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A functional model associated with a generalized Nevanlinna pair

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(*Presented by M. M. Malamud*)

Abstract. Let $\mathcal{L}$ be a Hilbert space and let $\mathcal{H}$ be a Pontryagin space. For every selfadjoint linear relation $\widetilde{A}$ in $\mathcal{H}\oplus\mathcal{L}$ the pair $\{I+\lambda\psi(\lambda), \psi(\lambda)\}$, where $\psi(\lambda)$ is the compressed resolvent of $\widetilde{A}$, is a normalized generalized Nevanlinna pair. Conversely, every normalized generalized Nevanlinna pair is shown to be associated with some selfadjoint linear relation $\widetilde{A}$ in the above sense. A functional model for this selfadjoint linear relation $\widetilde{A}$ is constructed.

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Introduction

In 1946 M. G. Krein introduced in [12] the notion of the Q-function of a symmetric operator A in a Hilbert space with finite deficiency indices (m, m), which plays an important role in the description of generalized resolvents of A. Later on M. G. Krein and H. Langer in [14] have generalized this notion to the case of a symmetric operator A with infinite indices acting in a Pontryagin space. In that paper it was shown that the Q-function uniquely determines a simple symmetric operator A up to unitary equivalence. Moreover, in [14] a functional model for a symmetric operator relied on ε-construction was introduced and investigated. This

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model has allowed them to solve an inverse problem for the Q-function, that is to find a criterion for a generalized Nevanlinna operator valued function to be the Q-function of a π-Hermitian operator.

Functional models for symmetric operators in Hilbert spaces in terms of the Q-function were constructed in [1, 15]. Different functional models for symmetric operators have been used by V. A. Derkach and M. M. Malamud in [10] (see also [16]) to solve the inverse problem for the Weyl function. Namely, starting with a uniformly strict R-function $M(\cdot)$ (i.e. R-function satisfying $0 \in \rho\big(\mathrm{Im}\, M(i)\big)$) the authors in [10] constructed a model symmetric operator A and a boundary triplet for A^* such that the corresponding Weyl function coincides with $M(\cdot)$. This result has also been extended to a wider class of strict R-functions (that is R-functions with $\ker \mathrm{Im}\, M(i) = \{0\}$) in order to realize any such R-function as the Weyl function corresponding to a generalized boundary triplet (see [10]).

Later on a concept a generalized boundary triplet was generalized in [7] where a notion of a boundary relation and the corresponding Weyl family was introduced. Using this notion the authors of [7] have realized arbitrary Nevanlinna pair $\{\varphi, \psi\}$ as the Weyl family of some symmetric operator corresponding to a boundary relation (a realization theorem). The proof in [7] was based on the Naimark dilation theorem and the so called main transform. Later on another proof of the realization theorem from [7] have been presented in [4, 5] where more general models for symmetric operators were introduced.

In the present paper given a generalized Nevanlinna pair $\{\varphi, \psi\}$ we construct a functional model for a selfadjoint linear relation $\widetilde{A}$ in Pontryagin space such that φ, ψ are recovered from $\widetilde{A}$ via (2.4). To make the paper clear for a wide audience we follow the scheme of [6] and use the notion of the selfadjoint linear relation $\widetilde{A}$ rather then the notion of the boundary relation Γ. In fact, one can treat $\widetilde{A}$ as the main transform of a boundary relation Γ and then the main result can be reformulated in terms of Γ.

The paper is organized as follows. In Section 1 definitions of N_κ-pairs and normalized N_κ-pairs are given. In Section 2 we consider a pair $\{\varphi, \psi\}$ generated by a selfadjoint relation $\widetilde{A}$ in a Pontryagin space and show that it is a normalized N_κ-pair. In Theorem 3.1 we prove the converse result. Moreover, a functional model for the selfadjoint relation $\widetilde{A}$ is constructed. In the rest of the paper properties of a generalized Fourier transform, associated with this model are studied. We also proved the unitary equivalence of an arbitrary $\mathcal{L}$-minimal selfadjoint linear relation $\widetilde{A}$ to the model relation $A(\varphi, \psi)$ in the reproducing kernel Pontryagin space.

1. Generalized Nevanlinna pairs

Let $\mathcal{L}$ be a Hilbert space. By a kernel is meant a function $\mathsf{K}_\omega(\lambda)$ on $\Omega\times\Omega$ with values in the space of continuous operators on a Hilbert space $\mathcal{L}$ ($\Omega\subset\mathbb{C}$). We say that the kernel $\mathsf{K}_\omega(\lambda)$ has κ negative squares and write $sq_-\mathsf{K}=\kappa$ if for any choice set of points $\omega_1,\dots,\omega_n$ in Ω, vectors $u_1,\dots,u_n$ in $\mathcal{L}$ and ξ_j in space $\mathbb{C}^n$ the quadratic form

$$\sum_{i,j=1}^{n}\big(\mathsf{K}_{\omega_j}(\omega_i)u_j,\,u_i\big)_{\mathcal{L}}\xi_j\overline{\xi}_i$$

has at most κ negative eigenvalues, and for some choice of n, ω_j, u_j such matrix has exactly κ negative squares ([2]).

Definition 1.1. A pair $\{\Phi,\Psi\}$ of $[\mathcal{L}]$-valued functions $\Phi(\cdot)$, $\Psi(\cdot)$ meromorphic on $\mathbb{C}\setminus\mathbb{R}$ with a common domain of homomorphy $\mathfrak{h}_{\Phi\Psi}$ is said to be a N_κ-pair (a generalized Nevanlinna pair) if:

(i) the kernel

$$\mathsf{N}_\omega^{\Phi\Psi}(\lambda)=\frac{\Psi(\bar\lambda)^*\Phi(\bar\omega)-\Phi(\bar\lambda)^*\Psi(\bar\omega)}{\lambda-\bar\omega},$$

has κ negative square on $\mathfrak{h}_{\Phi\Psi}$;

(ii) $\Psi(\bar\lambda)^*\Phi(\lambda)-\Phi(\bar\lambda)^*\Psi(\lambda)=0$ for all $\lambda\in\mathfrak{h}_{\Phi\Psi}$;

(iii) for all $\lambda\in\mathfrak{h}_{\Phi\Psi}\cap\mathbb{C}_+$ there is $\mu\in\mathbb{C}_+$ such that

$$0\in\rho\big(\Phi(\lambda)-\mu\Psi(\lambda)\big)\text{ and }0\in\rho\big(\Phi(\overline{\lambda})-\overline{\mu}\Psi(\overline{\lambda})\big).$$

Two N_κ-pairs $\{\Phi,\Psi\}$ and $\{\Phi_1,\Psi_1\}$ are said to be equivalent, if $\Phi_1(\lambda)=\Phi(\lambda)\chi(\lambda)$ and $\Psi_1(\lambda)=\Psi(\lambda)\chi(\lambda)$ for some operator function $\chi(\cdot)\in[\mathcal{H}]$, which is holomorphic and invertible on $\mathfrak{h}_{\Phi\Psi}$. The set of all equivalence classes of N_κ-pairs in $\mathcal{L}$ will be denoted by $\widetilde{N}_\kappa(\mathcal{L})$. We will write, for short, $\{\Phi,\Psi\}\in\widetilde{N}_\kappa(\mathcal{L})$ for the generalized Nevanlinna pair $\{\Phi,\Psi\}$.

