}\|_\lambda$$
$$\leq \|u - u_{M,N}\|_\lambda + \|u_{M,N} - u_{M,N,\delta}\|_\lambda + \|u_{M,N,\delta} - u_{M,N,\delta,\varepsilon}\|_\lambda$$
$$z_5 \left(\frac{N}{2}\right)^{\lambda-\mu} + 2d_\lambda z_4 e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}} \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} + \bar{c}_\lambda \left(\frac{N}{2}\right)^{\lambda-\alpha} \delta$$
$$+ z_6 \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} \varepsilon. \quad (4.1)$$

Further we consider the problem to select such discretization levels N and M to minimize the error bound (4.1). At the same time we consider both cases, namely smoothness parameter μ is known precisely (a priori case) and value μ is unknown (a posteriori case).

4.1.   A priori case

We group the right-hand side (4.1) in two parts:

$$f_1(N,M) := z_5 \left(\frac{N}{2}\right)^{\lambda-\mu} + 2d_\lambda z_4 e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}} \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu}$$

(decreasing for $N, M \to \infty$) and

$$f_2(N,M) := \bar{c}_\lambda \left(\frac{N}{2}\right)^{\lambda-\alpha} \delta + z_6 \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} \varepsilon$$

(increasing for $N, M \to \infty$). It is easy to see that the minimal value of the function $f_1 + f_2$ is attained in the points $\bar{N}$ and $\bar{M}$ that satisfy the conditions

$$z_5 \left(\frac{N}{2}\right)^{\lambda-\mu} = \bar{c}_\lambda \left(\frac{N}{2}\right)^{\lambda-\alpha} \delta,$$

$$2d_\lambda z_4 e^{-2\eta_2\left(\frac{M}{2}\right)^{1/\eta_1}} \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} = z_6 \left(\frac{M}{2}\right)^{\lambda''+\lambda_\nu} \varepsilon.$$

This allows us to find a priori rule for choosing the discretization parameters N, M that depend on the smoothness parameter μ.

Further we denote by $[q]$ the whole part of the number q and formulate the theorem for establishing an a priori rule for choosing discretization parameters.

Theorem 4.1. *Let the conditions* (1.3)–(1.6) *be satisfied and for the perturbation of the input data the inequalities* (3.6) *and* (3.5) *hold true. Then for any* $\nu > 1/2$, $\lambda \in [\alpha, \mu]$, $\mu > \alpha + 1/2$, *with choosing the discretization parameters according to the rules*

$$\bar{M} = \left[2 \left(\frac{1}{2\eta_2} \log \frac{z_8}{\varepsilon} \right)^{\eta_1} \right], \tag{4.2}$$

$$\bar{N} = \left[2 \left(\frac{\bar{c}_\lambda \delta}{z_5} \right)^{\frac{1}{\alpha - \mu}} \right] \tag{4.3}$$

the error bound of the method (3.7) *has the form*

$$\|u - u_{\bar{N}, \bar{M}, \delta, \varepsilon}\|_\lambda \leq 2(\bar{c}_\lambda)^{\frac{\lambda - \mu}{\alpha - \mu}} z_5^{\frac{\lambda - \alpha}{\mu - \alpha}} \delta^{\frac{\mu - \lambda}{\mu - \alpha}} + 2z_7(\lambda'', \lambda, \alpha) \varepsilon \log^{\eta_1(\lambda'' + \lambda_\nu)} \frac{z_8}{\varepsilon}, \tag{4.4}$$

$$z_7(\lambda'', \lambda, \alpha) = 2z_6 \left(\frac{1}{2\eta_2} \right)^{\eta_1(\lambda'' + \lambda_\nu)}, \quad z_8(\lambda'', \lambda_\nu) = \frac{z_1 \max\{\|a\|_{\eta_1, \eta_2}\}}{2c_3}.$$

Proof. Direct substitution of (4.2) and (4.3) into (4.1) gives the statement of Theorem. $\square$

