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Fundamental solutions of boundary problems and resolvents of differential operators

Vadim Mogilevskii

(*Presented by M. M. Malamud*)

Abstract. The main objects of our considerations are differential operators generated by a formally selfadjoint differential expression of an even order. The coefficients of this expression are operator valued functions defined on the interval $[0, b\rangle$ ($b \leq \infty$) with values in the set of all linear bounded operators in a separable Hilbert space H. Our approach is based on the concept of a decomposing D-boundary triplet, which enables to describe various properties of (regular and singular) differential operators immediately in terms of boundary conditions. First we complement and generalize known results on fundamental solutions of boundary problems with the boundary condition at the singular end b. Next by using Krein type formula for resolvents we obtain the representation of the resolvent $(\widetilde{A}-\lambda)^{-1}$ ($\widetilde{A}$ is a proper extension of the minimal operator L_0) in a form of the integral operator

$$((\widetilde{A}-\lambda)^{-1} f)(x) = \int\limits_0^b G(x,t,\lambda) f(t)\, dt \quad f = f(\cdot) \in \mathfrak{H}$$

with the operator valued Green function $G(x,t,\lambda)$. Unlike classical methods our approach enables to characterize spectrum of the extension $\widetilde{A}$ and represent the Green function immediately in terms of boundary conditions for $\widetilde{A}$ and fundamental solutions of the corresponding boundary problems. The above results are proved for differential operators with arbitrary (possibly unequal) deficiency indices in the case $\dim H \leq \infty$.

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1. Introduction

Let H be a separable Hilbert space, let $[H_1, H_2]([H])$ be the set of all bounded linear operators from H_1 to H_2 (in H) and let

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$$l[y] = l_H[y] = \sum_{k=1}^{n} (-1)^k ((p_{n-k} y^{(k)})^{(k)} - \tfrac{i}{2}[(q_{n-k}^* y^{(k)})^{(k-1)} + (q_{n-k} y^{(k-1)})^{(k)}]) + p_n y \quad (1.1)$$

be a formally selfadjoint differential expression of an even order $2n$ with operator-valued coefficients $p_k(\cdot), q_k(\cdot) : \Delta \to [H]$ defined on an interval $\Delta = [0, b\rangle$ ($b \le \infty$). Denote by L_0 and L minimal and maximal operators respectively, induced by the expression (1.1) in the Hilbert space $\mathfrak{H} := L_2(\Delta; H)$ and let $\mathcal{D}$ be the domain of L. As is known L_0 is a closed densely defined symmetric operator with not necessarily equal deficiency indices $n_\pm(L_0)$ and $L_0^* = L$. Moreover functions $y, z \in \mathcal{D}$ obey the Lagrange's identity

$$(Ly, z)_{\mathfrak{H}} - (y, Lz)_{\mathfrak{H}} = [y, z](b) - [y, z](0), \qquad y, z \in \mathcal{D}$$

where

$$[y, z](t) = (y^{(1)}(t), z^{(2)}(t))_{H^n} - (y^{(2)}(t), z^{(1)}(t))_{H^n},$$
$$[y, z](b) = \lim_{t \uparrow b} [y, z](t).$$

and $y^{(1)}(t),\ y^{(2)}(t)$ $(\in H^n)$ are vectors of the quasi-derivatives (see (3.2)).

Next recall that a closed operator $\widetilde{A}$ with the domain $\mathcal{D}(\widetilde{A})$ is called a proper extension of L_0 (and is referred to the class Ext_{L_0}) if $L_0 \subset \widetilde{A} \subset L$. As is known an important problem in the spectral theory of differential operators is a description of all selfadjoint boundary conditions or, equivalently, all selfadjoint extensions $\widetilde{A} \in Ext_{L_0}$. For a regular expression $l[y]$ (i.e., in the case $\Delta = [0, b],\ b < \infty$) this problem was solved in a compact form by F. S. Rofe-Beketov in [18]. In particular, it was shown in this paper that the set of all selfadjoint decomposing boundary conditions is described by the relation

$$\cos B_1\, y^{(1)}(0) + \sin B_1\, y^{(2)}(0) = 0, \quad \cos B_2\, y^{(1)}(b) + \sin B_2\, y^{(2)}(b) = 0, \tag{1.2}$$

where B_1, $B_2 \in [H^n]$ is a pair of selfadjoint operators obeying $-\frac{\pi}{2} I < B_1, B_2 \le \frac{\pi}{2} I$. Afterwards in [9] this result was extended to the case of a quasi-regular expression $l[y]$.

Next, in [19,20] for an arbitrary expression $l[y]$ the concept of a selfadjoint boundary condition at the singular end b was introduced as follows. Let $U : \mathcal{D} \to \mathcal{K}$ be a linear map with values in a Hilbert space $\mathcal{K}$ and let $L_b,\ L_b' \in Ext_{L_0}$ be extensions with domains

$$\mathcal{D}(L_b) = \{y \in \mathcal{D} :\ y^{(1)}(0) = y^{(2)}(0) = 0,\ Uy = 0\},$$
$$\mathcal{D}(L_b') = \{y \in \mathcal{D} :\ Uy = 0\}. \tag{1.3}$$

Then the equality

$$Uy = 0 \tag{1.4}$$

is called a selfadjoint boundary condition at the point b if: (i) (1.4) holds for each function $y \in \mathcal{D}$ finite at the point b; (ii) L_b is a symmetric operator and $L_b^* = L_b'$.

In the scalar case ($\dim H = 1$) for the operator L_0 with equal deficiency indices $m = n_\pm(L_0)$ each selfadjoint boundary condition (1.4) can be represented as

$$Uy := \big([y, z_1](b),\ [y, z_2](b), \dots, [y, z_p](b)\big) = 0, \tag{1.5}$$

where $p = m - n$ and $z_j \in \mathcal{D},\ j = 1 \div p$ are linearly independent modulo $\mathcal{D}(L_0)$ functions obeying $z_j^{(1)}(0) = z_j^{(2)}(0) = 0,\ j = 1 \div p$ and $[z_j, z_k](b) = 0,\ j \neq k$ (cf. [17, §18], [3, ch13.2]). In this connection note that formulas (1.4) and (1.5) are not convenient for applications, because unlike (1.2) they do not define explicit parametrization of all selfadjoint boundary conditions. This statement is especially evident in the case $\dim H = \infty$, when the finite representation (1.5) becomes impossible.

An attempt to extend the results of [18] to singular differential operators with arbitrary (possibly unequal) deficiency indices was carried out in our paper [16]. The method of [16] is based on the concept of a decomposing D-triplet for L, which is defined as follows. Let $\mathcal{H}_1'$ be a subspace in a Hilbert space $\mathcal{H}_0'$ and let $\Gamma_j' : \mathcal{D} \to \mathcal{H}_j',\ j \in \{0, 1\}$ be linear maps such that $\Gamma' = (\Gamma_0'\ \Gamma_1')^\top$ is a surjective linear map onto $\mathcal{H}_0' \oplus \mathcal{H}_1'$ and the following identity holds

$$[y, z](b) = (\Gamma_1' y, \Gamma_0' z) - (\Gamma_0' y, \Gamma_1' z) + i(P_2'\Gamma_0' y, P_2'\Gamma_0' z), \quad y, z \in \mathcal{D}.$$

(here P_2 is the orthoprojector in $\mathcal{H}_0'$ onto $\mathcal{H}_0' \ominus \mathcal{H}_1'$). Then a decomposing D-triplet for L is a collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, in which $\mathcal{H}_j = H^n \oplus \mathcal{H}_j'$ and $\Gamma_j : \mathcal{D} \to \mathcal{H}_j,\ j \in \{0, 1\}$ are linear maps given by

$$\begin{aligned} \Gamma_0 y &= \{y^{(2)}(0), \Gamma_0' y\}\ (\in H^n \oplus \mathcal{H}_0'), \\ \Gamma_1 y &= \{-y^{(1)}(0), \Gamma_1' y\}\ (\in H^n \oplus \mathcal{H}_1'), \end{aligned} \quad y \in \mathcal{D}. \tag{1.6}$$

In the case $\mathcal{H}_0' = \mathcal{H}_1' =: \mathcal{H}'$ ($\iff \mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$) a decomposing D-triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a decomposing boundary triplet for L.

Associated with the decomposing D-triplet Π for L is the Weyl function $M_+(\cdot)$ defined by

$$\Gamma_1 f_\lambda = M_+(\lambda)\Gamma_0 f_\lambda, \qquad f_\lambda \in \mathfrak{N}_\lambda(L_0) := \operatorname{Ker}(L^* - \lambda), \quad \lambda \in \mathbb{C}_+.$$

It turns out that a decomposing boundary triplet is a boundary triplet (boundary value space) in the sense of [8], while the function $M(\lambda) = M_+(\lambda)$ coincides with the Weyl function introduced by V. A. Derkach and M. M. Malamud [4] (see also [5] and references there in).

In the paper [16] we also defined defect numbers $n_{b\pm}$ of the expression (1.1) at the point b. In the case $\dim H < \infty$ this numbers obey the equalities

$$n_{b+} = n_+(L_0) - n \dim H, \qquad n_{b-} = n_-(L_0) - n \dim H.$$

Moreover, a decomposing D-triplet (boundary triplet) for L satisfies the relation $\dim \mathcal{H}_1' = n_{b-} \leq n_{b+} = \dim \mathcal{H}_0'$ (respectively, $n_{b-} = n_{b+} = \dim \mathcal{H}'$).

A decomposing D-triplet enables to describe various properties of differential operators in terms of boundary conditions. In particular it was shown in [16] that selfadjoint decomposing boundary conditions exist if and only if $n_{b+} = n_{b-}$. Moreover if this criterium is satisfied and $\Pi = \{H^n \oplus \mathcal{H}', \Gamma_0, \Gamma_1\}$ is a decomposing boundary triplet (1.6), then the set of all selfadjoint decomposing conditions is described by the relations

$$\cos B_1\, y^{(1)}(0) + \sin B_1\, y^{(2)}(0) = 0, \tag{1.7}$$

$$\cos B_2\, \Gamma_0' y + \sin B_2\, \Gamma_1' y = 0, \tag{1.8}$$

where $B_1 \in [H^n]$, $B_2 \in [\mathcal{H}']$ is a pair of selfadjoint operators. This implies that a selfadjoint boundary condition at the point b is defined by formula (1.8), which gives a parametrization of all such conditions by means of the selfadjoint parameter B_2 (cf. (1.4)). In this connection note that for the regular expression one can put $\Gamma_0' y = y^{(2)}(b)$, $\Gamma_1' y = y^{(1)}(b)$, in which case (1.7) and (1.8) turn into the boundary conditions (1.2). Moreover, for a singular expression $l[y]$ the boundary operators Γ_0' and Γ_1' can be explicitly defined in terms of limits of some regularizations of quasi-derivatives $y^{[k]}(t)$ at the point b (see Proposition 3.10 in [16]). Hence (1.8) defines a boundary condition in terms of boundary values $\Gamma_0' y$ and $\Gamma_1' y$ of the function $y \in \mathcal{D}$ at the singular end b.

Observe also that a decomposing boundary triplet (1.6) enables to define in a compact form boundary conditions of other classes. For example, an accumulative boundary condition at the point b is defined by the relation

$$N_0 \Gamma_0' y + N_1 \Gamma_1' y = 0 \tag{1.9}$$

with operators $N_0, N_1 \in [\mathcal{H}']$ generating an accumulative operator pair (linear relation) $\theta = \{(N_0, N_1); \mathcal{H}'\}$ or, equivalently, obeying $Im\, N_1 N_0^* \leq 0$ and $0 \in \rho(N_0 + iN_1)$. This observation can be useful in the theory of generalized resolvents of differential operators.

The present paper contains further investigations of differential operators by means of decomposing D-triplets. Here we first complement and generalize known results on fundamental solutions of boundary problems [19, 20] and then obtain the representation of the resolvent $(\widetilde{A}-\lambda)^{-1}$ ($\widetilde{A}\in Ext_{L_0}$) in a form of the integral operator

$$((\widetilde{A}-\lambda)^{-1}f)(x)=\int_0^b G(x,t,\lambda)f(t)\,dt \quad f=f(\cdot)\in\mathfrak{H}. \tag{1.10}$$

Moreover we show that the operator valued kernel $G(x,t,\lambda)(\in[H])$ in (1.10) can be explicitly defined in terms of boundary conditions for $\widetilde{A}$ and fundamental solutions of the corresponding boundary problems.

Recall that in [19, 20] a selfadjoint boundary problem is given by the equation

$$l[y]-\lambda y=0 \tag{1.11}$$

and the selfadjoint boundary condition (1.4). Moreover, an n-component operator function $v(\cdot,\lambda):\Delta\to[H^n,H]$ is called a fundamental solution of this problem if: (i) the equality

$$y=y(t,\lambda):=v(t,\lambda)\,\hat{h},\quad \hat{h}\in H^n$$

gives all vector solutions of (1.11) obeying the boundary condition (1.4); (ii) the operator

$$\widetilde{v}(0,\lambda):=(v^{(1)}(0,\lambda)\quad v^{(2)}(0,\lambda))^\top:H^n\to H^n\oplus H^n$$

is an injection, that is Ker $\widetilde{v}(0,\lambda)=\{0\}$; (iii) the range $\widetilde{v}(0,\lambda)H^n$ is a closed subspace in $H^n\oplus H^n$. A fundamental solution $v(t,\lambda)$ is said to be holomorphic on the set $E\subset\mathbb{C}$ if there is an open set $\Lambda\supset E$ such that $v(t,\lambda)$ id defined for all $\lambda\in\Lambda$ and the operator function $\widetilde{v}(0,\lambda)$ is holomorphic on Λ. It was proved in [19] that there exists a fundamental solution of the selfadjoint problem (1.11), (1.4) holomorphic on the set $\hat{\rho}_r(L_b)$ of all real regular type points of of the operator L_b (see (1.3)).