If $\Phi(\lambda)\equiv I_{\mathcal{L}}$ where $I_{\mathcal{L}}$ is the identity operator in the space $\mathcal{L}$ then the Definition 1.1 means that $\Psi(\lambda)$ is an $N_\kappa(\mathcal{L})$-function in the sense of [13]. Recall that the class $N_\kappa(\mathcal{L})$ consists of meromorphic in $\mathbb{C}_+\cup\mathbb{C}_-$ operator valued functions $\Psi(\lambda)$ such that $\Psi(\overline{\lambda})=\Psi(\lambda)^*$, and the kernel

$$\mathsf{N}_\omega^{\Psi}(\lambda)=\frac{\Psi(\lambda)-\Psi(\omega)^*}{\lambda-\overline{\omega}}$$

has κ negative squares on $\mathfrak{h}_\Psi$ — the domain of holomorphic Ψ. In this case the condition (iii) is satisfied automatically. Clearly, if $\{\Phi,\Psi\}$ is N_κ-pair such that $0\in\rho\big(\Phi(\lambda)\big)$ $\lambda\in\mathfrak{h}_{\Phi\Psi}$, then it is equivalent to the pair $\big\{I_{\mathcal{L}},\Psi(\lambda)\Phi(\lambda)^{-1}\big\}$, where $\Psi\Phi^{-1}\in N_\kappa(\mathcal{L})$.

Definition 1.2. An N_κ-pair $\{\phi, \psi\}$ is said to be a normalized N_κ-pair if:

(iii′) $\varphi(\lambda) - \lambda\psi(\lambda) \equiv I_{\mathcal{L}}$ for all $\lambda \in \mathfrak{h}_{\varphi\psi}$.

Clearly, every N_κ-pair $\{\Phi, \Psi\}$ such that $0 \in \rho\big(\Phi(\lambda) - \lambda\Psi(\lambda)\big)$ for $\lambda \in \mathfrak{h}_{\Phi\Psi}$ is equivalent to a unique normalized N_κ-pair $\{\varphi, \psi\}$ given by

$$\varphi(\lambda) = \Phi(\lambda)\big(\Phi(\lambda) - \lambda\Psi(\lambda)\big)^{-1}, \quad \psi(\lambda) = \Psi(\lambda)\big(\Phi(\lambda) - \lambda\Psi(\lambda)\big)^{-1}. \tag{1.1}$$

2. N_κ-pair corresponding to a selfadjoint linear relation and a scale

Let $\mathfrak{H}$ be a vector space with a Hermitian form $[\cdot, \cdot]_{\mathfrak{H}} : \mathfrak{H} \times \mathfrak{H} \to \mathbb{C}$. Two elements u and v of $\mathfrak{H}$ are said to be orthogonal if $[u, v]_{\mathfrak{H}} = 0$. Similarly, two subspaces of $\mathfrak{H}$ are said to be orthogonal if every element of the first is orthogonal to every element of the second. The linear space $\big(\mathfrak{H}, [\cdot, \cdot]\big)$ is called a Pontryagin space if there exists a direct orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$, where $\mathfrak{H}_+$ with the form $[\cdot, \cdot]_{\mathfrak{H}}$ is a Hilbert space and $\mathfrak{H}_-$ with the form $-[\cdot, \cdot]_{\mathfrak{H}}$ is a Hilbert space of finite dimension. The space $\mathfrak{H}$ is called Pontryagin space with κ negative squares (Π_κ-space) if the dimension of $\mathfrak{H}_-$ is $\kappa < \infty$ ([2]).

We will use the notion of a linear relation in a space $\mathfrak{H}$. Recall, that a subspace T of $\mathfrak{H}^2$ is called the linear relation in $\mathfrak{H}$. For a linear relation T in $\mathfrak{H}$ the symbols $\operatorname{dom} T$, $\ker T$, $\operatorname{ran} T$, and $\operatorname{mul} T$ stand for the domain, kernel, range, and the multivalued part, respectively. The adjoint T^+ is the closed linear relation in $\mathfrak{H}$ defined by (see [2])

$$T^+ = \{\, \{h, k\} \in \mathfrak{H}^2 : [k, f]_{\mathfrak{H}} = [h, g]_{\mathfrak{H}},\ \{f, g\} \in T \,\}. \tag{2.2}$$

Recall that a linear relation T in $\mathfrak{H}$ is called symmetric (selfadjoint) if $T \subset T^+$ ($T = T^+$, respectively).

Let $\mathcal{H}$ be a Pontryagin space and $\mathcal{L}$ be a Hilbert space.

Definition 2.1. A linear relation $\widetilde{A} = \widetilde{A}^*$ in $\mathcal{H} \oplus \mathcal{L}$ is said to be $\mathcal{L}$-minimal if

$$\mathcal{H}_0 := \overline{\operatorname{span}}\,\{P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}\mathcal{L} : \lambda \in \rho(\widetilde{A})\} = \mathcal{H}, \tag{2.3}$$

where $P_{\mathcal{H}}$ is the orthogonal projection onto the Pontryagin space $\mathcal{H}$.

Let $\widetilde{A}$ be a selfadjoint linear relation in $\mathcal{H} \oplus \mathcal{L}$ and let $P_{\mathcal{L}}$ be the orthogonal projection onto the scale space $\mathcal{L}$. Define the operator valued functions

$$\varphi(\lambda) := I_{\mathcal{L}} + \lambda P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1} \restriction_{\mathcal{L}}, \quad \psi(\lambda) := P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1} \restriction_{\mathcal{L}} \quad (\lambda \in \rho(\widetilde{A})). \tag{2.4}$$

Clearly,

$$\varphi(\lambda)^* = \varphi(\overline{\lambda}), \quad \psi(\lambda)^* = \psi(\overline{\lambda}) \quad (\lambda \in \rho(\widetilde{A})). \tag{2.5}$$

Proposition 2.1. Let $\mathcal{H}$ be a Π_κ-space, let $\mathcal{L}$ be a Hilbert space and let $\widetilde{A}$ be a selfadjoint linear relation in $\mathcal{H} \oplus \mathcal{L}$. The pair of operator valued functions $\{\varphi, \psi\}$ associated with $\widetilde{A}$ via (2.4) is a normalized $N_{\kappa'}$-pair where $0 \le \kappa' \le \kappa$. If, additionally, the linear relation $\widetilde{A}$ is $\mathcal{L}$-minimal then $\kappa' = \kappa$.