Remark 4.1. Let's check that condition (3.8) is fulfilled if M is chosen according to (4.2). It is evident that

$$z_8(\lambda'', \lambda_\nu) e^{-2\eta_2 \left(\frac{M}{2} \right)^{1/\eta_1}} = \varepsilon.$$

Therefrom we obtain that (3.8) is fulfilled for any $\varepsilon \leq \varepsilon_0$, where

$$\varepsilon_0 \log^{\eta_1(\lambda'' + \lambda_\nu)} \frac{z_8(\lambda'', \lambda_\nu)}{\varepsilon_0} < \frac{(2\eta_2)^{\lambda'' + \lambda_\nu}}{4c_3 \max\{1, d_\lambda\} z_1 z_2 z_3}.$$

On the other side the condition (3.10) is fulfilled for $M = \bar{M}$.

Remark 4.2. On the class of periodic integral equations (1.1) with conditions (1.3)–(1.6) the fully discrete projection method (3.7) for $M = N$ was investigated in [7]. It was obtained that choosing the discretization parameter as

$$N(\varepsilon, \delta) \asymp \min\{\delta^{-\frac{1}{\mu - \alpha}}, \varepsilon^{-1/(\mu - \lambda + \sigma \max(\lambda - \alpha, |\lambda|, \nu))}\}$$

the order accuracy of the method equals

$$O(\delta^{\frac{\mu-\lambda}{\mu-\alpha}} + \varepsilon^{\frac{\mu-\lambda}{\mu-\lambda+\sigma\max(\lambda-\alpha,|\lambda|,\nu)}}),$$

where $0 < \sigma < 1$ is some parameter. It is evident that in the a priori case regarding to the parameter ε the estimate (4.4) is better by the order then in [7].

Remark 4.3. Let us suppose that $\varepsilon \geq c\delta$ and calculate the amount of necessary discrete information for equation (1.1) to realize the proposed method (3.7) with the accuracy (4.4). It is evident that in that case M does not exceed the magnitude $O(\log(N))$. So, for the discretization (3.2) there should be used less then $O(\log^2 N)$ values of kernels $a_{p,\varepsilon}(t,s)$ in the points of the uniform grid. At the same time in the monograph [7] for the realization of the fully discrete projection method (3.7) for $M = N$ the order of discrete information was computed as $O(N \log N)$. The advantage of the method proposed in this paper in comparison with the classical one is evident.

4.2. A posteriori rule

In the case when the parameter μ is unknown the rule (4.3) for choosing N is not suitable. However there are methods that without knowledge of the smoothness μ allow to approximate the optimal value (4.3) while saving the error bound (4.4). One such rule is the balancing principle [5] that we will use for the determination of the best appropriate value of the discretization parameter. Further, we give the description of the mentioned approach in more detail.

Let's denote by

$$D_N = \{N : N = 1, \ldots, N_A, N_A = [\delta^{-\frac{1}{\lambda-\alpha}}]\},$$

the set of possible values of the discretization parameter N. Then according to the balancing principle as appropriate value of N we take the number $\hat{N}$ that satisfies the condition

$$\hat{N} = \min\{N : N \in D_N^+\}, \tag{4.5}$$

where

$$D_N^+ = \left\{ N \in D_N : \|u_{N,\bar{M},\delta,\varepsilon} - u_{j,\bar{M},\delta,\varepsilon}\|_\lambda \right.$$
$$\left. \leq 4\bar{c}_\lambda \left(\frac{j}{2}\right)^{\lambda-\alpha} \delta + 4z_7(\lambda'',\lambda,\alpha)\varepsilon \log^{\eta_1(\lambda''+\lambda_\nu)} \frac{z_8}{\varepsilon}, \quad \forall j > N \right\}.$$

Let's introduce the additional parameter

$$N_* := \min\{N : z_5 \left(\frac{N}{2}\right)^{\lambda-\mu} \leq \overline{c}_\lambda \left(\frac{N}{2}\right)^{\lambda-\alpha} \delta\}. \qquad (4.6)$$