In the present paper by using a decomposing D-triplet we complement this result and extend it to other classes of boundary problems. In particular, we show that the condition (iii) in the above definition is implied by (i) and (ii), so that it can be omitted. Moreover, the following statements are proved in the paper: 1) there exists a fundamental solution of the selfadjoint boundary problem (1.11), (1.8) holomorphic on $\hat{\rho}_r(L_b)\cup\mathbb{C}_+$. Moreover, for every $\lambda_0\in\hat{\rho}_r(L_b)$ there exists a fundamental solution of the same problem holomorphic on $(\lambda_0-\delta,\lambda_0+\delta)\cap\mathbb{C}_+\cap\mathbb{C}_-$ with some $\delta>0$; 2) the accumulative boundary problem (1.11), (1.9) has a fundamental solution holomorphic on $\mathbb{C}_+$; 3) the "free" boundary problem (1.11), (1.9) with $N_0 = N_1 = 0$ has fundamental solutions with the same properties as in the statement 1), but with $\hat{\rho}_r(L_0)$ instead of $\hat{\rho}_r(L_b)$.

By using a decomposing D-triplet (1.6) one can characterize various classes of proper extensions [16]. In particular, the extension $\widetilde{A} \in Ext_{L_0}$ is defined by decomposing boundary conditions (see definition in [7]) if and only if its domain is given by

$$\mathcal{D}(\widetilde{A}) = \{y \in \mathcal{D} : \ \hat{N}_1 y^{(1)}(0) + \hat{N}_2 y^{(2)}(0) = 0, \ N_0 \Gamma_0' y + N_1 \Gamma_1' y = 0\} \tag{1.12}$$

with operators $\hat{N}_1, \hat{N}_2 \in [H^n]$ and $N_j \in [\mathcal{H}_j, \mathcal{K}], \ j \in \{0, 1\}$.

Let $v(\cdot, \lambda)$ be a fundamental solution of the boundary problem (1.11), (1.9) with N_0 and N_1 taken from (1.12). We show in the paper that the extension (1.12) has the same spectral properties as the operator $\hat{N}_1 v^{(1)}(0, \lambda) + \hat{N}_2 v^{(2)}(0, \lambda)$ (in the particular case of selfadjoint boundary conditions (1.7), (1.8) this result was obtained in [19, 20]). Next by using the Krein type formula for resolvents [15] we obtain the representation (1.10) with the operator-valued Green function $G(\cdot, \cdot, \lambda) : \Delta \times \Delta \to [H]$ given by

$$G(x, t, \lambda) = \begin{cases} v(x, \lambda)\, \varphi^*(t, \overline{\lambda}), & x > t \\ \varphi_\times(x, \lambda)\, v^*(t, \overline{\lambda}), & x < t \end{cases}, \quad \lambda \in \rho(\widetilde{A}). \tag{1.13}$$

Here $\varphi(t, \overline{\lambda})$ is the operator solution of the equation $l[y] - \overline{\lambda} y = 0$ with the initial data

$$(\varphi^{(1)}(0, \overline{\lambda}) \quad \varphi^{(2)}(0, \overline{\lambda}))^\top = (-\hat{N}_2^* \quad \hat{N}_1^*)^\top (\hat{N}_1 v^{(1)}(0, \lambda) + \hat{N}_2 v^{(2)}(0, \lambda))^{-1*} \tag{1.14}$$

and $v(t, \overline{\lambda}), \ \varphi_\times(x, \lambda)$ are similar operator solutions for the adjoint boundary problem.

In the scalar case the representation (1.10) is a well known classical result [1, 3, 17]. Moreover, in [2, 13] formula (1.10) were partially extended to the case $\dim H = \infty$. In this connection note that unlike classical methods our approach enables to characterize spectrum of the extension $\widetilde{A} \in Ext_{L_0}$ and represent its resolvents immediately in terms of boundary conditions (1.12) and the fundamental solutions $v(t, \lambda), \ v(t, \overline{\lambda})$ of the corresponding boundary problems (see (1.10), (1.13) and (1.14)). Moreover, similar results are obtained in the paper for extensions $\widetilde{A} \in Ext_{L_0}$ defined by general (not necessarily decomposing) boundary conditions. Emphasize also that the above statements are proved for a differential expression (1.1) with $\dim H \le \infty$ and arbitrary deficiency indices $n_\pm(L_0)$.

In conclusion note that in the forthcoming paper the mentioned results will be applied to the theory of generalized resolvents and spectral functions of differential operators.

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2. Preliminaries

2.1. Notations

The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathcal{H}$ denote Hilbert spaces; $[\mathcal{H}_1, \mathcal{H}_2]$ is the set of all bounded linear operators defined on $\mathcal{H}_1$ with values in $\mathcal{H}_2$; $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$; $A \upharpoonright \mathcal{L}$ is the restriction of an operator A onto the linear manifold $\mathcal{L}$; $P_{\mathcal{L}}$ is the orthogonal projector in $\mathfrak{H}$ onto the subspace $\mathcal{L} \subset \mathfrak{H}$; $\mathbb{C}_+$ ($\mathbb{C}_-$) is the upper (lower) half-plain of the complex plain.

Recall that a closed linear relation from $\mathcal{H}_0$ to $\mathcal{H}_1$ is a closed subspace in $\mathcal{H}_0 \oplus \mathcal{H}_1$. The set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ (from $\mathcal{H}$ to $\mathcal{H}$) will be denoted by $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ ($\widetilde{\mathcal{C}}(\mathcal{H})$). A closed linear operator T from $\mathcal{H}_0$ to $\mathcal{H}_1$ is identified with its graph $\mathrm{gr}T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$.

For a relation $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we denote by $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\mathrm{Ker}T$ the domain, range and the kernel respectively. The inverse T^{-1} and adjoint T^* are relations defined by

$$T^{-1} = \{\{f', f\} : \{f, f'\} \in T\}, \quad T^{-1} \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$$
$$T^* = \{\{g, g'\} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : (f', g) = (f, g'),\ \{f, f'\} \in T\}, \qquad T^* \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0).$$

In the case $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we write:

$0 \in \rho(T)$ if $\mathrm{Ker}\, T = \{0\}$ and $\mathcal{R}(T) = \mathcal{H}_1$, or equivalently if $T^{-1} \in [\mathcal{H}_1, \mathcal{H}_0]$;

$0 \in \hat{\rho}(T)$ if $\mathrm{Ker}\, T = \{0\}$ and $\mathcal{R}(T)$ is a closed subspace in $\mathcal{H}_1$;

$0 \in \sigma_c(T)$ if $\mathrm{Ker}\, T = \{0\}$ and $\overline{\mathcal{R}(T)} = \mathcal{H}_1 \neq \mathcal{R}(T)$;

$0 \in \sigma_p(T)$ if $\mathrm{Ker}\, T \neq \{0\}$; $0 \in \sigma_r(T)$ if $\mathrm{Ker}\, T = \{0\}$ and $\overline{\mathcal{R}(T)} \neq \mathcal{H}_1$.

For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H})$ we denote by $\rho(T) = \{\lambda \in \mathbb{C} :\ 0 \in \rho(T - \lambda)\}$ and $\hat{\rho}(T) = \{\lambda \in \mathbb{C} :\ 0 \in \hat{\rho}(T - \lambda)\}$ the resolvent set and the set of regular type points of T respectively. Next, $\sigma(T) = \mathbb{C} \backslash \rho(T)$ stands for the spectrum of T. The spectrum $\sigma(T)$ admits the following classification:

$\sigma_c(T) = \{\lambda \in \mathbb{C} : 0 \in \sigma_c(T-\lambda)\}$ is the continuous spectrum; $\sigma_p(T) = \{\lambda \in \mathbb{C} : 0 \in \sigma_p(T-\lambda)\}$ is the point spectrum; $\sigma_r(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_c(T)) = \{\lambda \in \mathbb{C} : 0 \in \sigma_r(T-\lambda)\}$ is the residual spectrum.

Let $T \in \widetilde{\mathcal{C}}(\mathcal{H})$ be a densely defined operator. For any $\lambda \in \mathbb{C}$ we put

$$\mathfrak{N}_\lambda(T) := \operatorname{Ker}(T^* - \lambda)\,(= \mathcal{H} \ominus \mathcal{R}(T - \overline{\lambda})).$$

If $\overline{\lambda} \in \hat{\rho}(T)$, then $\mathfrak{N}_\lambda(T)$ is a defect subspace of the operator T.

2.2. Operator pairs and linear relations

Let $\mathcal{K}, \mathcal{H}_0, \mathcal{H}_1$ be Hilbert spaces and let $C_j \in [\mathcal{H}_j, \mathcal{K}],\ j \in \{0,1\}$ be a pair of operators. In what follows we identify such a pair with an operator

$$C = (C_0 \quad C_1) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{K}. \tag{2.1}$$

A pair (2.1) will be called admissible if $\mathcal{R}(C) = \mathcal{K}$. In the sequel all pairs (2.1) are admissible unless otherwise stated.

Definition 2.1. *Two operator pairs*

$$C^{(j)} = (C_0^{(j)} \ C_1^{(j)}) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{K}_j, \quad j \in \{1,2\}$$

are said to be equivalent if $C^{(2)} = XC^{(1)}$ *with an isomorphism* $X \in [\mathcal{K}_1, \mathcal{K}_2]$.

It is clear that the set of all operator pairs (2.1) falls into nonintersecting classes of equivalent pairs. Moreover each operator pair (2.1) generate a linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ by

$$\theta = \{(C_0, C_1); \mathcal{K}\} := \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : \, C_0 h_0 + C_1 h_1 = 0\} \tag{2.2}$$

Formula (2.2) gives a bijective correspondence between all $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ and all equivalence classes of operator pairs (2.1). Therefore we will denote by $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ both the set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ and the set of all equivalence classes of operator pairs (2.1) identifying them by means of (2.2).

The following lemma is immediate from Lemma 2.1 in [12].

Lemma 2.1. *Let* $\theta = \{(C_0, C_1); \mathcal{K}\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$, *let* $\theta^* = \{(C_{1*}, C_{0*}); \mathcal{K}_*\} \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$ *be the adjoint operator pair* (*linear relation*) *and let* $B \in [\mathcal{H}_1, \mathcal{H}_0]$. *Then* $0 \in \rho(C_{1*} + C_{0*}B) \iff 0 \in \rho(C_0^* + BC_1^*)$ *and the following equality holds*

$$C_1^*(C_0^* + BC_1^*)^{-1} = (C_{1*} + C_{0*}B)^{-1} C_{0*} (= -(\tau^* - B)^{-1}).$$

Next recall some results and definitions from our paper [14].

Let $\mathcal{H}_1$ be a subspace in a Hilbert space $\mathcal{H}_0$, let $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$ and let P_j be the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j \in \{1,2\}$. With every linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we associate a $\times$-adjoint linear relation $\theta^\times \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$, which is defined as the set of all vectors $\hat{k} = \{k_0, k_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1$ such that

$$(k_1, h_0) - (k_0, h_1) + i(P_2 k_0, P_2 h_0) = 0, \quad \{h_0, h_1\} \in \theta$$

Using this definition and the correspondence (2.2) we introduce the notion of a $\times$-adjoint operator pair (or more precisely a class of $\times$-adjoint operator pairs) $\theta^\times = \{(C_{0\times}, C_{1\times}); \mathcal{K}_\times\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ corresponding to an operator pair $\theta = \{(C_0, C_1); \mathcal{K}\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$. As was shown in [14] $(\theta^\times)^\times = \theta$. Moreover Proposition 3.1 in [14] implies that the adjoint operator pair θ^* admits the representation $\theta^* = \{(C_{1*}, C_{0*}); \mathcal{K}_\times\}$ with

$$C_{1*} = C_{0\times} \upharpoonright \mathcal{H}_1, \qquad C_{0*} = C_{1\times} P_1 - i C_{0\times} P_2. \tag{2.3}$$

It follows from (2.3) that $\theta^\times = \theta^*$ in the case $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$.

Definition 2.2 ([14]). *A linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to the class $Dis(\mathcal{H}_0, \mathcal{H}_1)$ $(Ac(\mathcal{H}_0, \mathcal{H}_1))$ if $\varphi_\theta(\hat{h}) := 2\,\mathrm{Im}(h_1, h_0) + \|P_2 h_0\|^2 \geq 0$ (resp. $\varphi_\theta(\hat{h}) \leq 0$) for all $\hat{h} = \{h_0, h_1\} \in \theta$ and there are no extensions $\widetilde{\theta} \supset \theta$, $\widetilde{\theta} \neq \theta$ with the same property.*

A linear relation $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ belongs to the class $Self(\mathcal{H}_0, \mathcal{H}_1)$ if $\theta = \theta^\times$

Note that in the case $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$ the classes $Dis(\mathcal{H}, \mathcal{H})$, $Ac(\mathcal{H}, \mathcal{H})$ and $Self(\mathcal{H}, \mathcal{H})$ coincide with the sets of all maximal dissipative, maximal accumulative and selfadjoint linear relations in $\mathcal{H}$ respectively. Moreover a description of above classes immediately in terms of the corresponding operator pairs (2.2) is contained in [14].

2.3. Dual pairs if linear operators

Definition 2.3 ([11,12]). *Closed densely defined operators A and B in a Hilbert space $\mathfrak{H}$ form a dual pair $\{A, B\}$ if $A \subset B^*$.*

A closed operator $\widetilde{A}$ in $\mathfrak{H}$ is called a proper extension of a dual pair $\{A, B\}$ and it is put in the class $Ext\{A, B\}$ if $A \subset \widetilde{A} \subset B^*$

For a dual pair $\{A, B\}$ we write $\lambda \in \hat{\rho}\{A, B\}$ if $\lambda \in \hat{\rho}(A)$ and $\overline{\lambda} \in \hat{\rho}(B)$.