Proof. In view of the properties (2.5) the kernel $\mathsf{N}_\omega^{\varphi\psi}(\lambda)$ for the pair $\{\varphi, \psi\}$ takes the form

$$\mathsf{N}_\omega^{\varphi\psi}(\lambda) = \frac{\psi(\lambda)\phi(\bar{\omega}) - \phi(\lambda)\psi(\bar{\omega})}{\lambda - \bar{\omega}}, \quad \lambda, \omega \in \rho(\widetilde{A}). \tag{2.6}$$

It follows from Definition (2.4) that

$$\begin{aligned}\mathsf{N}_\omega^{\varphi\psi}(\lambda) = \frac{\psi(\lambda) - \psi(\omega)^*}{\lambda - \bar{\omega}} - \psi(\lambda)\psi(\omega)^* \\ = P_\mathcal{L} \frac{R_\lambda - R_{\bar{\omega}}}{\lambda - \bar{\omega}} \restriction_\mathcal{L} - P_\mathcal{L} R_\lambda P_\mathcal{L} R_{\bar{\omega}} \restriction_\mathcal{L} \\ = P_\mathcal{L} R_\lambda P_\mathcal{H} R_{\bar{\omega}} \restriction_\mathcal{L},\end{aligned} \tag{2.7}$$

where $R_\lambda = (\widetilde{A} - \lambda)^{-1}$ is a resolvent of lineal relation $\widetilde{A}$. Let ω_j belongs to $\rho(A)$, u_j belongs to space $\mathcal{L}$ and ξ_j belongs to space $\mathbb{C}^n$ for $j = 1, \dots, n$. Then

$$\begin{aligned}\sum_{j,k=1}^n \big(\mathsf{N}_{\omega_j}^{\varphi\psi}(\omega_k) u_j,\, u_k\big)_\mathcal{L}\, \xi_j\, \overline{\xi}_k = \sum_{j,k=1}^n \big((P_\mathcal{L} R_{\omega_k} P_\mathcal{H} R_{\overline{\omega}_j|\mathcal{L}}) u_j,\, u_k\big)_\mathcal{L}\, \xi_j\, \overline{\xi}_k \\ = \sum_{j,k=1}^n [P_\mathcal{H} R_{\overline{\omega}_j} u_j,\, P_\mathcal{H} R_{\overline{\omega}_k} u_k]_\mathcal{H}\, \xi_j\, \overline{\xi}_k = \sum_{j,k=1}^n [g_j,\, g_k]_\mathcal{H}\, \xi_j\, \overline{\xi}_k,\end{aligned} \tag{2.8}$$

where $g_j = P_\mathcal{H} R_{\overline{\omega}_j} u_j$. Since $\mathcal{H}$ is Π_k-space and u_j $(j = 1, \dots, n)$ are arbitrary vectors in $\mathcal{L}$ then the quadratic form (2.8) has κ' negative squares, where $\kappa' \le \kappa$. Thus property (i) of Definition 1.1 is proved.

The property (ii) is easily checked. Obviously $\varphi(\lambda) - \lambda\psi(\lambda) \equiv I_\mathcal{L}$ for all $\lambda \in \rho(\widetilde{A})$ and, hence, the pair $\{\phi, \psi\}$ is a normalized $N_{\kappa'}$-pair.

If the relation $\widetilde{A}$ is $\mathcal{L}$-minimal then the set

$$\operatorname{span}\big\{P_\mathcal{H} R_\omega u :\ \omega \in \rho(\widetilde{A}),\, u \in \mathcal{L}\big\}$$

is dense in the space $\mathcal{H}$. In this case the quadratic form (2.8) has exactly κ negative squares and hence the kernel $\mathsf{N}_\omega^{\varphi\psi}(\lambda)$ has κ negative squares. Thus the pair $\{\phi, \psi\}$ is a normalized N_κ-pair. □

Definition 2.2. The pair of operator valued functions $\{\varphi, \psi\}$ determined by (2.4) will be called the N_κ-pair corresponding to the selfadjoint linear relation $\widetilde{A}$ and the scale $\mathcal{L}$.

Note that if the vector values functions $\varphi(\lambda)$ and $\psi(\lambda)$ are defined by (2.4) then $\mathfrak{h}_{\varphi\psi} = \mathfrak{h}_\varphi = \mathfrak{h}_\psi$.

3. Functional model of a selfadjoint linear relation

Consider the reproducing kernel Pontryagin space $\mathcal{H}(\phi, \psi)$, which is characterized by the properties:

(1) $\mathsf{N}_\omega^{\phi\psi}(\cdot)u \in \mathcal{H}(\phi, \psi)$ for all $\omega \in \mathfrak{h}_{\varphi\psi}$ and $u \in \mathcal{L}$;

(2) for every $f \in \mathcal{H}(\phi, \psi)$ the following identity holds

$$\left[f(\cdot), \mathsf{N}_\omega^{\phi\psi}(\cdot)u\right]_{\mathcal{H}(\phi,\psi)} = (f(\omega), u)_{\mathcal{L}}, \quad \omega \in \mathfrak{h}_{\varphi,\psi}, u \in \mathcal{L}. \tag{3.9}$$

It follows from (3.9) that the evaluation operator

$$E(\lambda) : f \mapsto f(\lambda) \quad (f \in \mathcal{H}(\phi, \psi))$$

is a bounded operator from $\mathcal{H}(\phi, \psi)$ to $\mathcal{L}$. Also note that the set of functions $\{\mathsf{N}_\omega^{\phi\psi}(\cdot)u : \ \omega \in \mathfrak{h}_{\varphi\psi}, \ u \in \mathcal{L}\}$ is total in $\mathcal{H}(\varphi, \psi)$ ([2]).

In the next theorem we give functional model of a selfadjoint linear relation $\widetilde{A}$ recovered from a N_κ-pair.