Theorem 4.2. *Let the conditions* (1.3)–(1.6) *and* (3.5), (3.6) *be fulfilled and* $a \in G_{\eta_1,\eta_2}$, $\eta_1 \geq 1, \eta_2 > 0$. *Moreover the discretization parameters* M *and* N *are selected according to* (4.2) *and* (4.5) *respectively. Then for any* $\nu > 1/2$, $\lambda \in [\alpha, \mu]$, $\mu > \alpha + 1/2$ *the error bound of FDPM* (3.7) *is the following*

$$\|u - u_{\hat{N},\bar{M},\delta,\varepsilon}\|_\lambda \leq z_9(\lambda, \alpha)\delta^{\frac{\mu-\lambda}{\mu-\alpha}} + 6z_7(\lambda'', \lambda, \alpha)\varepsilon \log^{\eta_1(\lambda''+\lambda_\nu)} \frac{z_8}{\varepsilon}, \quad (4.7)$$

where $z_9(\lambda, \alpha) = 6(\overline{c}_\lambda)^{\frac{\lambda-\mu}{\alpha-\mu}} z_5^{\frac{\lambda-\alpha}{\mu-\alpha}} \left(\frac{N_\star}{N_\star-1}\right)^{\lambda-\alpha}$.

Proof. Substituting the rule (4.2) into the estimate (4.1) leads to the following relation

$$\|u - u_{N,\bar{M},\delta,\varepsilon}\|_\lambda \leq z_5 \left(\frac{N}{2}\right)^{\lambda-\mu} + \overline{c}_\lambda \left(\frac{N}{2}\right)^{\lambda-\alpha} \delta$$

$$+ 2z_7(\lambda'', \lambda, \alpha)\varepsilon \log^{\eta_1(\lambda''+\lambda_\nu)} \frac{z_8(\lambda'', \lambda_\nu)}{\varepsilon}. \quad (4.8)$$

From (4.8) and (4.6) it follows

$$\|u_{j,\bar{M},\delta,\varepsilon} - u_{N_*,\bar{M},\delta,\varepsilon}\|_\lambda \leq \|u_{j,\bar{M},\delta,\varepsilon} - u\|_\lambda + \|u - u_{N_*,\bar{M},\delta,\varepsilon}\|_\lambda$$

$$\leq z_5 \left(\frac{j}{2}\right)^{\lambda-\mu} + \overline{c}_\lambda \left(\frac{j}{2}\right)^{\lambda-\alpha} \delta + + z_5 \left(\frac{N_*}{2}\right)^{\lambda-\mu} + \overline{c}_\lambda \left(\frac{N_*}{2}\right)^{\lambda-\alpha} \delta$$

$$+ 4z_7(\lambda'', \lambda, \alpha)\varepsilon \log^{\eta_1(\lambda''+\lambda_\nu)} \frac{z_8}{\varepsilon}. \quad (4.9)$$

Therefrom, due to (4.6) for all $j > N_*$, we have

$$\|u_{j,\bar{M},\delta,\varepsilon} - u_{N_*,\bar{M},\delta,\varepsilon}\|_\lambda \leq 2\overline{c}_\lambda \left(\frac{j}{2}\right)^{\lambda-\alpha} \delta + 2\overline{c}_\lambda \left(\frac{N_*}{2}\right)^{\lambda-\alpha} \delta$$

$$+ 4z_7(\lambda'', \lambda, \alpha)\varepsilon \log^{\eta_1(\lambda''+\lambda_\nu)} \frac{z_8}{\varepsilon}$$

$$\leq 4\overline{c}_\lambda \left(\frac{j}{2}\right)^{\lambda-\alpha} \delta + 4z_7(\lambda'', \lambda, \alpha)\varepsilon \log^{\eta_1(\lambda''+\lambda_\nu)} \frac{z_8}{\varepsilon}. \quad (4.10)$$

Taking into account the definition of the set D_N^+ from (4.10) we find that $\hat{N} \leq N_\star$ and $N_\star \in D_N^+$.