Definition 2.4 ([11]). *Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be Hilbert spaces and let $\Gamma^B = (\Gamma_0^B \ \ \Gamma_1^B)^\top : \mathcal{D}(B^*) \to \mathcal{H}_0 \oplus \mathcal{H}_1$ and $\Gamma^A = (\Gamma_0^A \ \ \Gamma_1^A)^\top : \mathcal{D}(A^*) \to \mathcal{H}_1 \oplus \mathcal{H}_0$*

be linear maps. A collection $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma^B,\Gamma^A\}$ *is called a boundary triplet for a dual pair* $\{A,B\}$ *if*

$$\Gamma^B B^*=\mathcal{H}_0\oplus\mathcal{H}_1,\quad \Gamma^A A^*=\mathcal{H}_1\oplus\mathcal{H}_0 \tag{2.4}$$

and the following Green identity holds

$$(B^*f,g)-(f,A^*g)=(\Gamma_1^Bf,\Gamma_0^Ag)-(\Gamma_0^Bf,\Gamma_1^Ag),\quad f\in\mathcal{D}(B^*),\ \ g\in\mathcal{D}(A^*). \tag{2.5}$$

Proposition 2.1 ([12]). *Let* $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma^B,\Gamma^A\}$ *be a boundary triplet for a dual pair* $\{A,B\}$ *and let* $\mathcal{D}(B^*)_+$ *be a Hilbert of all elements* $f\in\mathcal{D}(B^*)$ *with the inner product*

$$(f,g)_+=(f,g)_{\mathfrak{H}}+(B^*f,B^*g)_{\mathfrak{H}},\quad f,g\in\mathcal{D}(B^*).$$

Then $\Gamma^B\in[\mathcal{D}(B^*)_+,\mathcal{H}_0\oplus\mathcal{H}_1]$.

The following lemma will be useful in the sequel.

Lemma 2.2. *Assume that* $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma^B,\Gamma^A\}$ *is a boundary triplet for a dual pair* $\{A,B\}$, $\theta=\{(C_0,C_1);\mathcal{K}\}\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ *and* $\widetilde{A}\in Ext\{A,B\}$ *is an extension given by*

$$\mathcal{D}(\widetilde{A})=\{f\in\mathcal{D}(B^*):\ C_0\Gamma_0^Bf+C_1\Gamma_1^Bf=0\}. \tag{2.6}$$

Moreover let $\lambda\in\hat{\rho}\{A,B\}$, *let* $F\in[\mathcal{K}',\mathfrak{N}_\lambda(B)]$ *be an isomorphism of a Hilbert space* $\mathcal{K}'$ *onto* $\mathfrak{N}_\lambda(B)$ *and let* $T\in[\mathcal{K}',\mathcal{K}]$ *be an operator given by*

$$T=(C_0\Gamma_0^B+C_1\Gamma_1^B)F=C_0(\Gamma_0^BF)+C_1(\Gamma_1^BF). \tag{2.7}$$

Then the following relations hold

$$\lambda\in\rho(\widetilde{A})\iff 0\in\rho(T),\quad \lambda\in\sigma_j(\widetilde{A})\iff 0\in\sigma_j(T),\quad j=p,c,r, \tag{2.8}$$

$$\lambda\in\hat{\rho}(\widetilde{A})\iff 0\in\hat{\rho}(T),\quad \overline{\mathcal{R}(\widetilde{A}-\lambda)}=\mathcal{R}(\widetilde{A}-\lambda)\iff\overline{\mathcal{R}(T)}=\mathcal{R}(T). \tag{2.9}$$

Proof. As was shown in [12] (see [12, Proposition 3.17] and the proof of [12, Proposition 5.2]) the following equalities are valid

$$\overline{\mathcal{R}(\widetilde{A}-\lambda)}=\mathcal{R}(\widetilde{A}-\lambda)\iff\overline{\mathcal{D}(\widetilde{A})+\mathfrak{N}_\lambda(B)}=\mathcal{D}(\widetilde{A})+\mathfrak{N}_\lambda(B), \tag{2.10}$$

$$\overline{\mathcal{R}(\widetilde{A}-\lambda)}=\mathfrak{H}\iff\mathcal{D}(B^*)=\overline{\mathcal{D}(\widetilde{A})+\mathfrak{N}_\lambda(B)}, \tag{2.11}$$

$$\mathrm{Ker}(\widetilde{A}-\lambda)=\{0\}\iff\mathcal{D}(\widetilde{A})\cap\mathfrak{N}_\lambda(B)=\{0\} \tag{2.12}$$

where $\overline{\mathcal{D}(\widetilde{A})+\mathfrak{N}_\lambda(B)}$ denotes the closure of the set $\mathcal{D}(\widetilde{A})+\mathfrak{N}_\lambda(B)$ in $\mathcal{D}(B^*)_+$. Next, in view of Proposition 2.1 the equality

$$S = C_0\Gamma_0^B + C_1\Gamma_1^B$$

defines a bounded operator $S \in [\mathcal{D}(B^*)_+, \mathcal{K}]$. Moreover the equalities (2.6) and (2.4) imply that $\operatorname{Ker} S = \mathcal{D}(\widetilde{A})$ and $\mathcal{R}(S) = \mathcal{K}$. Therefore the following relations hold

$$\overline{\mathcal{D}(\widetilde{A})+\mathfrak{N}_\lambda(B)} = \mathcal{D}(\widetilde{A})+\mathfrak{N}_\lambda(B) \iff \overline{S\mathfrak{N}_\lambda(B)} = S\mathfrak{N}_\lambda(B), \tag{2.13}$$

$$\mathcal{D}(B^*) = \overline{\mathcal{D}(\widetilde{A})+\mathfrak{N}_\lambda(B)} \iff \overline{S\mathfrak{N}_\lambda(B)} = \mathcal{K}, \tag{2.14}$$

$$\mathcal{D}(\widetilde{A}) \cap \mathfrak{N}_\lambda(B) = \{0\} \iff \operatorname{Ker}(S \upharpoonright \mathfrak{N}_\lambda(B)) = \{0\}. \tag{2.15}$$

Observe also that in view of (2.7) one has

$$T = SF = S \upharpoonright \mathfrak{N}_\lambda(B) \cdot F \tag{2.16}$$

where F is an isomorphism of $\mathcal{K}'$ onto $\mathfrak{N}_\lambda(B)$. Now combining (2.10)-(2.12) with (2.13)- (2.15) one obtains (2.8) and (2.9) with $S \upharpoonright \mathfrak{N}_\lambda(B)$ in place of T. This and the equality (2.16) give the relations (2.8) and (2.9). □

Remark 2.1. Lemma 2.2 generalizes [12, Proposition 5.2]. Namely, let $A_0 \in Ext\{A, B\}$ be an extension with the domain $\mathcal{D}(A_0) = \operatorname{Ker}\Gamma_0$, let $\gamma_\Pi(\lambda) := (\Gamma_0^B \upharpoonright \mathfrak{N}_\lambda(B))^{-1}$ be the γ-field and let $M_\Pi(\lambda) := \Gamma_1^B\gamma_\Pi(\lambda)$ $(\lambda \in \rho(A_0))$ be the Weyl function for the triplet Π (see [12] for the precise definitions). Then letting in Lemma 2.2 $F := \gamma_\Pi(\lambda)$, one obtains that the relations (2.8) and (2.9) hold with $T = T(\lambda) := C_0 + C_1 M(\lambda)$. This statement was established by another way in [12] (see also [5] for the case of a symmetric operator A, that is for the dual pair $\{A, A\}$).

2.4. Boundary triplets for symmetric operators

Let A be a closed densely defined symmetric operator in $\mathfrak{H}$ with deficiency indices $n_\pm(A) := \dim \mathfrak{N}_\lambda(A)$ $(\lambda \in \mathbb{C}_\pm)$. Denote by Ext_A the set of all proper extensions of A, i.e., the set of all closed operators $\widetilde{A}$ in $\mathfrak{H}$ such that $A \subset \widetilde{A} \subset A^*$.

Let $\mathcal{H}_0$ be a Hilbert space, let $\mathcal{H}_1$ be a subspace in $\mathcal{H}_0$ and let $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$. Assume also that P_j is the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j \in \{1, 2\}$.

Definition 2.5 ([15]). *A collection* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, *where* Γ_j *are linear mappings from* $\mathcal{D}(A^*)$ *to* $\mathcal{H}_j$ $(j \in \{0, 1\})$, *is called a D-boundary* *triplet (or briefly D-triplet) for* A^*, *if* $\Gamma = (\Gamma_0 \ \ \Gamma_1)^\top : \mathcal{D}(A^*) \to \mathcal{H}_0 \oplus \mathcal{H}_1$ *is a surjective linear mapping onto* $\mathcal{H}_0 \oplus \mathcal{H}_1$ *and the following Green's identity holds*

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g) + i(P_2 \Gamma_0 f, P_2 \Gamma_0 g), \qquad f, g \in \mathcal{D}(A^*). \quad (2.17)$$

Proposition 2.2 ([15]). *Let* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *be a D-triplet for* A^*. *Then:*

1) *the operators* $\hat{\Gamma}^A = (\hat{\Gamma}_0^A \ \ \hat{\Gamma}_1^A)^\top : A^* \to \mathcal{H}_0 \oplus \mathcal{H}_1$, $\Gamma^A = (\Gamma_0^A \ \ \Gamma_1^A)^\top : A^* \to \mathcal{H}_1 \oplus \mathcal{H}_0$ *with*

$$\hat{\Gamma}_0^A = \Gamma_0, \quad \hat{\Gamma}_1^A = \Gamma_1, \quad \Gamma_0^A = P_1 \Gamma_0, \quad \Gamma_1^A = \Gamma_1 + i P_2 \Gamma_0. \quad (2.18)$$

form a boundary triplet $\hat{\Pi} = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \hat{\Gamma}^A, \Gamma^A\}$ *for the dual pair* $\{A, A\}$. *Therefore by Proposition* 2.1 $\Gamma_j \in [\mathcal{D}(A^*)_+, \mathcal{H}_j]$, $j \in \{0, 1\}$;

2) *the relation* $\dim \mathcal{H}_1 = n_-(A) \le n_+(A) = \dim \mathcal{H}_0$ *is valid;*

3) *the equalities*

$$\mathcal{D}(A_0) := \operatorname{Ker} \Gamma_0 = \{f \in \mathcal{D}(A^*) : \Gamma_0 f = 0\}, \quad A_0 = A^* \restriction \mathcal{D}(A_0) \quad (2.19)$$

define a maximal symmetric extension $A_0 \in Ext_A$ *with* $n_-(A_0) = 0$.

It turns out that for every $\lambda \in \mathbb{C}_+$ $(z \in \mathbb{C}_-)$ the map $\Gamma_0 \restriction \mathfrak{N}_\lambda(A)$ $(P_1 \Gamma_0 \restriction \mathfrak{N}_z(A))$ is an isomorphism. This makes it possible to introduce the operator functions (γ-fields) $\gamma_+(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_0, \mathfrak{H}]$, $\gamma_-(\cdot) : \mathbb{C}_- \to [\mathcal{H}_1, \mathfrak{H}]$ and the Weyl functions $M_+(\cdot) : \mathbb{C}_+ \to [\mathcal{H}_0, \mathcal{H}_1]$, $M_-(\cdot) : \mathbb{C}_- \to [\mathcal{H}_1, \mathcal{H}_0]$ by

$$\begin{aligned} \gamma_+(\lambda) &= (\Gamma_0 \restriction \mathfrak{N}_\lambda(A))^{-1}, \quad \lambda \in \mathbb{C}_+; \\ \gamma_-(z) &= (P_1 \Gamma_0 \restriction \mathfrak{N}_z(A))^{-1}, \quad z \in \mathbb{C}_-, \end{aligned} \quad (2.20)$$

$$\Gamma_1 \restriction \mathfrak{N}_\lambda(A) = M_+(\lambda) \Gamma_0 \restriction \mathfrak{N}_\lambda(A), \quad \lambda \in \mathbb{C}_+, \quad (2.21)$$

$$(\Gamma_1 + i P_2 \Gamma_0) \restriction \mathfrak{N}_z(A) = M_-(z) P_1 \Gamma_0 \restriction \mathfrak{N}_z(A), \quad z \in \mathbb{C}_-. \quad (2.22)$$

According to [15] all functions $\gamma_\pm$ and $M_\pm$ are holomorphic on their domains and $M_+^*(\lambda) = M_-(\overline{\lambda})$, $\lambda \in \mathbb{C}_+$.

Lemma 2.3. *Assume that* $\{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *is a D-triplet for* A^*, $\theta = \{(C_0, C_1); \mathcal{K}\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ *is an operator pair* (2.1), $\theta^\times = \{(C_{0\times}, C_{1\times}); \mathcal{K}_\times\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ *is the* $\times$*-adjoint pair and* C_{1*}, C_{0*} *are operators* (2.3). *Then*

$$(C_0\Gamma_0 + C_1\Gamma_1)\gamma_+(\lambda) = C_0 + C_1 M_+(\lambda), \quad \lambda \in \mathbb{C}_+ \tag{2.23}$$

$$(C_{0\times}\Gamma_0 + C_{1\times}\Gamma_1)\gamma_-(z) = C_{1*} + C_{0*}M_-(z), \quad z \in \mathbb{C}_-. \tag{2.24}$$

Proof. It follows from (2.20)–(2.22) that

$$\Gamma_0\gamma_+(\lambda) = I_{\mathcal{H}_0}, \quad \Gamma_1\gamma_+(\lambda) = M_+(\lambda), \quad \lambda \in \mathbb{C}_+ \tag{2.25}$$

$$\begin{gathered} P_1\Gamma_0\gamma_-(z) = I_{\mathcal{H}_1}, \quad P_2\Gamma_0\gamma_-(z) = -iP_2M_-(z), \\ \Gamma_1\gamma_-(z) = P_1M_-(z), \quad z \in \mathbb{C}_-. \end{gathered} \tag{2.26}$$

The equality (2.23) is immediate from (2.25). Moreover (2.26) yields

$$\begin{aligned} &(C_{0\times}\Gamma_0 + C_{1\times}\Gamma_1)\gamma_-(z) \\ &\quad = C_{0\times}P_1\Gamma_0\gamma_-(z) + C_{0\times}P_2\Gamma_0\gamma_-(z) + C_{1\times}\Gamma_1\gamma_-(z) \\ &\quad = C_{0\times} \upharpoonright \mathcal{H}_1 + (C_{1\times}P_1 - iC_{0\times}P_2)M_-(z) \\ &\quad = C_{1*} + C_{0*}M_-(z), \quad z \in \mathbb{C}_-. \end{aligned}$$

□

Combining [15, Proposition 4.1] with [15, Theorem 4.2] we arrive at the following theorem.

