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Fuzzy strongly pre-irresolute functions

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Abstract. In this paper, we have introduced and studied a new class of fuzzy irresolute functions, called fuzzy strongly pre-irresolute functions by using fuzzy preopen sets.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [9]. Based on the concept of fuzzy sets, Chang [2] introduced and developed the concept of fuzzy topological spaces. Since then various important notions in the classical topology such as continuous functions [2] have been extended to fuzzy topological spaces. Fuzzy irresolutness is one of the main topics in fuzzy topology. In this paper, we have introduced and studied a new class of fuzzy irresolute functions, called fuzzy strongly pre-irresolute functions by using fuzzy preopen sets. This concept can be extended to Intuitionstic fuzzy topology and Ideal fuzzy topology.

2. Preliminaries

Now, we recall some basic notions and results that are used in the sequel.

Definition 2.1. *A fuzzy topology on a nonempty set X is a family τ of fuzzy subsets of X which satisfies the following three conditions:*

(i) $0, 1 \in \tau$,

(ii) *If $g, h \in \delta$, their $g \wedge h \in \tau$,*

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(iii) If $f_i \in \tau$ for each $i \in I$, then $\vee_{i \in I} f_i \in \tau$.

The pair (X, τ) is called a fuzzy topological space [2].

Definition 2.2. *Members of τ are called fuzzy open sets* [2] *and the complement of fuzzy open sets are called fuzzy closed sets* [2], *where the complement of a fuzzy set A, denoted by A^c, is $1 - A$.*

Definition 2.3 ([7]). *The fuzzy subset x_a of a non-empty set X, with $x \in X$ and $0 < a \leq 1$ defined by*

$$x_a(p) = \begin{cases} a & \text{if } p = x \\ 0 & \text{if } p \neq x \end{cases}$$

is called a fuzzy point in X with suppost x and value a. The fuzzy point x_1 is called crisp point.

Definition 2.4 ([7]). *Let λ be fuzzy set in X and x_a a fuzzy point in X. We say that $x_a \in \lambda$ if and only if $x_a \leq \lambda$.*

Definition 2.5. *A fuzzy set B is a quasi neighbourhood* [7] *(q-neighbourhood, for short) of A if and only if there exists a fuzzy open set U such that $AqU \leq B$.*

Definition 2.6. *A fuzzy set λ is a pre-q-neighbourhood (for short, pre-q-nbd) of μ if and only if there exists a fuzzy preopen set γ such that $\lambda q \gamma$ and $\gamma \leq \mu$.*

Definition 2.7. *Let λ be any fuzzy set in the fuzzy topological space (X, τ). Then we define the fuzzy preinterior* [8] *of $\lambda = p\,\mathrm{Int}(\lambda) = \vee\{\mu : \mu$ is fuzzy preopen and $\mu \leq \lambda\}$ and the fuzzy preclosure of $\lambda = p\,\mathrm{Cl}(\lambda) = \wedge\{\mu : \mu$ is fuzzy preclosed and $\mu \leq \lambda\}$.*

Remark 2.1. For any fuzzy λ in a fuzzy topological space (X, τ), $1 - p\,\mathrm{Cl}(\lambda) = p\,\mathrm{Int}(1 - \lambda)$ and $p\,\mathrm{Cl}(\lambda) = 1 - p\,\mathrm{Int}(\lambda)$.

Definition 2.8 ([2]). *Let X and Y be two fuzzy topological spaces. Let $A \in I^X$, $B \in I^Y$. Then $f(A)$ is a fuzzy subset of Y, defined by $f(A) : Y \to [0, 1]$*

$$f(A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(\{y\})} A(x) & \text{if } f^{-1}(\{y\}) \neq \varnothing \\ 0 & \text{if } f^{-1}(\{y\}) = \varnothing \end{cases}$$

and $f^{-1}(B)$ is a fuzzy subset of X, defined by $f^{-1}(B)(x) = B(f(x))$.

Lemma 2.1 ([1]). *Let $f : X \to Y$ be a function and λ_α $\alpha \in I$ be a family of fuzzy sets of Y, then*

(i) $f^{-1}(\cup\lambda_\alpha) = \cup f^{-1}(\lambda_\alpha)$ *and*

(ii) $f^{-1}(\cap\lambda_\alpha) = \cap f^{-1}(\lambda_\alpha)$.