Theorem 3.1. Let $\mathcal{L}$ be a Hilbert space and let $\{\phi, \psi\}$ be a normalized N_κ-pair. Then the linear relation

$$A(\phi, \psi) = \left\{ \left\{ \begin{bmatrix} f \\ u \end{bmatrix}, \begin{bmatrix} f' \\ u' \end{bmatrix} \right\} : \begin{array}{l} f, f' \in \mathcal{H}(\phi, \psi), \ u, u' \in \mathcal{L}, \\ f'(\lambda) - \lambda f(\lambda) = \phi(\lambda)u - \psi(\lambda)u', \ \lambda \in \mathfrak{h}_{\varphi\psi} \end{array} \right\} \tag{3.10}$$

is a selfadjoint linear relation in $\mathcal{H}(\phi, \psi) \oplus \mathcal{L}$ and the normalized pair $\{\phi, \psi\}$ is the N_κ-pair corresponding to $A(\phi, \psi)$ and $\mathcal{L}$.

Proof. Step 1. Let us show that $A(\phi, \psi)$ contains vectors of the form

$$\{F_\omega v, F'_\omega v\} := \left\{ \begin{bmatrix} \mathsf{N}_\omega(\cdot)v \\ \psi(\bar{\omega})v \end{bmatrix}, \begin{bmatrix} \bar{\omega}\mathsf{N}_\omega(\cdot)v \\ \phi(\bar{\omega})v \end{bmatrix} \right\}, \quad v \in \mathcal{L}, \ \omega \in \mathfrak{h}_{\varphi\psi}, \tag{3.11}$$

where $\mathsf{N}_\omega(\cdot) := \mathsf{N}_\omega^{\phi\psi}(\cdot)$ and

$$A' := \operatorname{span}\Big\{\{F_\omega v, F'_\omega v\}:\ v \in \mathcal{L},\ \omega \in \mathfrak{h}_{\varphi\psi}\Big\}$$

is a symmetric linear relation.

Indeed, it follows from (3.10) and the equality

$$(\bar\omega - \lambda)\mathsf{N}_\omega(\lambda)v = \phi(\bar\lambda)^*\psi(\bar\omega)v - \psi(\bar\lambda)^*\phi(\bar\omega)v$$

that $\{F_\omega v, F'_\omega v\} \in A(\phi,\psi)$.

For arbitrary $\omega_j \in \mathfrak{h}_{\varphi\psi}$, $v_j \in \mathcal{L}$ $(j = 1,2)$ one obtains

$$\begin{aligned}
&\Big[\bar\omega_1\mathsf{N}_{\omega_1}(\cdot)v_1, \mathsf{N}_{\omega_2}(\cdot)v_2\Big]_{\mathcal{H}(\phi,\psi)} - \Big[\mathsf{N}_{\omega_1}(\cdot)v_1, \bar\omega_2\mathsf{N}_{\omega_2}(\cdot)v_2\Big]_{\mathcal{H}(\phi,\psi)}\\
&\quad + \Big(\phi(\bar\omega_1)v_1, \psi(\bar\omega_2)v_2\Big)_{\mathcal{L}} - \Big(\psi(\bar\omega_1)v_1, \phi(\bar\omega_2)v_2\Big)_{\mathcal{L}}\\
&\quad = (\bar\omega_1 - \omega_2)\Big(\mathsf{N}_{\omega_1}(\omega_2)v_1, v_2\Big)_{\mathcal{L}}\\
&\qquad - \Big(\big(\phi(\bar\omega_2)^*\psi(\bar\omega_1) - \psi(\bar\omega_2)^*\phi(\bar\omega_1)\big)v_1, v_2\Big)_{\mathcal{L}} = 0,
\end{aligned}$$

therefore, A' is symmetric in $\mathcal{H}(\phi,\psi)\oplus\mathcal{L}$.

Step 2. Let us show that $\operatorname{ran}(A'-\lambda)$ is dense in $\mathcal{H}(\phi,\psi)\oplus\mathcal{L}$ for $\lambda \in \mathfrak{h}_{\varphi\psi}$. Choose the vector $\{F_\omega v, F'_\omega v\}$ with $\omega = \bar\lambda$. Since $\phi(\lambda) - \lambda\psi(\lambda) = I_{\mathcal{L}}$ then

$$\begin{aligned}
\big\{F_{\bar\lambda}v, F'_{\bar\lambda}v - \lambda F_{\bar\lambda}v\big\} &= \left\{\begin{bmatrix}\mathsf{N}_{\bar\lambda}(\cdot)v\\ \psi(\lambda)v\end{bmatrix}, \begin{bmatrix}0\\ \phi(\lambda)v - \lambda\psi(\lambda)v\end{bmatrix}\right\}\\
&= \left\{\begin{bmatrix}\mathsf{N}_{\bar\lambda}(\cdot)v\\ \psi(\lambda)v\end{bmatrix}, \begin{bmatrix}0\\ v\end{bmatrix}\right\} \in A' - \lambda. \qquad (3.12)
\end{aligned}$$

Hence $0\oplus\mathcal{L} \subset \operatorname{ran}(A'-\lambda)$. Taking $\{F_\omega v, F'_\omega v\}$ with $\omega \neq \bar\lambda$ one obtains from (3.11)

$$\left\{\begin{bmatrix}\mathsf{N}_\omega(\cdot)v\\ \psi(\bar\omega)v\end{bmatrix}, \begin{bmatrix}(\bar\omega-\lambda)\mathsf{N}_\omega(\cdot)v\\ \phi(\bar\omega)v - \lambda\psi(\bar\omega)v\end{bmatrix}\right\} \in A' - \lambda$$

and, hence, $\begin{bmatrix}\mathsf{N}_\omega(\cdot)v\\ 0\end{bmatrix} \in \operatorname{ran}(A'-\lambda)$ for all $\omega \neq \bar\lambda$. Due to the properties (1) and (2) of $\mathcal{H}(\phi,\psi)$ one obtains the statement. Thus A' is an essentially selfadjoint lineal relation and hence $(A')^+$ is a selfadjoint lineal relation in $\mathcal{H}(\varphi,\psi)\oplus\mathcal{L}$.