Theorem 2.1. *Suppose that* $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *is a D-triplet for* A^*, $\theta = \{(C_0, C_1); \mathcal{K}\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ *is an operator pair and* $\widetilde{A} \in Ext_A$ *is an extension defined by the abstract boundary condition*

$$\mathcal{D}(\widetilde{A}) = \{f \in \mathcal{D}(A^*) : C_0\Gamma_0 f + C_1\Gamma_1 f = 0\}, \quad \widetilde{A} = A^* \upharpoonright \mathcal{D}(\widetilde{A}) \tag{2.27}$$

Then $\lambda \in \rho(\widetilde{A}) \cap \mathbb{C}_+ \iff 0 \in \rho(C_0 + C_1M_+(\lambda))$ *and the following Krein type formula for canonical resolvents holds*

$$(\widetilde{A} - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma_+(\lambda)(C_0 + C_1M_+(\lambda))^{-1}C_1\gamma_-^*(\overline{\lambda}), \quad \lambda \in \rho(\widetilde{A}) \cap \mathbb{C}_+. \tag{2.28}$$

Remark 2.2. 1) If a D-triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ satisfies the relation $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$ ($\Leftrightarrow A_0 = A_0^*$), then it is a boundary triplet. More precisely this means that the collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet (boundary value space) for A^* in the sense of [8]. In this case the relations

$$\gamma(\lambda) = (\Gamma_0 \upharpoonright \mathfrak{N}_\lambda(A))^{-1}, \quad \Gamma_1 \upharpoonright \mathfrak{N}_\lambda(A) = M(\lambda)\Gamma_0 \upharpoonright \mathfrak{N}_\lambda(A), \quad \lambda \in \rho(A_0) \tag{2.29}$$

define the operator function (γ-field) $\gamma(\cdot) : \rho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and the Weyl function $M(\cdot) : \rho(A_0) \to [\mathcal{H}]$ introduced by V. A. Derkach and M. M. Malamud [4] (see also [5] and the references therein). It is clear that $\gamma(\cdot)$ and $M(\cdot)$ are connected with the operator functions (2.20)–(2.22) by means of the following relations $\gamma(\lambda) = \gamma_\pm(\lambda)$ and $M(\lambda) = M_\pm(\lambda)$, $\lambda \in \mathbb{C}_\pm$.

2) In the case of a symmetric operator A with $n_+(A) = n_+(A)$ and an ordinary boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* formula (2.28) coincides with [5, Proposition 2.2]. In turn, if $\widetilde{A} = \widetilde{A}^*$ (hence the corresponding linear relation $\theta = \{(C_0, C_1); \mathcal{K}\} \in \widetilde{\mathcal{C}}(\mathcal{H})$ is selfadjoint), then (2.28) coincides with the known Krein–Naimark formula for canonical resolvents of the operator A (see for instance [10]). Note also that a simple proof of the Krein–Naimark formula as well as its connection with a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* (and, in particular, with boundary conditions (2.27)) has been discovered in [5] (for other proofs of the Krein–Naimark formula see also recent publication [6] and references there in).

3. Differential operators and decomposing boundary triplets

3.1. Differential operators

Let $\Delta = [0, b\rangle$ $(b \leq \infty)$ be an interval on the real axis (in the case $b < \infty$ the point b may or may not belong to Δ), let H be a separable Hilbert space with $\dim H \leq \infty$ and let

$$l[y] = l_H[y] = \sum_{k=1}^{n} (-1)^k ((p_{n-k} y^{(k)})^{(k)} - \frac{i}{2}[(q_{n-k}^* y^{(k)})^{(k-1)} + (q_{n-k} y^{(k-1)})^{(k)}]) + p_n y, \tag{3.1}$$

be a differential expression of an even order $2n$ with smooth enough operator-valued coefficients $p_k(\cdot), q_k(\cdot) : \Delta \to [H]$ such that $p_k(t) = p_k^*(t)$ and $0 \in \rho(p_0(t))$ for all $t \in \Delta$ and $k = 0 \div n$. Denote by $y^{[k]}(\cdot)$, $k = 0 \div 2n$ the quasi-derivatives of a vector-function $y(\cdot) : \Delta \to H$, corresponding to the expression (3.1). Moreover for every operator function $Y(\cdot) : \Delta \to$

$[\mathcal{K}, H]$ ($\mathcal{K}$ is a Hilbert space) introduce quasi-derivatives $Y^{[k]}(\cdot)$ by the same formulas as $y^{[k]}$ (see [17, 18]).

Let $\mathcal{D}(l)$ be the set of all functions $y(\cdot)$ such that $y^{[k]}(\cdot),\ k = 0 \div (2n-2)$ has a continuous derivative on Δ and $y^{[2n-1]}$ is absolutely continuous on Δ. Furthermore for a given Hilbert space $\mathcal{K}$ denote by $\mathcal{D}_{\mathcal{K}}(l)$ the set of all operator-functions $Y(\cdot)$ with values in $[\mathcal{K}, H]$ such that $Y^{[k]}(\cdot),\ k = 0 \div (2n-1)$ has a continuous derivative on Δ. Clearly for every $y \in \mathcal{D}(l)$ and $Y \in \mathcal{D}_{\mathcal{K}}(l)$ the functions $y^{[k]}(\cdot) : \Delta \to H,\ k = 0 \div (2n-1)$ and $Y^{[k]}(\cdot) : \Delta \to [\mathcal{K}, H],\ k = 0 \div 2n$ are continuous on Δ, the function $y^{[2n]}(t)(\in H)$ is defined almost everywhere on Δ and

$$l[y] = y^{[2n]}(t), \quad y \in \mathcal{D}(l); \qquad l[Y] = Y^{[2n]}(t), \quad Y \in \mathcal{D}_{\mathcal{K}}(l).$$

This makes it possible to introduce the vector functions $y^{(j)}(\cdot) : \Delta \to H^n,\ j \in \{1, 2\}$ and $\widetilde{y}(\cdot) : \Delta \to H^n \oplus H^n$,

$$y^{(1)}(t) := \{y^{[k-1]}(t)\}_{k=1}^n (\in H^n), \quad y^{(2)}(t) := \{y^{[2n-k]}(t)\}_{k=1}^n (\in H^n), \tag{3.2}$$

$$\widetilde{y}(t) = \{y^{(1)}(t), y^{(2)}(t)\} (\in H^n \oplus H^n), \quad t \in \Delta, \tag{3.3}$$

which correspond to every $y \in \mathcal{D}(l)$. Similarly with each $Y \in \mathcal{D}_{\mathcal{K}}(l)$ we associate the operator-functions $Y^{(j)}(\cdot) : \Delta \to [\mathcal{K}, H^n]$ and $\widetilde{Y}(\cdot) : \Delta \to [\mathcal{K}, H^n \oplus H^n]$ given by

$$Y^{(1)}(t) = (Y(t)\ \ Y^{[1]}(t) \cdots Y^{[n-1]}(t))^\top,$$
$$Y^{(2)}(t) = (Y^{[2n-1]}(t)\ \ Y^{[2n-2]}(t) \cdots Y^{[n]}(t))^\top,$$
$$\widetilde{Y}(t) = (Y^{(1)}(t)\ \ Y^{(2)}(t))^\top : \mathcal{K} \to H^n \oplus H^n, \quad t \in \Delta.$$

Next for a given $\lambda \in \mathbb{C}$ consider the equation

$$l[y] - \lambda y = 0. \tag{3.4}$$

As is known this equation has the unique vector solution $y \in \mathcal{D}(l)$ (operator solution $Y \in \mathcal{D}_{\mathcal{K}}(l)$) with the given initial data $y_{j0} = y^{(j)}(0)$ (respectively, $Y_{j0} = Y^{(j)}(0)$), $j \in \{1, 2\}$. We distinguish the two "canonical" operator solutions $c(\cdot, \lambda)$ and $s(\cdot, \lambda) : \Delta \to [H^n, H],\ \lambda \in \mathbb{C}$ of the equation (3.4) with the initial data

$$\begin{aligned} c^{(1)}(0,\lambda) &= I_{H^n}, \quad c^{(2)}(0,\lambda) = 0, \\ s^{(1)}(0,\lambda) &= 0, \quad s^{(2)}(0,\lambda) = I_{H^n}, \end{aligned} \qquad \lambda \in \mathbb{C}. \tag{3.5}$$

The following lemma is well known (see for instance [9, 17]).

Lemma 3.1. *Let $f \in L_{1,loc}(\Delta; H)$ and let $Y(\cdot) : \Delta \to [\mathcal{K}, H]$ be an operator solution of* (3.4). *Next assume that an absolutely continuous function $C(\cdot) : \Delta \to \mathcal{K}$ satisfies*

$$\widetilde{Y}(x)C'(x) = \hat{f}(x) \quad (\bmod \mu), \tag{3.6}$$

where the vector function $\hat{f}(\cdot) : \Delta \to H^n \oplus H^n$ is given by $\hat{f}(x) = \{\underbrace{0, \dots, 0,}_{n} -f(x), \underbrace{0, \dots, 0}_{n-1}\}$ and μ is the Lebesgue measure on Δ. Then the vector function $y(x) := Y(x)C(x)$ belongs to $\mathcal{D}(l)$ and obeys the relations

$$l[y] - \lambda y = f, \qquad \widetilde{y}(x) = \widetilde{Y}(x)C(x). \tag{3.7}$$

In what follows we denote by $\mathfrak{H}(= L_2(\Delta; H))$ the Hilbert space of all measurable functions $f(\cdot) : \Delta \to H$ such that $\int_0^b \|f(t)\|^2 \, dt < \infty$. Moreover, $L_2'[\mathcal{K}, H]$ stands for the set of all operator-functions $Y(\cdot) : \Delta \to [\mathcal{K}, H]$ such that $Y(t)h \in \mathfrak{H}$ for all $h \in \mathcal{K}$.

It is known [17, 18] that the expression (3.1) generate the maximal operator L in $\mathfrak{H}$, defined on the domain $\mathcal{D} = \mathcal{D}(L) := \{y \in \mathcal{D}(l) \cap \mathfrak{H} : l[y] \in \mathfrak{H}\}$ by the equality $Ly = l[y]$, $y \in \mathcal{D}$. Moreover the Lagrange's identity

$$(Ly, z)_{\mathfrak{H}} - (y, Lz)_{\mathfrak{H}} = [y, z](b) - [y, z](0), \quad y, z \in \mathcal{D} \tag{3.8}$$

holds with

$$[y, z](t) = (y^{(1)}(t), z^{(2)}(t))_{H^n} - (y^{(2)}(t), z^{(1)}(t))_{H^n},$$
$$[y, z](b) = \lim_{t \uparrow b} [y, z](t).$$

The minimal operator L_0 is defined as a restriction of L onto the domain $D_0 = \mathcal{D}(L_0)$ of all functions $y \in \mathcal{D}$ such that $\widetilde{y}(0) = 0$ and $[y, z](b) = 0$ for all $z \in \mathcal{D}$. As is known [17, 18] L_0 is a closed densely defined symmetric operator in $\mathfrak{H}$ and $L_0^* = L$. Moreover the subspace $\mathfrak{N}_\lambda(L_0)$ $(= \text{Ker}(L - \lambda))$ is the set of all solutions of (3.4) belonging to $\mathfrak{H}$.

Definition 3.1 ([7, 20]). *An extension $\widetilde{A} \in Ext_{L_0}$ is said to be defined by decomposing boundary conditions if for every $y \in \mathcal{D}(\widetilde{A})$ there exists $z \in \mathcal{D}(\widetilde{A})$ such that $\widetilde{z}(0) = 0$ and $z(t) = y(t)$ on some interval $(\eta, b) \in \Delta$.*

3.2. Decomposing boundary triplets

Assume that $\mathcal{H}_1'$ is a subspace in a Hilbert space $\mathcal{H}_0'$, $\mathcal{H}_2' := \mathcal{H}_0' \ominus \mathcal{H}_1'$, $\Gamma_0' : \mathcal{D} \to \mathcal{H}_0'$ and $\Gamma_1' : \mathcal{D} \to \mathcal{H}_1'$ are linear maps and P_j' is the orthoprojector in $\mathcal{H}_0'$ onto $\mathcal{H}_j'$, $j \in \{1, 2\}$. Moreover let $\mathcal{H}_0 = H^n \oplus$

$\mathcal{H}_0'$, $\mathcal{H}_1 = H^n \oplus \mathcal{H}_1'$ and let $\Gamma_j : \mathcal{D} \to \mathcal{H}_j$, $j \in \{0,1\}$ be linear maps given by

$$\begin{aligned} \Gamma_0 y &= \{y^{(2)}(0), \Gamma_0' y\} \; (\in H^n \oplus \mathcal{H}_0'), \\ \Gamma_1 y &= \{-y^{(1)}(0), \Gamma_1' y\} \; (\in H^n \oplus \mathcal{H}_1'), \end{aligned} \qquad y \in \mathcal{D}. \tag{3.9}$$

Definition 3.2 ([16]). *A collection $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where Γ_0 and Γ_1 are linear maps* (3.9)*, is said to be a decomposing D-triplet for L if the map $\Gamma' = (\Gamma_0' \; \Gamma_1')^\top : \mathcal{D} \to \mathcal{H}_0' \oplus \mathcal{H}_1'$ is surjective and the following identity holds*

$$[y, z](b) = (\Gamma_1' y, \Gamma_0' z) - (\Gamma_0' y, \Gamma_1' z) + i(P_2' \Gamma_0' y, P_2' \Gamma_0' z), \quad y, z \in \mathcal{D}. \tag{3.10}$$

In the case $\mathcal{H}_0' = \mathcal{H}_1' =: \mathcal{H}'$ ($\iff$ $\mathcal{H}_0 = \mathcal{H}_1 =: \mathcal{H}$) a decomposing D-triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a decomposing boundary triplet for L. For such a triplet the identity (3.10) takes the form

$$[y, z](b) = (\Gamma_1' y, \Gamma_0' z) - (\Gamma_0' y, \Gamma_1' z), \quad y, z \in \mathcal{D}. \tag{3.11}$$

As was shown in [16, Lemma 3.4] a decomposing D-triplet (a decomposing boundary triplet) for L is a D-triplet (a boundary triplet) in the sense of Definition 2.5 and Remark 2.2. Moreover a decomposing D-triplet (boundary triplet) for L exists if and only if $n_{b-} \le n_{b+}$ (respectively, $n_{b-} = n_{b+}$), where $n_{b\pm}$ are deficiency indices of the expression (3.1) at the point b [16]. Therefore in the sequel we suppose (without loss of generality) that $n_{b-} \le n_{b+}$ and, consequently, $n_-(L_0) \le n_+(L_0)$.