From (4.5), (4.6), (4.8) and in virtue of definition of the set D_N^+ we obtain

$$
\begin{aligned}
\|u - u_{\hat{N},\bar{M},\delta,\varepsilon}\|_\lambda &\leq \|u - u_{N_*,\bar{M},\delta,\varepsilon}\|_\lambda + \|u_{\hat{N},\bar{M},\delta,\varepsilon} - u_{N_*,\bar{M},\delta,\varepsilon}\|_\lambda \\
&\leq z_5 \left(\frac{N_*}{2}\right)^{\lambda-\mu} + \bar{c}_\lambda \left(\frac{N_*}{2}\right)^{\lambda-\alpha} \delta + 4\bar{c}_\lambda \left(\frac{N_*}{2}\right)^{\lambda-\alpha} \delta \\
&\quad + 6z_7(\lambda'',\lambda,\alpha)\varepsilon \log^{\eta_1(\lambda''+\lambda_\nu)} \frac{z_8}{\varepsilon} \\
&\leq 6\bar{c}_\lambda \left(\frac{N_*}{2}\right)^{\lambda-\alpha} \delta + 6z_7(\lambda'',\lambda,\alpha)\varepsilon \log^{\eta_1(\lambda''+\lambda_\nu)} \frac{z_8}{\varepsilon}. \quad (4.11)
\end{aligned}
$$

Due to (4.3) and (4.6) we have $N_\star - 1 \leq \bar{N}$. Then by the last inequality from (4.11) it follows

$$
\begin{aligned}
\|u - u_{\hat{N},\bar{M},\delta,\varepsilon}\|_\lambda &\leq 6\bar{c}_\lambda \left(\frac{N_\star - 1}{N_\star - 1}\right)^{\lambda-\alpha} \left(\frac{N_*}{2}\right)^{\lambda-\alpha} \delta \\
&\quad + 6z_7(\lambda'',\lambda,\alpha)\varepsilon \log^{\eta_1(\lambda''+\lambda_\nu)} \frac{z_8}{\varepsilon} \\
&\leq 6\bar{c}_\lambda \left(\frac{N_\star}{N_\star - 1}\right)^{\lambda-\alpha} \left(\frac{\bar{N}}{2}\right)^{\lambda-\alpha} \delta + 6z_7(\lambda'',\lambda,\alpha)\varepsilon \log^{\eta_1(\lambda''+\lambda_\nu)} \frac{z_8}{\varepsilon}.
\end{aligned}
$$

Therefrom, taking into account (4.3), we finally obtain

$$
\begin{aligned}
\|u - u_{\hat{N},\bar{M},\delta,\varepsilon}\|_\lambda &\leq 6(\bar{c}_\lambda)^{\frac{\lambda-\mu}{\alpha-\mu}} z_5^{\frac{\lambda-\alpha}{\mu-\alpha}} \left(\frac{N_\star}{N_\star - 1}\right)^{\lambda-\alpha} \delta^{\frac{\lambda-\mu}{\alpha-\mu}} \\
&\quad + 6z_7(\lambda'',\lambda,\alpha)\varepsilon \log^{\eta_1(\lambda''+\lambda_\nu)} \frac{z_8}{\varepsilon}.
\end{aligned}
$$

The theorem is proved.   □

Remark 4.4. Thus our approach to solve equation (1.1) allows to reduce the error bounds with respect to a perturbation ε in comparison with previous estimates in [7] for both a priori and a posteriori cases. Moreover, applying the balancing principle for choosing the discretization parameter gives the appropriate value for the parameter N without additional information about the smoothness. At the same time comparing the estimates (4.4) and (4.7) we can conclude that the order for the error bounds is the same.

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