Lemma 2.2 ([1]). *For functions $f_i : X_i \to Y_i$ and fuzzy sets λ_i of Y_i, $i = 1, 2$, we have $(f_1 \times f_2)^{-1}(\lambda_1 \times \lambda_2) = f_1^{-1}(\lambda_1) \times f_2^{-1}(\lambda_2)$.*

Lemma 2.3 ([1]). *Let $g : X \to X \times Y$ be the graph of a function $f : X \to Y$. Then, if λ is a fuzzy set of X and μ is a fuzzy set of Y, $g^{-1}(\lambda \times \mu) = \lambda \wedge f^{-1}(\mu)$.*

Definition 2.9. *A fuzzy singleton x_α is said to be fuzzy pre-θ-cluster* [5] *point of a fuzzy subset A of X if the fuzzy preclosure if every fuzzy pre-q-nbd of x_α is q-coincident with A. The union of all fuzzy pre-θ-cluster points of A is called the pre-θ-closure* [5] *of A and is denoted by $p\,\mathrm{Cl}_\theta(A)$. A fuzzy set A of X is called fuzzy pre-θ-closed* [5] *if $A = p\,\mathrm{Cl}_\theta(A)$, and the complement of a fuzzy pre-θ-closed set is called fuzzy pre-θ-open* [5]. *The union of all fuzzy pre-θ-open sets of X contained in A is called the fuzzy pre-θ-interior of A and is denoted by $p\,\mathrm{Int}_\theta(A)$.*

Remark 2.2. For a fuzzy set A of X, $A \le p\,\mathrm{Cl}(A) \le p\,\mathrm{Cl}_\theta(A)$ and hence each fuzzy pre-θ-closed set is fuzzy preclosed.

Lemma 2.4. *For a fuzzy set A of X, we have that*

(i) $p\,\mathrm{Int}_\theta(1 - A) = 1 - p\,\mathrm{Cl}_\theta(A)$;

(ii) $p\,\mathrm{Cl}_\theta(1 - A) = 1 - p\,\mathrm{Int}_\theta(A)$.

Definition 2.10. *A function $f : X \to Y$ is said to be fuzzy pre-irresolute* [3] *if $f^{-1}(V) \in FPO(X)$ for each $V \in FPO(Y)$.*

Definition 2.11. *A function $f : X \to Y$ is said to be fuzzy weakly pre-irresolute* [6] *if for any fuzzy singleton x_α in X and each fuzzy preopen set V of Y containing $f(x_\alpha)$, there exists a fuzzy preopen set U containing x_α such that $f(U) \le p\,\mathrm{Cl}(V)$.*

3. Fuzzy strongly pre-irresolute functions

Definition 3.1. *A function $f : X \to Y$ is said to be fuzzy strongly pre-irresolute if for any fuzzy singleton x_α in X and each fuzzy preopen set V containing $f(x_\alpha)$, there exists a fuzzy preopen set U containing x_α such that $f(p\,\mathrm{Cl}(U)) \le V$.*

It is clear that , every fuzzy strongly pre-irresolute function is fuzzy pre-irresolute. But the converse is not true in general as it can be seen from the following example.

Example 3.1. Let (X,τ) as in Example 2.7 of [4] and $\tau = \{0,1\}$. Define a function $f : X \to Y$ by $f(x) = y$ and $f(y) = x$. Then f is fuzzy pre-irresolute but not fuzzy strongly pre-irresolute.

Theorem 3.1. *For a function $f : X \to Y$, the following properties are equivalent:*

(i) f is fuzzy strongly pre-irresolute;

(ii) for each fuzzy singleton x_α of X and each fuzzy preopen set V of Y containing $f(x_\alpha)$, there exists a fuzzy regular preopen set U of X containing x_α such that $f(p\,\mathrm{Cl}_\theta(U)) \leq V$;

(iii) for each fuzzy singleton x_α of X and each fuzzy preopen set V of Y containing $f(x_\alpha)$, there exists a fuzzy regular preopen set U in X containing x_α such that $f(U) \leq U$;

(iv) for each fuzzy singleton x_α of X and each fuzzy preopen set V of Y containing $f(x_\alpha)$, there exists a fuzzy pre-θ-open set U in X containing x_α such that $f(U) \leq V$;

(v) $f^{-1}(V)$ is fuzzy pre-θ-open in X for every $V \in FPO(Y)$;

(vi) $f^{-1}(F)$ is fuzzy pre-θ-closed in X for every $F \in FPC(Y)$;

(vii) $f(p\,\mathrm{Cl}_\theta(A)) \leq p\,\mathrm{Cl}(f(A))$ for every fuzzy subset A of X;

(viii) $p\,\mathrm{Cl}_\theta(f^{-1}(B)) \leq f^{-1}(p\,\mathrm{Cl}(B))$ for every fuzzy subset B of Y.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$ are obivious.

$(iv) \Rightarrow (v)$: Let $V \in FPO(Y)$. Suppose that $x_\alpha \in f^{-1}(V)$. Therefore, $f(x) \in V$ and then there exists a fuzzy pre-θ-open set U in X containing x_α such that $f(U) \leq V$. Therefore, we have $x_\alpha \in U \leq f^{-1}(V)$. Since the union of fuzzy pre-θ-open sets is fuzzy pre-θ-open, $f^{-1}(V)$ is fuzzy pre-θ-open in X.