Theorem 3.1 ([16]). *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet* (3.9) *and let*

$$M_+(\lambda) = \begin{pmatrix} m(\lambda) & M_{2+}(\lambda) \\ M_{3+}(\lambda) & M_{4+}(\lambda) \end{pmatrix} : H^n \oplus \mathcal{H}_0' \to H^n \oplus \mathcal{H}_1', \quad \lambda \in \mathbb{C}_+, \tag{3.12}$$

$$M_-(z) = \begin{pmatrix} m(z) & M_{2-}(z) \\ M_{3-}(z) & M_{4-}(z) \end{pmatrix} : H^n \oplus \mathcal{H}_1' \to H^n \oplus \mathcal{H}_0', \quad z \in \mathbb{C}_-, \tag{3.13}$$

be the block-matrix representations of the Weyl functions (2.21), (2.22) *for Π. Then:*

1) *the maximal symmetric extension $A_0 \in Ext_{L_0}$ (see* (2.19)*) has the domain*

$$\mathcal{D}(A_0) = \{y \in \mathcal{D} : y^{(2)}(0) = 0, \; \Gamma_0' y = 0\}; \tag{3.14}$$

2) *for every* $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ *there exists the unique operator function* $v_0(\cdot,\lambda) \in L_2'[H^n, H]$, *satisfying the equation* (3.4)*and the boundary conditions*

$$v_0^{(2)}(0,\lambda) = I_{H^n}, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \tag{3.15}$$

$$\begin{aligned} \Gamma_0'(v_0(t,\lambda)\hat{h}) = 0, \quad \lambda \in \mathbb{C}_+; \\ P_1'\Gamma_0'(v_0(t,z)\hat{h}) = 0, \quad z \in \mathbb{C}_-, \ \hat{h} \in H^n. \end{aligned} \tag{3.16}$$

Moreover, for every $\lambda \in \mathbb{C}_+$ $(z \in \mathbb{C}_-)$ *there exists the unique operator function* $u_+(\cdot,\lambda) \in L_2'[\mathcal{H}_0', H]$ $(u_-(\cdot,z) \in L_2'[\mathcal{H}_1', H])$, *satisfying* (3.4) *and the boundary conditions*

$$u_+^{(2)}(0,\lambda) = 0, \quad \Gamma_0'(u_+(t,\lambda)h_0') = h_0', \quad \lambda \in \mathbb{C}_+, \ h_0' \in \mathcal{H}_0'; \tag{3.17}$$

$$u_-^{(2)}(0,z) = 0, \quad P_1'\Gamma_0'(u_-(t,z)h_1') = h_1', \ z \in \mathbb{C}_-, \ h_1' \in \mathcal{H}_1'. \tag{3.18}$$

3) *let* $Z_+(\cdot,\lambda) \in L_2'[\mathcal{H}_0, H]$ *and* $Z_-(\cdot,z) \in L_2'[\mathcal{H}_1, H]$ *be operator solutions of* (3.4) *defined by the block-matrix representations*

$$Z_+(t,\lambda) = (v_0(t,\lambda) \ \ u_+(t,\lambda)) : H^n \oplus \mathcal{H}_0' \to H, \quad \lambda \in \mathbb{C}_+, \tag{3.19}$$

$$Z_-(t,z) = (v_0(t,z) \ \ u_-(t,z)) : H^n \oplus \mathcal{H}_1' \to H, \quad z \in \mathbb{C}_-. \tag{3.20}$$

Then

$$\widetilde{Z}_+(0,\lambda) = \begin{pmatrix} v_0^{(1)}(0,\lambda) & u_+^{(1)}(0,\lambda) \\ v_0^{(2)}(0,\lambda) & u_+^{(2)}(0,\lambda) \end{pmatrix} = \begin{pmatrix} -m(\lambda) & -M_{2+}(\lambda) \\ I_{H^n} & 0 \end{pmatrix}, \quad \lambda \in \mathbb{C}_+ \tag{3.21}$$

$$\widetilde{Z}_-(0,z) = \begin{pmatrix} v_0^{(1)}(0,z) & u_-^{(1)}(0,z) \\ v_0^{(2)}(0,z) & u_-^{(2)}(0,z) \end{pmatrix} = \begin{pmatrix} -m(z) & -M_{2-}(z) \\ I_{H^n} & 0 \end{pmatrix}, \quad z \in \mathbb{C}_-. \tag{3.22}$$

and the γ*-fields* (2.20) *for* Π *obey the relations*

$$(\gamma_+(\lambda)h_0)(t) = Z_+(t,\lambda)h_0, \quad \lambda \in \mathbb{C}_+, \ h_0 \in \mathcal{H}_0 \tag{3.23}$$

$$(\gamma_-(z)h_1)(t) = Z_-(t,z)h_1, \quad z \in \mathbb{C}_-, \ h_1 \in \mathcal{H}_1 \tag{3.24}$$

This implies that $Z_+(\cdot,\lambda)$ *and* $Z_-(\cdot,\lambda)$ *are holomorphic fundamental solutions of* (3.4) (*see Definition* 4.3 *below*).

4. Fundamental solutions and spectra of proper extensions

4.1. Boundary conditions

Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet (3.9) for L, let $\theta = \{(C_0, C_1); \mathcal{K}\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ be an operator pair and let $\theta^\times = \{(C_{0\times}, C_{1\times}); \mathcal{K}_\times\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ be a $\times$-adjoint operator pair (this means that $\mathcal{K}$ and $\mathcal{K}_\times$ are Hilbert spaces and $C_j \in [\mathcal{H}_j, \mathcal{K}]$, $C_{j\times} \in [\mathcal{H}_j, \mathcal{K}_\times]$, $j \in \{0,1\}$). Since $\mathcal{H}_j = H^n \oplus \mathcal{H}'_j$, the operators C_j and $C_{j\times}$ admit the block-matrix representations

$$\begin{aligned} C_0 &= (\hat{C}_2\ C'_0) : H^n \oplus \mathcal{H}'_0 \to \mathcal{K}, \\ C_1 &= (-\hat{C}_1\ C'_1) : H^n \oplus \mathcal{H}'_1 \to \mathcal{K}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} C_{0\times} &= (\hat{C}_{2\times}\ C'_{0\times}) : H^n \oplus \mathcal{H}'_0 \to \mathcal{K}_\times, \\ C_{1\times} &= (-\hat{C}_{1\times}\ C'_{1\times}) : H^n \oplus \mathcal{H}'_1 \to \mathcal{K}_\times \end{aligned} \tag{4.2}$$

The description of all proper extensions of the minimal operator L_0 in terms of boundary conditions is contained in the following theorem.

Theorem 4.1 ([16]). *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet* (3.9) *for L. Then:*

1) *the equalities* (*the boundary conditions*)

$$\mathcal{D}(\widetilde{A}) = \{y \in \mathcal{D} :\ \hat{C}_1 y^{(1)}(0) + \hat{C}_2 y^{(2)}(0) + C'_0\Gamma'_0 y + C'_1\Gamma'_1 y = 0\}, \quad \widetilde{A} = L \restriction \mathcal{D}(\widetilde{A}) \tag{4.3}$$

establish a bijective correspondence between all proper extensions $\widetilde{A} \in Ext_{L_0}$ and all operator pairs $\theta = \{(C_0, C_1); \mathcal{K}\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ defined by (4.1). *Moreover the adjoint $\widetilde{A}^*$ to the extension* (4.3) *has the domain*

$$\mathcal{D}(\widetilde{A}^*) = \{y \in \mathcal{D} : \hat{C}_{1\times} y^{(1)}(0) + \hat{C}_{2\times} y^{(2)}(0) + C'_{0\times}\Gamma'_0 y + C'_{1\times}\Gamma'_1 y = 0\} \tag{4.4}$$

where the operators $\hat{C}_{1\times}$, $\hat{C}_{2\times}$, $C'_{0\times}$ and $C'_{1\times}$ are defined by (4.2)

2) *the equalities*

$$\mathcal{D}(\widetilde{A}) = \{y \in \mathcal{D} : \hat{N}_1 y^{(1)}(0) + \hat{N}_2 y^{(2)}(0) = 0,\ N_0\Gamma'_0 y + N_1\Gamma'_1 y = 0\}, \quad \widetilde{A} = L \restriction \mathcal{D}(\widetilde{A}) \tag{4.5}$$

give a bijective correspondence between all extensions $\widetilde{A} \in Ext_{L_0}$ defined by decomposing boundary conditions (see Definition 3.1) and all all collections of two operator pairs $\theta_0 = \{(\hat{N}_2, -\hat{N}_1); \mathcal{K}_0\} \in \widetilde{\mathcal{C}}(H^n)$ and $\theta' = \{(N_0, N_1); \mathcal{K}_1\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0', \mathcal{H}_1')$.

Remark 4.1. It is clear that the boundary conditions (4.3) can be written as

$$\mathcal{D}(\widetilde{A}) = \{y \in \mathcal{D} : \ C_0\Gamma_0 y + C_1\Gamma_1 y = 0\}, \tag{4.6}$$

while the decomposing boundary conditions (4.5) are equivalent to (4.6) with

$$\begin{aligned} C_0 &= \begin{pmatrix} \hat{N}_2 & 0 \\ 0 & N_0 \end{pmatrix} : H^n \oplus \mathcal{H}_0' \to \mathcal{K}_0 \oplus \mathcal{K}_1, \\ C_1 &= \begin{pmatrix} -\hat{N}_1 & 0 \\ 0 & N_1 \end{pmatrix} : H^n \oplus \mathcal{H}_1' \to \mathcal{K}_0 \oplus \mathcal{K}_1. \end{aligned} \tag{4.7}$$

4.2. Fundamental solutions

The following proposition will be systematically used in the sequel.

Proposition 4.1. *Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet (3.9) for L,*

$$\theta' = \{(N_0, N_1); \mathcal{K}_1\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0', \mathcal{H}_1') \tag{4.8}$$

be an operator pair with $N_j \in [\mathcal{H}_j', \mathcal{K}_1]$ $(j \in \{0,1\})$, $\theta'^{\times} = \{(N_{0\times}, N_{1\times}); \mathcal{K}_{1\times}\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0', \mathcal{H}_1')$ be a $\times$-adjoint operator pair with $N_{j\times} \in [\mathcal{H}_j', \mathcal{K}_{1\times}]$ $(j \in \{0,1\})$ and let L_b, $L_{b\times} \in Ext_{L_0}$ be extensions with the domains

$$\mathcal{D}(L_{b\times}) = \{y \in \mathcal{D} : \widetilde{y}(0) = 0, \ N_0\Gamma_0' y + N_1\Gamma_1' y = 0\}, \tag{4.9}$$

$$\mathcal{D}(L_b) = \{y \in \mathcal{D} : \widetilde{y}(0) = 0, \ N_{0\times}\Gamma_0' y + N_{1\times}\Gamma_1' y = 0\}. \tag{4.10}$$

Then:

1) *the adjoint extensions L_b^* and $L_{b\times}^*$ have the domains*

$$\begin{aligned} \mathcal{D}(L_b^*) &= \{y \in \mathcal{D} : N_0\Gamma_0' y + N_1\Gamma_1' y = 0\}, \\ \mathcal{D}(L_{b\times}^*) &= \{y \in \mathcal{D} : N_{0\times}\Gamma_0' y + N_{1\times}\Gamma_1' y = 0\}; \end{aligned} \tag{4.11}$$

2) *the subspace $\mathfrak{N}_\lambda(L_b)(= \mathrm{Ker}(L_b^* - \lambda))$, $\lambda \in \mathbb{C}$ is defined by*

$$\mathfrak{N}_\lambda(L_b) = \{y \in \mathfrak{N}_\lambda(L_0) : N_0\Gamma_0' y + N_1\Gamma_1' y = 0\}; \tag{4.12}$$

3) *$\{L_{b\times}, L_b\}$ is a dual pair and (4.5) give a bijective correspondence between all extensions $\widetilde{A} \in Ext\{L_{b\times}, L_b\}$ and all operator pairs $\theta_0 = \{(\hat{N}_2, -\hat{N}_1); \mathcal{K}_0\} \in \widetilde{\mathcal{C}}(H^n)$;*

4) *the operators* $\Gamma^{L_b} = (\Gamma_0^{L_b} \ \ \Gamma_1^{L_b})^\top : \mathcal{D}(L_b^*) \to H^n \oplus H^n$ *and* $\Gamma^{L_{b\times}} = (\Gamma_0^{L_{b\times}} \ \ \Gamma_1^{L_{b\times}})^\top : \mathcal{D}(L_{b\times}^*) \to H^n \oplus H^n$ *given by*

$$\Gamma_0^{L_b} y = y^{(2)}(0), \quad \Gamma_1^{L_b} y = -y^{(1)}(0), \quad y \in \mathcal{D}(L_b^*) \qquad (4.13)$$

$$\Gamma_0^{L_{b\times}} z = z^{(2)}(0), \quad \Gamma_1^{L_{b\times}} z = -z^{(1)}(0), \quad z \in \mathcal{D}(L_{b\times}^*) \qquad (4.14)$$

form a boundary triplet $\widetilde{\Pi} = \{H^n \oplus H^n, \Gamma^{L_b}, \Gamma^{L_{b\times}}\}$ *for the dual pair* $\{L_{b\times}, L_b\}$*;*

5) *if* $\lambda \in \hat{\rho}\{L_{b\times}, L_b\}$*, then there exists an extension* $\widetilde{A} \in Ext\{L_{b\times}, L_b\}$ *with* $\lambda \in \rho(\widetilde{A})$*.*

Proof. The statement 1) is implied by Theorem 4.1, 1), while the statements 2) and 3) are obvious.