Step 3. Let us show that $A(\phi,\psi) = (A')^+$. Indeed, for every vector

$$\widehat{F} := \{F, F'\} = \left\{\begin{bmatrix}f(\cdot)\\ u\end{bmatrix}, \begin{bmatrix}f'(\cdot)\\ u'\end{bmatrix}\right\} \in A(\phi,\psi)$$

where $f, f' \in \mathcal{H}(\phi,\psi)$ and $u, u' \in \mathcal{L}$ and arbitrary $\omega \in \mathfrak{h}_{\varphi\phi}$, $v \in \mathcal{L}$ it follows from (3.10) that

$$\begin{aligned}\big[F', F_\omega v\big]_{\mathcal{H}(\phi,\psi)\oplus\mathcal{L}} &- \big[F, F'_\omega v\big]_{\mathcal{H}(\phi,\psi)\oplus\mathcal{L}} \\ &= \big[f', \mathsf{N}_\omega(\cdot)v\big]_{\mathcal{H}(\phi,\psi)} - \big[f, \bar\omega \mathsf{N}_\omega(\cdot)v\big]_{\mathcal{H}(\phi,\psi)} \\ &\quad + \big(u', \psi(\bar\omega)v\big)_{\mathcal{L}} - \big(u, \phi(\bar\omega)v\big)_{\mathcal{L}} \\ &\quad = \big(f'(\omega) - \omega f(\omega) + \psi(\bar\omega)^* u' - \phi(\bar\omega)^* u, v\big)_{\mathcal{L}} = 0.\end{aligned}$$

Hence $\widehat{F} \in (A')^+$ and $A(\phi,\psi) \subset (A')^+$. Conversely, if

$$\big[f', \mathsf{N}_\omega(\cdot)v\big]_{\mathcal{H}(\phi,\psi)} - \big[f, \bar\omega \mathsf{N}_\omega(\cdot)v\big]_{\mathcal{H}(\phi,\psi)} + \big(u', \psi(\bar\omega)v\big)_{\mathcal{L}} - \big(u, \phi(\bar\omega)v\big)_{\mathcal{L}} = 0$$

for some $f, f' \in \mathcal{H}(\varphi,\psi)$, $u, u' \in \mathcal{L}$ and all $\omega \in \mathfrak{h}_{\varphi,\psi}$, $v \in \mathcal{L}$, then

$$f'(\omega) - \omega f(\omega) - \big(\phi(\omega)u - \psi(\omega)u'\big) = 0$$

and, hence, $\widehat{F} \in A(\phi,\psi)$. This proves that $(A')^+ \subset A(\phi,\psi)$, and, hence, $(A')^+ = A(\phi,\psi)$. Therefore, $A(\phi,\psi)$ is a selfadjoint lineal relation.

Step 4. Finally, we show that $\{\varphi,\psi\}$ is a pair corresponding to the selfadjoint linear relation $\widetilde{A}$ and the scale $\mathcal{L}$. Indeed, it follows from (3.12) and Definition 1.2 (iii′) that

$$\begin{aligned} P_{\mathcal{L}}\big(\widetilde{A}(\phi,\psi) - \lambda\big)^{-1} \restriction_{\mathcal{L}} &= \psi(\lambda), \\ I_{\mathcal{L}} + \lambda P_{\mathcal{L}}\big(\widetilde{A}(\phi,\psi) - \lambda\big)^{-1} \restriction_{\mathcal{L}} &= \varphi(\lambda). \end{aligned}$$

Therefore, the pair $\{\varphi,\psi\}$ is a normalized N_κ-pair corresponding to the linear relation $A(\phi,\psi)$ and the scale $\mathcal{L}$. □

Remark 3.1. It follows from (3.12) that the linear relation $A(\varphi,\psi)$ given by (3.10) is $\mathcal{L}$-minimal.

Remark 3.2. For every normalized N_κ pair $\{\varphi,\psi\}$ and $h \in \mathcal{H}(\varphi,\psi)$ the following identity holds

$$P_{\mathcal{L}}(A(\varphi,\psi) - \lambda)^{-1}\begin{bmatrix} h \\ 0 \end{bmatrix} = h(\lambda), \quad (\lambda \in \mathfrak{h}_{\varphi\psi}). \tag{3.13}$$

Indeed, it follows from (3.12) that for every $v \in \mathcal{L}$ one obtains

$$\begin{aligned}\left(P_{\mathcal{L}}\big(A(\varphi,\psi) - \lambda\big)^{-1}\begin{bmatrix} h \\ 0 \end{bmatrix}, v\right)_{\mathcal{L}} \\ = \left[\begin{bmatrix} h \\ 0 \end{bmatrix}, \big(A(\varphi,\psi) - \bar\lambda\big)^{-1}\begin{bmatrix} 0 \\ v \end{bmatrix}\right]_{\mathcal{H}(\varphi,\psi)\oplus\mathcal{L}}\end{aligned}$$

$$= \big[h, \mathsf{N}_\lambda(\cdot)v\big]_{\mathcal{H}(\varphi,\psi)} = \big(h(\lambda), v\big)_{\mathcal{L}}.$$

Therefore, $E(\lambda) = P_{\mathcal{L}}\big(A(\varphi,\psi) - \lambda\big)^{-1} \upharpoonright_{\mathcal{H}(\varphi,\psi)}$ is the evaluation operator in $\mathcal{H}(\varphi,\psi)$.

We define the lineal space $\widetilde{\mathfrak{N}}_\omega$ via the formula

$$\widetilde{\mathfrak{N}}_\omega := \left\{\mathsf{N}_{\overline{\omega}}^{\varphi\psi}(\cdot)u,\ u \in \mathcal{L}\right\}. \tag{3.14}$$

Proposition 3.1. Let $\{\varphi,\psi\}$ be a normalized N_κ-pair in the space $\mathcal{L}$. Then

(i) the space $\widetilde{\mathfrak{N}}_{\overline{\omega}}$ is a positive subspace in $\mathcal{H}(\varphi,\psi)$ if and only if $\mathsf{N}_\omega^{\varphi\psi}(\omega)$ is a strictly positive operator in $\mathcal{L}$.

(ii) if additionally $\bigcap_\lambda \ker \mathsf{N}_\omega^{\varphi\psi}(\lambda) = \{0\}$ then the space $\widetilde{\mathfrak{N}}_{\overline{\omega}}$ is a degenerate subspace in $\mathcal{H}(\varphi,\psi)$ if and only if 0 is an eigenvalue of $\mathsf{N}_\omega^{\varphi\psi}(\omega)$.

Proof. Denote $\mathsf{N}_\omega(\cdot) := \mathsf{N}_\omega^{\varphi\psi}(\cdot)$. Let us prove the first statement. Since

$$\big[\mathsf{N}_\omega(\cdot)u, \mathsf{N}_\omega(\cdot)u\big]_{\mathcal{H}(\varphi,\psi)} = \big(\mathsf{N}_\omega(\omega)u, u\big)_{\mathcal{L}} \quad (u \in \mathcal{L})$$

then a conditions $\mathsf{N}_\omega(\omega) > 0$ is equivalent to the inequality $\big(\mathsf{N}_\omega(\cdot)u, \mathsf{N}_\omega(\cdot)u\big)_{\mathcal{H}(\varphi,\psi)} > 0$ which holds for all $(0 \neq)u \in \mathcal{L}$.