$(v) \Rightarrow (vi)$: This is obvious.

$(vi) \Rightarrow (vii)$: Let B be any fuzzy subset of Y. Since $p\,\mathrm{Cl}(A)$ is fuzzy preclosed in Y, by (vi), $f^{-1}(p\,\mathrm{Cl}(f(A)))$ is fuzzy pre-θ-closed in X and we have

$$\begin{aligned} p\,\mathrm{Cl}_\theta(A) &\leq p\,\mathrm{Cl}_\theta(f^{-1}(f(A))) \\ &\leq p\,\mathrm{Cl}_\theta(f^{-1}(p\,\mathrm{Cl}(f(A)))) = f^{-1}(p\,\mathrm{Cl}(f(A))). \end{aligned}$$

Therefore, we obtain $f(p\,\mathrm{Cl}_\theta(A)) \leq p\,\mathrm{Cl}(f(A))$.

$(vii) \Rightarrow (viii)$: Let B be any fuzzy subset of Y. By (vii), we obtain

$$f(p\,\mathrm{Cl}_\theta(f^{-1}(B))) \leq p\,\mathrm{Cl}(f(f^{-1}(B))) \leq p\,\mathrm{Cl}(B)$$

and hence

$$p\,\mathrm{Cl}_\theta(f^{-1}(B)) \leq f^{-1}(p\,\mathrm{Cl}(B)).$$

$(viii) \Rightarrow (i)$: Let x_α be a fuzzy singleton of X and V be a fuzzy preopen set of Y containing $f(x_\alpha)$. Since $1 - V \in FPC(Y)$, we have

$$p\,\mathrm{Cl}_\theta(f^{-1}(1 - V)) \leq f^{-1}(p\,\mathrm{Cl}(1 - V)) = f^{-1}(1 - V).$$

Therefore, $f^{-1}(1 - V)$ is fuzzy pre-θ-closed in X and $f^{-1}(V)$ is a fuzzy pre-θ-open set containing x_α. Then there exists a fuzzy preopen set U of X containing x_α such that $p\,\mathrm{Cl}(U) \leq f^{-1}(V)$; hence $f(p\,\mathrm{Cl}(U)) \leq V$. This shows that f is fuzzy strongly pre-irresolute. □

Definition 3.2. *A fuzzy space (X, τ) is said to be fuzzy pre-regular if for each fuzzy singleton x_α and each fuzzy preopen set A there exists a fuzzy preopen set B such that $x_\alpha \in B \leq p\,\mathrm{Cl}(B) \leq A$.*

Theorem 3.2. *A fuzzy pre-irresolute function $f : X \to Y$ is fuzzy strongly pre-irresolute if and only if X is fuzzy pre-regular.*

Proof. Necessity: Let $f : X \to Y$ be the identity function. Then f is fuzzy pre-irresolute and fuzzy strongly fuzzy pre-irresolute by the hypothesis. For any $U \in FPO(X)$ and any fuzzy singleton x_α of U, we have $f(x_\alpha) = x_\alpha \in U$ and there exists a fuzzy preopen set G of X containing x_α such that $f(p\,\mathrm{Cl}(G)) \leq U$. Therefore, we have $x \in G \leq p\,\mathrm{Cl}(G) \leq U$. By definition, X is fuzzy pre-regular. □

Corollary 3.1. *Let X be a fuzzy pre-regular space. Then $f : X \to Y$ is fuzzy strongly pre-irresolute if and only if f is fuzzy pre-irresolute.*

Proof. This follows immediately from Theorem 3.2. □

Theorem 3.3. *If $f : X \to Y$ is fuzzy weakly pre-irresolute and Y is fuzzy pre-regular, then f is fuzzy strongly pre-irresolute.*

Proof. Let x_α be a fuzzy singleton of X and let V be a fuzzy preopen set containing $f(x_\alpha)$. Then there exists a fuzzy preopen set G containing $f(x_\alpha)$ such that $f(x_\alpha) \in G \leq p\,\mathrm{Cl}(G) \leq V$. By Theorem 3.1, there exists a fuzzy preopen set U containing x_α such that $f(p\,\mathrm{Cl}(U)) \leq p\,\mathrm{Cl}(G)$. Thus, $f(p\,\mathrm{Cl}(G)) \leq V$ and f is fuzzy strongly pre-irresolute. □

Corollary 3.2. *If $f : X \to Y$ is a function and Y is fuzzy pre-regular, then the following statements are equivalent:*