4) For every $y \in \mathcal{D}(L_b^*)$ and $z \in \mathcal{D}(L_{b\times}^*)$ one has $\{\Gamma_0' y, \Gamma_1' y\} \in \theta'$ and $\{\Gamma_0' z, \Gamma_1' z\} \in \theta'^\times$, which in view of (3.10) yields $[y,z](b) = 0$. This and (3.8) give the identity (2.5) for operators (4.13) and (4.14).

Next for every $h_1, h_2 \in H^n$ there is $y \in \mathcal{D}$ such that $y^{(1)}(0) = h_1$, $y^{(2)}(0) = h_2$ and $\Gamma_0' y = \Gamma_1' y = 0$ (see formula (3.16) in [16]). Since by (4.11) $y \in \mathcal{D}(L_b^*) \cap \mathcal{D}(L_{b\times}^*)$, this proves the equality (2.4) for the triplet $\widetilde{\Pi}$.

Finally, the statement 5) follows from Corollary 3.16 in [12] □

For a given operator pair (4.8) consider a boundary problem

$$l[y] - \lambda y = 0 \qquad (4.15)$$

$$N_0 \Gamma_0' y + N_1 \Gamma_1' y = 0 \qquad (4.16)$$

with a boundary condition (4.16) at the right end b of the interval Δ.

Definition 4.1. *A function* $y(\cdot) : \Delta \to H$ *will be called a (vector) solution of the boundary problem* (4.15), (4.16) *if it belongs to* $\mathcal{D}$ *and obeys the equalities* (4.15), (4.16).

It follows from (4.12) that the set of all solutions of the problem (4.15), (4.16) coincides with $\mathfrak{N}_\lambda(L_b)$.

Definition 4.2. *Let* $\lambda \in \mathbb{C}$ *and let* $\mathcal{K}_0'$ *be a Hilbert space. An operator function*

$$v(\cdot, \lambda) : \Delta \to [\mathcal{K}_0', H] \qquad (4.17)$$

will be called a fundamental solution of the boundary problem (4.15), (4.16) *if* $v(\cdot,\lambda)$ *is an operator solution of the equation* (4.15) *and the equality*

$$y (= y(t)) = v(t, \lambda) h_0' \qquad (4.18)$$

gives a bijective correspondence between all vector solutions $y(\cdot)$ *of the problem* (4.15), (4.16) *and all* $h_0' \in \mathcal{K}_0'$.

In the case $N_0 = N_1 = 0 (\iff L_b = L_0)$ *a fundamental solution*

$$Z(\cdot, \lambda) : \Delta \to [\mathcal{K}', H] \tag{4.19}$$

of the problem (4.15), (4.16) *will be called a fundamental solution of the equation* (4.15).

It follows from Definition 4.2 that a fundamental solution of the problem (4.15), (4.16) belongs to $L_2'[\mathcal{K}', H]$. Moreover, an operator function (4.19) is a fundamental solution of the equation (4.15) if and only if it is an operator solution of this equation and the equality $y = Z(t, \lambda)h'$ gives a bijective correspondence between all functions $y \in \mathfrak{N}_\lambda(L_0)$ and all $h' \in \mathcal{K}'$.

Let $\mathcal{D}_+$ be a Hilbert space of all functions $y \in \mathcal{D}$ with the inner product

$$(y, z)_+ = (y, z)_{\mathfrak{H}} + (Ly, Lz)_{\mathfrak{H}}, \quad y, z \in \mathcal{D}_+$$

and let $\delta : \mathcal{D} \to H^n \oplus H^n$ be a linear map given by

$$\delta y = \widetilde{y}(0), \quad y \in \mathcal{D}. \tag{4.20}$$

It follows from (3.9) and Proposition 2.2, 1) that $\delta \in [\mathcal{D}_+, H^n \oplus H^n]$.

Lemma 4.1. *Let assumptions be as in Proposition* 4.1 *and let* $\lambda \in \mathbb{C}$. *Then:*

1) *for every operator* $Z \in [\mathcal{K}', \mathfrak{N}_\lambda(L_0)]$ *the relation*

$$Z(t, \lambda)h' = (Zh')(t), \quad h' \in \mathcal{K}' \tag{4.21}$$

defines an operator solution $Z(\cdot, \lambda) \in L_2'[\mathcal{K}', H]$ *of the equation* (4.15). *Conversely, for each such a solution there exists an operator* $Z \in [\mathcal{K}', \mathfrak{N}_\lambda(L_0)]$ *obeying* (4.21).

2) *the equality*

$$v(t, \lambda)h_0' = (vh_0')(t), \quad h_0' \in \mathcal{K}_0' \tag{4.22}$$

establishes a bijective correspondence between all fundamental solutions (4.17) *of the problem* (4.15), (4.16) *and all isomorphisms* $v \in [\mathcal{K}_0', \mathfrak{N}_\lambda(L_b)]$.

3) *let* $Z(\cdot, \lambda) \in L_2'[\mathcal{K}', H]$ *be an operator solution of* (4.15) *and let* $Z \in [\mathcal{K}', \mathfrak{N}_\lambda(L_0)]$ *be the corresponding operator* (4.21) *considered as acting to the Hilbert space* $\mathfrak{H}$. *Then*

$$Z^* f = \int_0^b Z^*(t, \lambda) f(t)\, dt := \lim_{\eta \uparrow b} \int_0^\eta Z^*(t, \lambda) f(t)\, dt, \quad f = f(t) \in \mathfrak{H}. \tag{4.23}$$

Proof. 1) For a given $Z \in [\mathcal{K}', \mathfrak{N}_\lambda(L_0)]$ consider the operator solution $Z(\cdot,\lambda) : \Delta \to [\mathcal{K}', H]$ of the equation (4.15) with the initial data $\widetilde{Z}(0,\lambda) = \delta_\lambda Z$, where $\delta_\lambda = \delta \upharpoonright \mathfrak{N}_\lambda(L_0) \in [\mathfrak{N}_\lambda(L_0), H^n \oplus H^n]$. It is clear that for every fixed $h' \in \mathcal{K}'$ the vector function

$$y = y(t,h') := (Zh')(t) \tag{4.24}$$

is a solution of (4.15) with $\widetilde{y}(0) = \delta_\lambda Zh' = \widetilde{Z}(0,\lambda)h'$. Therefore $y = Z(t,\lambda)h'$ and by (4.24) the introduced operator function $Z(\cdot,\lambda)$ satisfies (4.21). Hence $Z(\cdot,\lambda) \in L_2'[\mathcal{K}', H]$.

Conversely, let $Z(\cdot,\lambda) \in L_2'[\mathcal{K}', H]$ be an operator solution of (4.15). Then the equality (4.21) defines a linear map $Z : \mathcal{K}' \to \mathfrak{N}_\lambda(L_0)$ obeying $\delta_\lambda Z = \widetilde{Z}(0,\lambda)$ with bounded operators δ_λ and $\widetilde{Z}(0,\lambda)$. This and the equality $\operatorname{Ker} \delta_\lambda = \{0\}$ imply that the operator Z is closed and, consequently, bounded.

The statement 2) is immediate from 1).

3) The proof of (4.23) is similar to that of the formula (3.70) in [16] □

The following theorem is immediate from Lemma 4.1, 2).

Theorem 4.2. 1) *Let the assumptions of Proposition* 4.1 *be satisfied and let* $\lambda \in \mathbb{C}$. *Then for every Hilbert space* $\mathcal{K}_0'$ *with*

$$\dim \mathcal{K}_0' = \dim \mathfrak{N}_\lambda(L_b) \tag{4.25}$$

there exists a fundamental solution (4.17) *of the boundary problem* (4.15), (4.16). *Conversely, for every fundamental solution* (4.17) *the equality* (4.25) *holds.*

2) *Let* $v_0(\cdot,\lambda) : \Delta \to [\mathcal{K}_0', H]$ *be a fundamental solution of the boundary problem* (4.15), (4.16). *Then the equality*

$$v(t,\lambda) = v_0(t,\lambda)X$$

gives a bijective correspondence between all fundamental solutions (4.17) *of the same problem and all bounded isomorphisms* $X \in [\mathcal{K}_0']$.

Definition 4.3. *Let* Λ *be an open set in* $\mathbb{C}$ *and let* $\mathcal{K}_0'$ *be a Hilbert space. An operator function* $v(\cdot,\cdot) : \Delta \times \Lambda \to [\mathcal{K}_0', H]$ *will be called a holomorphic (on the set* Λ*) fundamental solution of the boundary problem* (4.15), (4.16) *if:*

(i) *for every* $\lambda \in \Lambda$ *the operator function* $v(\cdot,\lambda)$ *is a fundamental solution of the problem* (4.15), (4.16)*;*

(ii) *for some (and, hence, for each) fixed* $t \in \Delta$ *all quasi-derivatives* $v^{[k]}(t, \cdot)$, $k = 0 \div (2n-1)$ *are holomorphic on* Λ.

In the following theorem we specify conditions, which guarantee existence of holomorphic fundamental solutions.

Theorem 4.3. 1) *Let* $\widetilde{A}$ *be a proper extension of* L_0 *with nonempty resolvent set* $\rho(\widetilde{A})$. *Then there exists a fundamental solution* (4.19) *of the equation* (4.15) *holomorphic on* $\rho(\widetilde{A})$.

2) *Let* θ' *be an operator pair* (4.8), *let* (4.15), (4.16) *be the corresponding boundary problem and let* $\widetilde{A} \in Ext_{L_0}$ *be an extension defined by decomposing boundary conditions* (4.5) *with some operator pair* $\theta_0 = \{(\hat{N}_2, -\hat{N}_1); \mathcal{K}_0)\} \in \widetilde{\mathcal{C}}(H^n)$. *If* $\rho(\widetilde{A}) \neq \emptyset$, *then there exists a fundamental solution* (4.17) *of the problem* (4.15), (4.16) *holomorphic on* $\rho(\widetilde{A})$.

Proof. 1) Let $\lambda_0 \in \rho(\widetilde{A})$ and let $\mathcal{Z}_0$ be an isomorphism of a Hilbert space $\mathcal{K}'$ onto $\mathfrak{N}_{\lambda_0}(L_0)$. Since the resolvent $(\widetilde{A} - \lambda)^{-1}$ $(\lambda \in \rho(\widetilde{A}))$ is a holomorphic operator function with values in $[\mathfrak{H}, \mathcal{D}_+]$, the equality

$$\mathcal{Z}(\lambda) := \mathcal{Z}_0 + (\lambda - \lambda_0)(\widetilde{A} - \lambda)^{-1}\mathcal{Z}_0, \quad \lambda \in \rho(\widetilde{A}) \tag{4.26}$$

defines a holomorphic operator function $\mathcal{Z}(\cdot) : \rho(\widetilde{A}) \to [\mathcal{K}', \mathcal{D}_+]$. Moreover one can easily verify that $\mathcal{Z}(\lambda)\mathcal{K}' = \mathfrak{N}_\lambda(L_0)$ and $\operatorname{Ker} \mathcal{Z}(\lambda) = \{0\}$. Therefore by Lemma 4.1, 2) the relation

$$\mathcal{Z}(t, \lambda)h' := (\mathcal{Z}(\lambda)h')(t), \quad h' \in \mathcal{K}', \quad t \in \Delta \tag{4.27}$$

defines a family of fundamental solutions of the equation (4.15) with $\widetilde{\mathcal{Z}}(0, \lambda) = \delta\mathcal{Z}(\lambda)$. Hence the operator function $\widetilde{\mathcal{Z}}(0, \cdot)$ is holomorphic on $\rho(\widetilde{A})$, so that $\mathcal{Z}(t, \lambda)$ is a holomorphic fundamental solution.

2) Let $L_b^* \in Ext_{L_0}$ be the extension (4.11), let v_0 be an isomorphism of a Hilbert space $\mathcal{K}_0'$ onto $\mathfrak{N}_{\lambda_0}(L_b)$ $(\lambda_0 \in \rho(\widetilde{A}))$ and let $v(\cdot) : \rho(\widetilde{A}) \to [\mathcal{K}_0', \mathcal{D}_+]$ be a holomorphic operator function given by

$$v(\lambda) := v_0 + (\lambda - \lambda_0)(\widetilde{A} - \lambda)^{-1}v_0, \quad \lambda \in \rho(\widetilde{A}).$$

Using the inclusion $\widetilde{A} \subset L_b^*$ one can easily prove that $v(\lambda)\mathcal{K}_0' = \mathfrak{N}_\lambda(L_b)$ and $\operatorname{Ker}\, v(\lambda) = \{0\}$. Now applying Lemma 4.1, 2) one obtains the required statement in the same way as the previous one. □

Combining this theorem with statements 3) and 5) of Proposition 4.1 we arrive at the following corollary.

Corollary 4.1. *Let* θ' *be an operator pair* (4.8) *and let* $\{L_{b\times}, L_b\}$ *be the corresponding dual pair of operators* (4.9), (4.10). *If* $\lambda_0 \in \hat{\rho}\{L_{b\times}, L_b\}$, *then there exists a fundamental solution of the boundary problem* (4.15), (4.16) *holomorphic in some neighborhood of the point* λ_0.