Now we prove the second statement. Let at first the space $\widetilde{\mathfrak{N}}_{\overline{\omega}}$ is a degenerate subspace. Then exist $(0 \neq)u_0 \in \mathcal{L}$ such that

$$0 = \big[\mathsf{N}_\omega(\cdot)u_0, \mathsf{N}_\omega(\cdot)v\big]_{\mathcal{H}(\varphi,\psi)} = \big(\mathsf{N}_\omega(\omega)u_0, v\big)_{\mathcal{L}}$$

which holds for all $v \in \mathcal{L}$. Therefore $\mathsf{N}_\omega(\omega)u_0 = 0$ and hence 0 is an eigenvalue of $\mathsf{N}_\omega(\omega)$.

Conversely, let $\mathsf{N}_\omega(\omega)u_0 = 0$ where $(0 \neq)u_0 \in \mathcal{L}$. Then

$$0 = \big(\mathsf{N}_\omega(\omega)u_0, v\big)_{\mathcal{L}} = \big[\mathsf{N}_\omega(\cdot)u_0, \mathsf{N}_\omega(\cdot)v\big]_{\mathcal{H}(\varphi,\psi)},$$

therefore $\mathsf{N}_\omega(\cdot)u_0$ is orthogonal to the space $\widetilde{\mathfrak{N}}_\omega$. Since $\mathsf{N}_\omega(\cdot)u_0 \not\equiv 0$ then it is a nontrivial isotropic vector in the space $\widetilde{\mathfrak{N}}_\omega$. □

Proposition 3.2. Let $\widetilde{A}$ be a selfadjoint linear relation in $\mathcal{H} \oplus \mathcal{L}$ and let $\{\varphi,\psi\}$ be the normalized Nevanlinna pair given by (2.4). Let the operator valued function $\gamma(\lambda) : \mathcal{L} \to \mathcal{H}$ be defined by

$$\gamma(\lambda) := P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{L}} \quad (\lambda \in \rho(\widetilde{A})). \tag{3.15}$$

Then the following identity holds

$$\mathsf{N}_\omega^{\varphi\psi}(\lambda) = \gamma(\bar{\lambda})^*\gamma(\bar{\omega}). \tag{3.16}$$

Proof. Indeed, it follows from (2.7) that the kernel $\mathsf{N}_\omega^{\varphi\psi}(\lambda)$ takes the form

$$\mathsf{N}_\omega^{\varphi\psi}(\lambda) = (P_{\mathcal{L}} R_\lambda P_{\mathcal{H}})(P_{\mathcal{H}} R_{\bar\omega}|_{\mathcal{L}}) = \gamma(\bar\lambda)^*\gamma(\bar\omega).$$

□

Proposition 3.3. Let $\{\varphi, \psi\}$ be a normalized N_κ-pair in the space $\mathcal{L}$. Then $\widetilde{\mathfrak{N}}_\omega$ is a closed space if and only if $\mathsf{N}_\omega(\omega)$ is normally solvable.

Proof. Denote $B := \mathsf{N}_\omega^{\varphi\psi}(\omega)$ and consider its spectral decomposition

$$B = B_+ \oplus B_- \oplus B_0 \tag{3.17}$$

and the corresponding decomposition of the Hilbert space $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_- \oplus \mathcal{L}_0 \tag{3.18}$$

where $B_+ > 0$, $B_- < 0$, and $B_0 = 0_{\mathcal{L}_0}$. It follows from (3.12) and (3.15) that

$$\gamma(\omega)v = \mathsf{N}_{\bar\omega}^{\varphi\psi}(\cdot)v \quad \forall v \in \mathcal{L}.$$

Since $\widetilde{\mathfrak{N}}_\omega = \gamma(\omega)\mathcal{L}$ then $\widetilde{\mathfrak{N}}_\omega$ can be decomposed as

$$\widetilde{\mathfrak{N}}_\omega = \mathfrak{N}_\omega^+[+]\mathfrak{N}_\omega^-[+]\mathfrak{N}_\omega^0 \tag{3.19}$$

where $\mathfrak{N}_\omega^\pm = \gamma(\omega)\mathcal{L}_\pm$ and $\mathfrak{N}_\omega^0 = \gamma(\omega)\mathcal{L}_0$. This decomposition is orthogonal since $\mathsf{N}_\omega(\omega) = \mathsf{N}_{\bar\omega}(\bar\omega)$. For instance, if $v_+ \in \mathcal{L}_+$, $v_- \in \mathcal{L}_-$ then

$$\begin{aligned}\left[\gamma(\omega)v_+,\, \gamma(\omega)v_-\right]_{\mathcal{H}(\varphi,\psi)} &= \left[\mathsf{N}_{\overline{\omega}}(\cdot)v_+, \mathsf{N}_{\overline{\omega}}(\cdot)v_-\right]_{\mathcal{H}(\varphi,\psi)} \\ &= \left(\mathsf{N}_{\overline{\omega}}(\overline{\omega})v_+,\, v_-\right)_{\mathcal{L}} = \left(\mathsf{N}_\omega(\omega)v_+, v_-\right)_{\mathcal{L}} = 0.\end{aligned}$$

Let $B = \mathsf{N}_\omega^{\varphi\psi}(\omega)$ be normally solvable $\operatorname{ran} B$ is closed in $\mathcal{L}$. Since $\mathcal{L}_-$ and $\mathcal{L}_0$ are finite-dimensional subspaces in $\mathcal{L}$ then $\operatorname{ran} B_+$ is closed and by Banach Theorem there is $c > 0$ such that

$$(B_+v,\, v)_{\mathcal{L}} \geq c^2||v||^2_{\mathcal{L}} \quad (v \in \mathcal{L}^+). \tag{3.20}$$

Due to Proposition 3.2 it can be rewritten as

$$\left[\gamma(\omega)v,\, \gamma(\omega)v\right]_{\mathcal{H}(\varphi,\psi)} \geq c^2\|v\|^2_{\mathcal{L}} \quad (v \in \mathcal{L}^+). \tag{3.21}$$

Thus $\mathfrak{N}_\omega^+ = \gamma(\omega)\mathcal{L}_+$ is closed. Since $\dim \mathfrak{N}_\omega^- \leq \kappa$ and $\dim \mathfrak{N}_\omega^0 \leq \kappa$ this implies that $\widetilde{\mathfrak{N}}_\omega$ is closed.