(*i*) *f is fuzzy strongly pre-irresolute;*

(*ii*) *f is fuzzy irresolute;*

(*iii*) *f is fuzzy weakly pre-irresolute.*

Theorem 3.4. *Let $f : X \to Y$ be a function, where X is product related to Y, and $g : X \to X \times Y$, the graph function of g. If g is fuzzy strongly pre-irresolute, then f is so and X is fuzzy pre-regular.*

Proof. Suppose that g is fuzzy strongly pre-irresolute. First, we show that f is fuzzy strongly pre-irresolute. Let x_α be a fuzzy singleton of X and V be a fuzzy preopen set of Y containing $f(x_\alpha)$. Then $1 \times V$ is a fuzzy preopen set of $X \times Y$ containing $g(x_\alpha)$. Since g is fuzzy strongly pre-irresolute, there exists a fuzzy preopen set U of X containing x_α such that $g(p\,\mathrm{Cl}(U)) \leq 1 \times V$. Therefore, we obtain $f(p\,\mathrm{Cl}(U)) \leq V$. This shows that f is fuzzy strongly pre-irresolute. Next, we show that X is fuzzy pre-regular. Let U be a fuzzy preopen set of X containing x_α. Since $g(x) \in U \times 1$ and $U \times 1$ is fuzzy preopen in $X \times Y$, there exists a fuzzy preopen set G of X containing x_α such that $g(p\,\mathrm{Cl}(G)) < U \times 1$. Therefore, we obtain $x_\alpha \in G \leq p\,\mathrm{Cl}(G) \leq U$ and hence X is fuzzy pre-regular. □

Definition 3.3. *A function $f : X \to Y$ is said to be fuzzy strongly preopen if $f(U) \in FPO(Y)$ for each $U \in FPO(X)$.*

Lemma 3.1. *If $f : X \to Y$ is fuzzy pre-irresolute and V is fuzzy pre-θ-open in Y, then $f^{-1}(V)$ is fuzzy pre-θ-open in X.*

Proof. Let V be a fuzzy pre-θ-open set of Y and $x_\alpha \in f^{-1}(V)$. Then there exists $W \in FPO(Y)$ such that $f(x_\alpha) \in W \leq p\,\mathrm{Cl}(W) \leq V$. Since f is fuzzy pre-irresolute, we have $f^{-1}(W) \in FPO(X)$ and $p\,\mathrm{Cl}(f^{-1}(W)) \leq f^{-1}(p\,\mathrm{Cl}(W))$. Therefore, we have $x_\alpha \in f^{-1}(W) \leq p\,\mathrm{Cl}(f^{-1}(W)) \leq f^{-1}(V)$ and $f^{-1}(V)$ is fuzzy pre-θ-open in X. □

Theorem 3.5. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the following properties hold:*

(*i*) *If f is fuzzy strongly pre-irresolute and g is fuzzy pre-irresolute, then $g \circ f : X \to Z$ is fuzzy strongly pre-irresolute;*

(*ii*) *If f is fuzzy pre-irresolute and g is fuzzy strongly pre-irresolute, then $g \circ f : X \to Z$ is fuzzy strongly pre-irresolute;*

(*iii*) *If f is fuzzy strongly pre-irresolute and g is fuzzy weakly pre-irresolute, then $g \circ f : X \to Z$ is fuzzy weakly pre-irresolute;*

(*iv*) *If f is fuzzy strongly preopen bijection and $g \circ f : X \to Z$ is fuzzy strongly pre-irresolute, then g is fuzzy strongly pre-irresolute.*

Proof. (i): This is obvious.

(ii): This follows immediately from Theorem 3.1 and Lemma 3.1.

(iii): Let $W \in FPO(Z)$. Since $g \circ f$ is fuzzy strongly pre-irresolute, $(g \circ f)^{-1}(W)$ is fuzzy pre-θ-open in X. Since f is fuzzy strongly preopen and bijective, f^{-1} is fuzzy pre-irresolute and by Lemma 3.1, we have $g^{-1}(W) = f((g \circ f)^{-1}(W))$ us fuzzy pre-θ-open in Y. Therefore,Hence by Theorem 3.1, g is fuzzy strongly pre-irresolute.

(iv): Let x_α be a fuzzy singleton of X and $y_\beta = f(x_\alpha)$. Let V be a fuzzy regular preopen in Z containing $g(y_\beta) = g(f(x_\alpha))$. Since g is fuzzy weakly pre-irresolute, there exists a fuzzy regular preopen set U containing y_β such that $f(p\,\mathrm{Cl}(G)) \leq U$. Hence

$$(g \circ f)(p\,\mathrm{Cl}(G)) = g(f(p\,\mathrm{Cl}(G))) \leq g(U) \leq V.$$

Therefore, $g \circ f$ is fuzzy weakly pre-irresolute. □

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