4.3. Spectra of proper extensions

In this subsection we describe spectrum of a proper extension $\widetilde{A} \in Ext_{L_0}$ in terms of boundary conditions and fundamental solutions of the boundary problem (4.15), (4.16).

Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet (3.9) for L, let $\theta = \{(C_0, C_1); \mathcal{K}\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ be an operator pair with operators $C_j \in [\mathcal{H}_j, \mathcal{K}]$, $j \in \{0,1\}$ and let $Z(\cdot, \lambda) : \Delta \to [\mathcal{K}', H]$ $(\lambda \in \mathbb{C})$ be a fundamental solution of the equation (4.15). Introduce the operator $T \in [\mathcal{K}', \mathcal{K}]$ by

$$T = (C_0\Gamma_0 + C_1\Gamma_1)Z, \tag{4.28}$$

where $Z \in [\mathcal{K}', \mathfrak{N}_\lambda(L_0)]$ is the corresponding isomorphism (4.21) (see Lemma 4.1, 2)). Using the block-matrix representation (4.1) one rewrites (4.28) as

$$Th' = (\hat{C}_1 Z^{(1)}(0,\lambda) + \hat{C}_2 Z^{(2)}(0,\lambda))h' \\ + (C_0'\Gamma_0' + C_1'\Gamma_1')(Z(t,\lambda)h'), \quad h' \in \mathcal{K}'.$$

Theorem 4.4. *Let under the above suppositions $\widetilde{A} \in Ext_{L_0}$ be an extension* (4.3) *corresponding to the operator pair θ. Then for every $\lambda \in \hat{\rho}(L_0)$ the relations* (2.8) *and* (2.9) *hold with the operator T given by* (4.28).

Proof. Let $\hat{\Pi} = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \hat{\Gamma}^{L_0}, \Gamma^{L_0}\}$ be a boundary triplet (2.18) for the dual pair $\{L_0, L_0\}$ corresponding to the decomposing D-triplet Π for L (see Proposition 2.2, 1)). Applying Lemma 2.2 to the triplet $\hat{\Pi}$ (with $F := Z$) and taking (4.6) into account we arrive at the desired statement. □

Next assume that $\theta_0 = \{(\hat{N}_2, -\hat{N}_1); \mathcal{K}_0\} \in \widetilde{\mathcal{C}}(H^n)$ is an operator pair with operators $\hat{N}_1, \hat{N}_2 \in [H^n, \mathcal{K}_0]$ and θ' is an operator pair (4.8). Consider the boundary problem (4.15), (4.16) corresponding to the given θ' and let $v(\cdot, \lambda) : \Delta \to [\mathcal{K}_0', H]$ $(\lambda \in \mathbb{C})$ be a fundamental solution of this problem. Introduce the operator $\hat{T} \in [\mathcal{K}_0', \mathcal{K}_0]$ by setting

$$\hat{T} = \hat{N}_1 v^{(1)}(0,\lambda) + \hat{N}_2 v^{(2)}(0,\lambda). \tag{4.29}$$

Theorem 4.5. *Let under the above assumptions $\widetilde{A} \in Ext_{L_0}$ be an extension* (4.5) *corresponding to the operator pairs θ_0 and θ' and let $\{L_{b\times}, L_b\}$ be a dual pair of operators* (4.9), (4.10). *Then for every $\lambda \in \hat{\rho}\{L_{b\times}, L_b\}$ the relations* (2.8) *and* (2.9) *hold with the operator $\hat{T}$ instead of T.*

Proof. Let $\widetilde{\Pi} = \{H^n \oplus H^n, \Gamma^{L_b}, \Gamma^{L_{b\times}}\}$ be the boundary triplet (4.13), (4.14) for the dual pair $\{L_{b\times}, L_b\}$. Then in view of Proposition 4.1, 3) the domain $\mathcal{D}(\widetilde{A})$ is given by

$$\mathcal{D}(\widetilde{A}) = \{y \in \mathcal{D}(L_b^*) : \hat{N}_2\Gamma_0^{L_b}y - \hat{N}_1\Gamma_1^{L_b}y = 0\}. \tag{4.30}$$

Next assume that $v \in [\mathcal{K}_0', \mathfrak{N}_\lambda(L_b)]$ is the isomorphism (4.22) corresponding to the fundamental solution $v(\cdot,\lambda)$. Then the operator (4.29) can be written as

$$\hat{T} = \hat{N}_2(\Gamma_0^{L_b}v) - \hat{N}_1(\Gamma_1^{L_b}v). \tag{4.31}$$

Now applying Lemma 2.2 to the triplet $\widetilde{\Pi}$ and taking (4.30), (4.31) into account one obtains the required statement. □

Corollary 4.2. *Assume that θ' is an operator pair* (4.8)*, $\{L_{b\times}, L_b\}$ is the dual pair* (4.9)*,* (4.10) *and $A_{\theta'} \in Ext_{L_0}$ is the extension with the domain*

$$\mathcal{D}(A_{\theta'}) = \{y \in \mathcal{D} :\ y^{(2)}(0) = 0,\ N_0\Gamma_0'y + N_1\Gamma_1'y = 0\}. \tag{4.32}$$

If $\lambda \in \hat{\rho}\{L_{b\times}, L_b\}$ and $v(\cdot,\lambda) : \Delta \to [\mathcal{K}_0', H]$ is a fundamental solution of the problem (4.15), (4.16) *(corresponding to the given θ'), then the set*

$$\widetilde{v}(0,\lambda)\mathcal{K}_0' = \{\widetilde{y}(0) : y(\cdot) \text{ is a vector solution of the problem (4.15), (4.16)}\}$$

is a closed subspace in $H^n \oplus H^n$.

If in addition $\lambda \in \overline{\rho(A_{\theta'})}$, then $\dim \mathcal{K}_0' = \dim H^n$ *and, therefore, there exists a fundamental n-component solution $v(\cdot,\lambda) : \Delta \to [H^n, H]$ of the problem* (4.15), (4.16).

Proof. If $\lambda \in \hat{\rho}\{L_{b\times}, L_b\}$, then according to Proposition 4.1 there is an operator pair $\theta_0 = \{(\hat{N}_2, -\hat{N}_1); \mathcal{K}_0\} \in \widetilde{\mathcal{C}}(H^n)$ such that the extension $\widetilde{A} \in Ext\{L_{b\times}, L_b\}$ with the domain (4.5) obeys the inclusion $\lambda \in \rho(\widetilde{A})$. Let $\hat{T}$ be the corresponding operator (4.29) and let $\hat{N} = (\hat{N}_1 \quad \hat{N}_2) \in [H^n \oplus H^n, \mathcal{K}_0]$. Then $\hat{T} = \hat{N}\,\widetilde{v}(0,\lambda)$ and by Theorem 4.5 $0 \in \rho(\hat{T})$. Hence $\widetilde{v}(0,\lambda)\mathcal{K}_0'$ is a closed subspace in $H^n \oplus H^n$ and $\dim \mathcal{K}_0' = \dim \mathcal{K}_0$.

If in addition $\lambda \in \overline{\rho(A_{\theta'})}$, then $\rho(A_{\theta'}) \cap \rho(\widetilde{A}) \neq \emptyset$. Moreover, $A_{\theta'} \in Ext\{L_{b\times}, L_b\}$ and the boundary triplet (4.13), (4.14) for $\{L_{b\times}, L_b\}$ obeys the equality $\mathcal{D}(A_{\theta'}) = \operatorname{Ker}\Gamma_0^{L_b}$. Therefore by Corollary 5.3, ii) in [12] one can put $\mathcal{K}_0 = H^n$, which implies that $\dim \mathcal{K}_0' = \dim H^n$ □

In what follows we denote by $Sol(H^n)$ the set of all fundamental n-component solutions $v(\cdot,\lambda) : \Delta \to [H^n, H]$ of the problem (4.15), (4.16) obeying $\overline{\widetilde{v}(0,\lambda)H^n} = \widetilde{v}(0,\lambda)H^n$.

Definition 4.4. *A boundary condition* (4.16) *will be called dissipative, accumulative or selfadjoint if the corresponding operator pair (linear relation) θ' given by* (4.8) *belongs to the class $Dis(\mathcal{H}_0', \mathcal{H}_1')$, $Ac(\mathcal{H}_0', \mathcal{H}_1')$ or $Self(\mathcal{H}_0', \mathcal{H}_1')$ respectively.*

Corollary 4.3. 1) *Every boundary problem* (4.15), (4.16) *with a dissipative (accumulative) boundary condition has a fundamental solution $v(\cdot, \lambda) \in Sol(H^n)$ holomorphic on $\mathbb{C}_-$ (respectively, $\mathbb{C}_+$).*

2) *Let under assumptions of Theorem* 4.5 *$\theta' \in Dis(\mathcal{H}_0', \mathcal{H}_1')$ ($\theta' \in Ac(\mathcal{H}_0', \mathcal{H}_1')$). Then the statement of this theorem holds for all $\lambda \in \mathbb{C}_-$ ($\lambda \in \mathbb{C}_+$).*

Proof. 1) If $\theta' \in Dis(\mathcal{H}_0', \mathcal{H}_1')$, then $\theta'^{\times} \in Ac(\mathcal{H}_0', \mathcal{H}_1')$ and by (3.10) $L_{b\times}$ (L_b) is a closed dissipative (accumulative) operator in $\mathfrak{H}$. Therefore the operator pair $\{L_{b\times}, L_b\}$ admits a maximal dissipative proper extension $\widetilde{A}$, which in view of Proposition 4.1, 3) is given by (4.5). Moreover, by Theorem 4.1, 3) from [16] the equality (4.32) defines a maximal dissipative extension $A_{\theta'}$. Hence $\mathbb{C}_- \subset \rho(\widetilde{A}) \cap \rho(A_{\theta'})$ and Theorem 4.3, 2) together with Corollary 4.2 give the required statement.

The statement 2) is implied by Theorem 4.5 and the obvious inclusion $\mathbb{C}_- \subset \hat{\rho}\{L_{b\times}, L_b\}$. □

For an operator $T \in \widetilde{\mathcal{C}}(\mathfrak{H})$ denote by $\hat{\rho}_r(T) := \hat{\rho}(T) \cap \mathbb{R}$ the set of all real regular type points of T.

Corollary 4.4. 1) *Let* (4.15), (4.16) *be a boundary problem with a selfadjoint boundary condition* (4.16) *and let $L_b \in Ext_{L_0}$ be the corresponding extension* (4.10). *Then:*

(i) *the operator $L_b (= L_{b\times})$ is symmetric;*

(ii) *for every point $\lambda_0 \in \hat{\rho}_r(L_b)$ there exists a fundamental solution $v(\cdot, \lambda) \in Sol(H^n)$ holomorphic on $U(\lambda_0) \cup \mathbb{C}_+ \cup \mathbb{C}_-$ (here $U(\lambda_0) \subset \mathbb{R}$ is a real neighborhood of λ_0);*

(iii) *there exists a fundamental solution $v(\cdot, \lambda) \in Sol(H^n)$ holomorphic on $\mathbb{C}_+ \cup \hat{\rho}_r(L_b)$ (i.e., on some domain Λ containing the set $\mathbb{C}_+ \cup \hat{\rho}_r(L_b)$).*

2) *The following statements are valid:*

(i) *for every point $\lambda_0 \in \hat{\rho}_r(L_0)$ there exists a fundamental solution of the equation* (4.15) *holomorphic on $U(\lambda_0) \cup \mathbb{C}_+ \cup \mathbb{C}_-$;*

(ii) *there exists a fundamental solution of* (4.15) *holomorphic on $\mathbb{C}_+ \cup \hat{\rho}_r(L_0)$.*

Proof. 1) If $\lambda_0 \in \hat{\rho}_r(L_b)$, then there is a selfadjoint extension $\widetilde{A} \in Ext_{L_b}$ with $\lambda_0 \in \rho(\widetilde{A})$. Moreover, there exists a maximal accumulative extension $\widetilde{A}' \in Ext_{L_b}$ with $(\mathbb{C}_+ \cup \hat{\rho}_r(L_b)) \subset \rho(\widetilde{A}')$ (for example, so is the extension $\widetilde{A}'$ with the domain $\mathcal{D}(\widetilde{A}') = \mathcal{D}(L_b) \dotplus \mathfrak{N}_{-i}(L_b))$. Observe also that the extensions $\widetilde{A}$ and $\widetilde{A}'$ are defined by decomposing boundary conditions (4.5) and the operator (4.32) is selfadjoint, so that $\overline{\rho(A_{\theta'})} = \mathbb{C}$. Now applying Theorem 4.3, 2) and Corollary 4.2 we obtain the statement 1).

The statement 2) can be proved similarly. □

The following corollary is immediate from Theorem 4.5.

Corollary 4.5. *Let under suppositions of Theorem* 4.5 $\theta' \in Self(\mathcal{H}_0', \mathcal{H}_1')$ *and let* L_b *be the corresponding* (*symmetric*) *extension* (4.10). *Then the statement of the mentioned theorem holds for all* $\lambda \in \hat{\rho}_r(L_b) \cup \mathbb{C}_+ \cup \mathbb{C}_-$.