Conversely, if $\widetilde{\mathfrak{N}}_\omega$ is closed, then $\mathfrak{N}_\omega^+$ is also closed. Since $\gamma(\omega) \upharpoonright_{\mathcal{L}_+}$ is invertible, then there is $c > 0$ such that (3.21) holds. In view of Proposition 3.2 this implies that $\operatorname{ran} B_+$ is closed in $\mathcal{L}$. Since B_- is finite-dimensional then $\operatorname{ran} B$ is closed. This proves the statement. □

Remark 3.3. If m is a function from the class $N_\kappa(\mathcal{L})$ such that $I-\lambda m(\lambda)$ is invertible for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ then the pair $\{I_\mathcal{L},\, m(\lambda)\}$ is equivalent to the normalized N_κ-pair

$$\{\varphi,\psi\} = \left\{\left(I_\mathcal{L} - \lambda m(\lambda)\right)^{-1},\, m(\lambda)\left(I_\mathcal{L} - \lambda m(\lambda)\right)^{-1}\right\}$$

and the corresponding model operator can be rewritten as

$$A(\phi,\psi)=\left\{\left\{\begin{bmatrix} f \\ u \end{bmatrix}, \begin{bmatrix} f' \\ u' \end{bmatrix}\right\} : \begin{array}{l} f, f' \in \mathcal{H}(\phi,\psi), \quad u, u' \in \mathcal{L}; \\ f'(\lambda) - \lambda f(\lambda) = u - m(\lambda)u',\ \lambda \in \mathbb{C}\backslash\mathbb{R} \end{array}\right\}. \tag{3.22}$$

Considering the projection of this model to the space $\mathcal{H}(\varphi,\psi)$, one obtains the model for a symmetric operator S with the abstract Weyl function $m(\lambda)$, given in [10] in the Hilbert space case and in [8] in the Pontryagin space case. In particular, a model for a selfadjoint extension A_0 of S can be derived from (3.22) in the form

$$A_0 = \left\{\{f,\, f'\} \in \mathcal{H}(\phi,\psi)^2 : f'(\lambda) - \lambda f(\lambda) \equiv u \text{ for some } u \in \mathcal{L}\right\}. \tag{3.23}$$

This reproducing kernel space model appeared originally in [1].

4. Generalized Fourier transform

In this section we show that every $\mathcal{L}$-minimal selfadjoint linear relation A is unitarily equivalent to its functional model $A(\varphi,\psi)$, constructed in Theorem 3.1. The operator $\mathcal{F} : \mathcal{H} \to \mathcal{H}(\varphi,\psi)$ given by the formula

$$h \mapsto (\mathcal{F}h)(\lambda) = \gamma(\bar\lambda)^* h = P_\mathcal{L}(\widetilde{A} - \lambda)^{-1} h \quad (h \in \mathcal{H}) \tag{4.24}$$

is called the generalized Fourier transform associated with $\widetilde{A}$ and the scale $\mathcal{L}$.

Theorem 4.1. Let $\widetilde{A}$ be a selfadjoint linear relation in $\mathcal{H} \oplus \mathcal{L}$ and let $\{\varphi,\psi\}$ be the corresponding N_κ-pair given by (2.4). Then:

1) The generalized Fourier transform $\mathcal{F}$ maps isometrically the subspace $\mathcal{H}_0$ onto $\mathcal{H}(\varphi,\psi)$ and $\mathcal{F}$ is identically equal to 0 on $\mathcal{H} \ominus \mathcal{H}_0$;

2) For every $\left\{\begin{bmatrix} f \\ u \end{bmatrix}, \begin{bmatrix} f' \\ u' \end{bmatrix}\right\} \in \widetilde{A}$ the following identity holds

$$E(\lambda)\mathcal{F}(f' - \lambda f) = \begin{bmatrix}\varphi(\lambda) & -\psi(\lambda)\end{bmatrix} \begin{bmatrix} u \\ u' \end{bmatrix}. \tag{4.25}$$

Proof. 1) For every vector $h = \gamma(\bar{\omega})v$ ($\omega \in \rho(\widetilde{A})$, $v \in \mathcal{L}$) it follows from Proposition 3.2 that

$$(\mathcal{F}h)(\lambda) = \gamma(\bar{\lambda})^*\gamma(\bar{\omega})v = \mathsf{N}_\omega^{\varphi\psi}(\lambda)v.$$

Therefore, $\mathcal{F}$ maps the linear space $\operatorname{span}\{\gamma(\bar{\omega})\mathcal{L} : \omega \in \rho(\widetilde{A})\}$ which is dense in $\mathcal{H}_0$ onto the linear space $\operatorname{span}\{\mathsf{N}_\omega^{\varphi\psi}(\cdot)\mathcal{L} : \omega \in \rho(\widetilde{A})\}$ which is dense in $\mathcal{H}(\varphi, \psi)$. Moreover, this mapping is isometric, since

$$\begin{aligned}[][\mathcal{F}h, \mathcal{F}h]_{\mathcal{H}(\varphi,\psi)} &= [\mathsf{N}_\omega^{\varphi\psi}(\cdot)v, \mathsf{N}_\omega^{\varphi\psi}(\cdot)v]_{\mathcal{H}(\varphi,\psi)} \\ &= (\mathsf{N}_\omega^{\varphi\psi}(\omega)v, v)_{\mathcal{L}} = [h, h]_{\mathcal{H}}. \qquad (4.26)\end{aligned}$$

This proves the first statement. It is clear from (4.24) that $\mathcal{F}h \equiv 0$ for $h \in \mathcal{H} \ominus \mathcal{H}_0$.

2) Let $h = \gamma(\bar{\omega})v = P_{\mathcal{H}}(\widetilde{A} - \bar{\omega})^{-1}v$, $v \in \mathcal{L}$. It follows from (2.4), (3.15) that

$$\begin{bmatrix} h \\ \psi(\bar{\omega})v \end{bmatrix} = (\widetilde{A} - \bar{\omega})^{-1}\begin{bmatrix} 0 \\ v \end{bmatrix}, \quad \begin{bmatrix} \bar{\omega}h \\ \varphi(\bar{\omega})v \end{bmatrix} = \left(I + \bar{\omega}(\widetilde{A} - \bar{\omega})^{-1}\right)\begin{bmatrix} 0 \\ v \end{bmatrix}$$

and hence

$$\left\{\begin{bmatrix} h \\ \psi(\bar{\omega})v \end{bmatrix}, \begin{bmatrix} \bar{\omega}h \\ \varphi(\bar{\omega})v \end{bmatrix}\right\} \in \widetilde{A}.$$

Since $\widetilde{A} = \widetilde{A}^*$ one obtains for all $\left\{\begin{bmatrix} f \\ u \end{bmatrix}, \begin{bmatrix} f' \\ u' \end{bmatrix}\right\} \in \widetilde{A}$

$$[f', h]_{\mathcal{H}} - [f, \bar{\omega}h]_{\mathcal{H}} + \left(u', \psi(\bar{\omega})\right)_{\mathcal{L}} - \left(u, \varphi(\bar{\omega})v\right)_{\mathcal{L}} = 0, \quad v \in \mathcal{L}.$$

This implies

$$\gamma(\bar{\omega})^*(f' - \bar{\omega}f) = \varphi(\omega)u - \psi(\omega)u', \quad \omega \in \rho(\widetilde{A}). \qquad (4.27)$$

This proves the identity (4.25). □

Corollary 4.1. In the case, when the linear relation $\widetilde{A}$ is $\mathcal{L}$-minimal it is unitary equivalent to the linear relation $A(\varphi, \psi)$ via the formula

$$A(\varphi, \psi) = \left\{\left\{\begin{bmatrix} \mathcal{F}f \\ u \end{bmatrix}, \begin{bmatrix} \mathcal{F}f' \\ u' \end{bmatrix}\right\} : \left\{\begin{bmatrix} f \\ u \end{bmatrix}, \begin{bmatrix} f' \\ u' \end{bmatrix}\right\} \in \widetilde{A}\right\}. \qquad (4.28)$$

The operator $\mathcal{F} \oplus I_{\mathcal{L}}$ establishes this unitary equivalence.

Corollary 4.2. It follows from (3.13) that the Fourier transform $\mathcal{F}$ associated with the operator $A(\varphi, \psi)$ is identical, since

$$(\mathcal{F}h)(\lambda) = P_{\mathcal{L}}(A(\varphi, \psi) - \lambda)^{-1}\begin{bmatrix} h \\ 0 \end{bmatrix} = h(\lambda) \quad \text{for every } h \in \mathcal{H}(\varphi, \psi).$$

Lemma 4.1. Let $\widetilde{A}$ be a selfadjoint linear relation in $\mathcal{H}\oplus\mathcal{L}$, let $\{\varphi,\psi\}$ be the normalized N_κ-pair given by (2.4). Then the following implications hold

(i) $\ker\psi(\lambda)=\{0\}$ for some $\lambda\in\rho(\widetilde{A})\Rightarrow P_{\mathcal{L}}\operatorname{dom}\widetilde{A}$ is dense in $\mathcal{L}$;

(ii) $\ker\varphi(\lambda)=\{0\}$ for some $\lambda\in\rho(\widetilde{A})\Rightarrow P_{\mathcal{L}}\operatorname{ran}\widetilde{A}$ is dense in $\mathcal{L}$.

If, in addition, the relation $\widetilde{A}$ is $\mathcal{L}$-minimal, and $\mathsf{N}_\omega^{\varphi\psi}(\omega)>0$ for some $\omega\in\rho(\widetilde{A})$ then

$$\ker\varphi(\omega)=\{0\},\quad \ker\psi(\omega)=\{0\}.$$

Proof. Let us prove the first statement. The set $P_{\mathcal{L}}\operatorname{dom}\widetilde{A}$ consists of the vectors $u\in\mathcal{L}$ such that

$$\left\{\begin{bmatrix}f\\u\end{bmatrix},\begin{bmatrix}f'\\u'\end{bmatrix}\right\}\in\widetilde{A}\quad\text{for some}\quad f,f'\in\mathcal{H},\ u'\in\mathcal{L}.$$

If there is a vector $v\in\mathcal{L}$ such that $v\perp u$ for all $u\in P_{\mathcal{L}}\operatorname{dom}\widetilde{A}$ then

$$\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\v\end{bmatrix}\right\}\in\widetilde{A},$$

and then $\psi(\lambda)v=0$, due to (2.4). But $\ker\psi(\lambda)=\{0\}$ therefore $v=0$.

The proof of the second statement is similar.

Assume now that $\mathsf{N}_\omega^{\varphi\psi}(\omega)>0$ for some $\omega\in\rho(\widetilde{A})$ and that $\psi(\omega)v=0$. Then in view of (2.4) $\varphi(\omega)v=v$ and

$$\begin{aligned}\big(\mathsf{N}_\omega(\omega)v,v\big)_{\mathcal{L}}&=\big(\mathsf{N}_{\overline{\omega}}(\overline{\omega})v,v\big)_{\mathcal{L}}\\&=\frac{1}{\omega-\bar{\omega}}\Big(\big(\psi(\bar{\omega})\varphi(\omega)-\varphi(\bar{\omega})\psi(\omega)\big)v,v\Big)_{\mathcal{L}}\\&=\frac{1}{\omega-\bar{\omega}}\big(v,\varphi(\omega)v\big)_{\mathcal{L}}=0.\end{aligned}$$

This implies $v=0$. □

Criterions for the right parts in (i) and (ii) to be true are given in the following lemma.

Lemma 4.2. Let $\widetilde{A}$ be a $\mathcal{L}$-minimal selfadjoint linear relation in $\mathcal{H}\oplus\mathcal{L}$, let $\{\varphi,\psi\}$ be the normalized N_κ-pair given by (2.4). Then

(i) $\bigcap_{\lambda\in\rho(\widetilde{A})}\ker\psi(\lambda)=\{0\}$ if and only if $P_{\mathcal{L}}\operatorname{ran}\widetilde{A}$ is dense in $\mathcal{L}$;

(ii) $\bigcap_{\lambda\in\rho(\widetilde{A})}\ker\varphi(\lambda)=\{0\}$ if and only if $P_{\mathcal{L}}\operatorname{ran}\widetilde{A}$ is dense in $\mathcal{L}$.

Proof. The necessity of (i) and (ii) follows from Lemma 4.1. To prove the sufficiency let us consider the linear relation $A(\varphi,\psi)$ given by (3.10). By Corollary 4.1 $A(\varphi,\psi)$ is unitary equivalent to the linear relation $\widetilde{A}$ and, hence, we may prove the statement for the linear relation $A(\varphi,\psi)$.

Assume that $\psi(\lambda)v = 0$ for some $v \in \mathcal{L}$ and for all $\lambda \in \rho\big(A(\varphi,\psi)\big)$. Then in view of (2.4) $\varphi(\lambda)v = v$ and

$$\mathsf{N}^{\varphi\psi}_{\bar{\lambda}}(\omega)v = \frac{1}{\omega-\lambda}\big(\psi(\omega)\varphi(\lambda) - \varphi(\omega)\psi(\lambda)\big)v = 0$$

for all λ, $\omega \in \rho\big(A(\varphi,\psi)\big)$. Now it follows from (3.11) that

$$\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\v\end{bmatrix}\right\} = \left\{\begin{bmatrix}\mathsf{N}^{\varphi\psi}_{\omega}(\cdot)v\\0\end{bmatrix}, \begin{bmatrix}\bar{\omega}\mathsf{N}^{\varphi\psi}_{\omega}(\cdot)v\\v\end{bmatrix}\right\} \in A(\varphi,\psi).$$

and, hence, $v \perp P_{\mathcal{L}} \operatorname{dom} \widetilde{A}$. □

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