5. Resolvents of proper extensions

Let $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing D-triplet (3.9) for L, let $\theta = \{(C_0, C_1); \mathcal{K}\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ be an operator pair (4.1) and let $\widetilde{A} \in Ext_{L_0}$ be the corresponding extension (4.3). Assume that $\lambda \in \rho(\widetilde{A})$ and let $Z(\cdot, \lambda) : \Delta \to [\mathcal{K}', H]$ be a fundamental solution of the equation (4.15) (such a solution exists in view of Theorem 4.2). It follows from Theorem 4.4 that the corresponding operator $T \in [\mathcal{K}', \mathcal{K}]$ (see (4.28)) is invertible. This allows us to introduce the operator function $Y_\theta(\cdot, \overline{\lambda}) : \Delta \to [\mathcal{K}', H]$ as the operator solution of the equation $l[y] - \overline{\lambda} y = 0$ with the initial data

$$Y_\theta^{(1)}(0, \overline{\lambda}) = -\hat{C}_2^* T^{-1*}, \qquad Y_\theta^{(2)}(0, \overline{\lambda}) = \hat{C}_1^* T^{-1*}. \tag{5.1}$$

Next assume that $\theta^\times = \{(C_{0\times}, C_{1\times}); \mathcal{K}_\times\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ is a $\times$-adjoint operator pair (4.2). Then the adjoint extension $\widetilde{A}^*$ is defined by (4.4) and $\overline{\lambda} \in \rho(\widetilde{A}^*)$. Let $Z(\cdot, \overline{\lambda}) : \Delta \to [\mathcal{K}_\times', H]$ be a fundamental solution of the equation $l[y] - \overline{\lambda} y = 0$ and let $T_\times \in [\mathcal{K}_\times', \mathcal{K}_\times]$ be the operator given by

$$\begin{aligned} T_\times h' = (\hat{C}_{1\times} Z^{(1)}(0, \overline{\lambda}) + \hat{C}_{2\times} Z^{(2)}(0, \overline{\lambda}))h' \\ + (C_{0\times}' \Gamma_0' + C_{1\times}' \Gamma_1')(Z(t, \overline{\lambda})h'), \quad h' \in \mathcal{K}_\times'. \end{aligned} \tag{5.2}$$

Then in view of Theorem 4.4 $0 \in \rho(T_\times)$, which makes it possible to define the operator function $Y_{\theta^\times}(\cdot, \lambda) : \Delta \to [\mathcal{K}_\times', H]$ as the operator solution of (3.4) with the initial data

$$Y_{\theta^\times}^{(1)}(0, \lambda) = -\hat{C}_{2\times}^* T_\times^{-1*}, \qquad Y_{\theta^\times}^{(2)}(0, \lambda) = \hat{C}_{1\times}^* T_\times^{-1*}. \tag{5.3}$$

One can easily verify that for a given fundamental solution $Z(\cdot,\lambda)$ $(Z(\cdot,\overline{\lambda}))$ the operator function $Y_\theta(\cdot,\overline{\lambda})$ $(Y_{\theta^\times}(\cdot,\lambda))$ is uniquely defined by θ, i.e., it does not depend on the choice of the equivalent representation $\theta=\{(C_0,C_1);\mathcal{K}\}$ (respectively, $\theta^\times=\{(C_{0\times},C_{1\times});\mathcal{K}_\times\}$).

Definition 5.1. *The operator function $G_\theta(\cdot,\cdot,\lambda):\Delta\times\Delta\to[H]$, given by*

$$G_\theta(x,t,\lambda)=\begin{cases} Z(x,\lambda)\,Y_\theta^*(t,\overline{\lambda}), & x>t \\ Y_{\theta^\times}(x,\lambda)\,Z^*(t,\overline{\lambda}), & x<t \end{cases}, \quad \lambda\in\rho(\widetilde{A}) \tag{5.4}$$

will be called the Green function, corresponding to an operator pair $\theta\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$.

It is easily seen that for a given operator pair θ the Green function (5.4) does not depend on the choice of fundamental solutions $Z(\cdot,\lambda)$ and $Z(\cdot,\overline{\lambda})$.

Next assume that $\theta_0=\{(\hat{N}_2,-\hat{N}_1);\mathcal{K}_0\}\in\widetilde{\mathcal{C}}(H^n)$ is an operator pair with $\hat{N}_1,\hat{N}_2\in[H^n,\mathcal{K}_0]$, θ' is an operator pair (4.8) and $\widetilde{A}\in Ext_{L_0}$ is the corresponding extension defined by the decomposing boundary conditions (4.5). Suppose that $\lambda\in\rho(\widetilde{A})$ and let $v(\cdot,\lambda):\Delta\to[\mathcal{K}_0',H]$ be a fundamental solution of the boundary problem (4.15), (4.16) with the given operator pair θ'. It follows from Theorem 4.5 that $0\in\rho(\hat{T})$ where $\hat{T}\in[\mathcal{K}_0',\mathcal{K}_0]$ is the corresponding operator (4.29). This enables to introduce the operator solution $\varphi_\theta(\cdot,\overline{\lambda}):\Delta\to[\mathcal{K}_0',H]$ of the equation $l[y]-\overline{\lambda}y=0$ with the initial data

$$\varphi_\theta^{(1)}(0,\overline{\lambda})=-\hat{N}_2^*\hat{T}^{-1*}, \qquad \varphi_\theta^{(2)}(0,\overline{\lambda})=\hat{N}_1^*\hat{T}^{-1*}. \tag{5.5}$$

Furthermore, let $\theta_0^*=\{(\hat{N}_{2*},-\hat{N}_{1*});\mathcal{K}_{0*}\}\in\widetilde{\mathcal{C}}(H^n)$ be the adjoint operator pair (linear relation) to θ_0 and let $\theta'^\times=\{(N_{0\times},N_{1\times});\mathcal{K}_{1\times}\}\in\widetilde{\mathcal{C}}(\mathcal{H}_0',\mathcal{H}_1')$ be a $\times$-adjoint operator pair to θ'. Then the adjoint $\widetilde{A}^*$ is defined by the decomposing boundary conditions

$$\mathcal{D}(\widetilde{A}^*)=\{y\in\mathcal{D}:,\hat{N}_{1*}y^{(1)}(0)+\hat{N}_{2*}y^{(2)}(0)=0,\ N_{0\times}\Gamma_0'y+N_{1\times}\Gamma_1'y=0\} \tag{5.6}$$

and $\overline{\lambda}\in\rho(\widetilde{A}^*)$. Let $v(\cdot,\overline{\lambda}):\Delta\to[\mathcal{K}_{0\times}',H]$ be a fundamental solution of the $\times$-adjoint boundary problem

$$l[y]-\overline{\lambda}y=0 \tag{5.7}$$

$$N_{0\times}\Gamma_0'y+N_{1\times}\Gamma_1'y=0 \tag{5.8}$$

and let $\hat{T}_\times\in[\mathcal{K}_{0\times}',\mathcal{K}_{0*}]$ be the operator given by

$$\hat{T}_\times=\hat{N}_{1*}v^{(1)}(0,\overline{\lambda})+\hat{N}_{2*}v^{(2)}(0,\overline{\lambda}). \tag{5.9}$$

Then according to Theorem 4.5 $0 \in \rho(\hat{T}_\times)$, which makes it possible to define the operator solution $\varphi_{\theta\times}(\cdot,\lambda) : \Delta \to [\mathcal{K}'_{0\times}, H]$ of (3.4) with the initial data

$$\varphi_{\theta\times}^{(1)}(0,\lambda) = -\hat{N}_{2*}^*\hat{T}_\times^{-1*}, \qquad \varphi_{\theta\times}^{(2)}(0,\lambda) = \hat{N}_{1*}^*\hat{T}_\times^{-1*}. \tag{5.10}$$

In the following proposition we show that in the case of decomposing boundary conditions the Green function can be represented in a rather simpler form.

Proposition 5.1. *Let the extension* $\widetilde{A} \in Ext_{L_0}$ *be defined by decomposing boundary conditions* (4.5) *and let* $v(\cdot,\lambda)$ $(v(\cdot,\overline{\lambda}))$ *be a fundamental solution of the boundary problem* (4.15)-(4.16)*(respectively,*(5.7)-(5.8)*). Then the corresponding Green function* (5.4) *admits the representation*

$$G_\theta(x,t,\lambda) = \begin{cases} v(x,\lambda)\,\varphi_\theta^*(t,\overline{\lambda}), & x > t \\ \varphi_{\theta^\times}(x,\lambda)\,v^*(t,\overline{\lambda}), & x < t \end{cases}, \quad \lambda \in \rho(\widetilde{A}). \tag{5.11}$$

Proof. First observe that formula (5.11) defines the same function $G_\theta(x,t,\lambda)$ independently on the choice of fundamental solutions $v(\cdot,\lambda)$ and $v(\cdot,\overline{\lambda})$. Therefore it is sufficient to prove (5.11) only for a certain pair of such solutions.

According to Remark 4.1 the extension $\widetilde{A}$ is defined by (4.6) and (4.7). This and Theorem 4.4 imply that for each $\lambda \in \rho(\widetilde{A})$ there exists a fundamental solution

$$Z_c(t,\lambda) = (v_c(t,\lambda) \quad u_c(t,\lambda)) : \mathcal{K}_0 \oplus \mathcal{K}_1 \to H \tag{5.12}$$

of the equation (4.15) such that $T = I_{\mathcal{K}_0\oplus\mathcal{K}_1}$ (here T is the operator (4.28)). Therefore by (4.7) one has

$$\begin{gathered} \hat{N}_1 v_c^{(1)}(0,\lambda) + \hat{N}_2 v_c^{(2)}(0,\lambda) = I_{\mathcal{K}_0}, \\ (N_0\Gamma'_0 + N_1\Gamma'_1)(v_c(t,\lambda)h) = 0, \quad h \in \mathcal{K}_0. \end{gathered} \tag{5.13}$$

Similarly by using (5.6) one proves the existence of a fundamental solution

$$Z_c(t,\overline{\lambda}) = (v_c(t,\overline{\lambda}) \quad u_c(t,\overline{\lambda})) : \mathcal{K}_{0*} \oplus \mathcal{K}_{1\times} \to H \tag{5.14}$$

of the equation $l[y] - \overline{\lambda}y = 0$ such that the operator (5.2) obeys $T_\times = I_{\mathcal{K}_{0*}\oplus\mathcal{K}_{1\times}}$ and

$$\begin{gathered} \hat{N}_{1*} v_c^{(1)}(0,\overline{\lambda}) + \hat{N}_{2*} v_c^{(2)}(0,\overline{\lambda}) = I_{\mathcal{K}_{0*}}, \\ (N_{0\times}\Gamma'_0 + N_{1\times}\Gamma'_1)(v_c(t,\overline{\lambda})h) = 0, \quad h \in \mathcal{K}_{0*}. \end{gathered} \tag{5.15}$$

It follows from the second equality in (5.13) ((5.15)) that $v_c(\cdot,\lambda)$ ($v_c(\cdot,\overline{\lambda})$) is a fundamental solution of the boundary problem (4.15)–(4.16) (respectively, (5.7)–(5.8)). Moreover, the first equalities in (5.13) and (5.15) show that $\hat{T}=I_{\mathcal{K}_0}$ and $\hat{T}_\times=I_{\mathcal{K}_{0*}}$, where $\hat{T}$ and $\hat{T}_\times$ are given by (4.29) and (5.9) respectively. Therefore (5.5) and (5.10) take the form

$$\begin{aligned}\varphi_\theta^{(1)}(0,\overline{\lambda})&=-\hat{N}_2^*, \quad \varphi_\theta^{(2)}(0,\overline{\lambda})=\hat{N}_1^*;\\ \varphi_{\theta^\times}^{(1)}(0,\lambda)&=-\hat{N}_{2*}^*, \quad \varphi_{\theta^\times}^{(2)}(0,\lambda)=\hat{N}_{1*}^*.\end{aligned} \tag{5.16}$$

Next, in view of (4.7) the initial data (5.1) can be written as

$$Y_\theta^{(1)}(0,\overline{\lambda})=(-\hat{N}_2^* \quad 0):\mathcal{K}_0\oplus\mathcal{K}_1\to H^n,$$

$$Y_\theta^{(2)}(0,\overline{\lambda})=(\hat{N}_1^* \quad 0):\mathcal{K}_0\oplus\mathcal{K}_1\to H^n$$

and similarly (5.3) gives

$$Y_{\theta^\times}^{(1)}(0,\lambda)=(-\hat{N}_{2*}^* \quad 0):\mathcal{K}_{0*}\oplus\mathcal{K}_{1\times}\to H^n,$$

$$Y_{\theta^\times}^{(2)}(0,\lambda)=(\hat{N}_{1*}^* \quad 0):\mathcal{K}_{0*}\oplus\mathcal{K}_{1\times}\to H^n.$$

Combining these equalities with (5.16) one obtains the block-matrix representations

$$\begin{aligned}Y_\theta(t,\overline{\lambda})&=(\varphi_\theta(t,\overline{\lambda}) \quad 0):\mathcal{K}_0\oplus\mathcal{K}_1\to H,\\ Y_{\theta^\times}(t,\lambda)&=(\varphi_{\theta^\times}(t,\lambda) \quad 0):\mathcal{K}_{0*}\oplus\mathcal{K}_{1\times}\to H.\end{aligned} \tag{5.17}$$

Now by using definition (5.4) (with $Z(x,\lambda):=Z_c(x,\lambda)$, $Z(t,\overline{\lambda}):=Z_c(t,\overline{\lambda})$) and taking (5.12), (5.14) and (5.17) into account we arrive at the equality (5.11) with $v(x,\lambda):=v_c(x,\lambda)$ and $v(t,\overline{\lambda}):=v_c(t,\overline{\lambda})$. □

Now we are ready to prove the main result of the paper — the representation of the resolvent of a proper extension $\widetilde{A}\in Ext_{L_0}$ in a form of the integral operator.

Theorem 5.1. *Suppose that $\Pi=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ is a decomposing D-triplet* (3.9) *for L, $\theta=\{(C_0,C_1);\mathcal{K}\}\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ is the operator pair* (4.1)*, $\widetilde{A}\in Ext_{L_0}$ is the corresponding extension* (4.3)*, $\lambda\in\rho(\widetilde{A})$ and $G_\theta(x,t,\lambda)$ is the Green function* (5.4)*. Then the resolvent $(\widetilde{A}-\lambda)^{-1}\in[\mathfrak{H}]$ is the integral operator, given by*

$$((\widetilde{A}-\lambda)^{-1}f)(x)=\int_0^b G_\theta(x,t,\lambda)f(t)\,dt$$

$$:=\lim_{\eta\uparrow b}\int_0^\eta G_\theta(x,t,\lambda)f(t)\,dt, \quad f=f(\cdot)\in\mathfrak{H}. \tag{5.18}$$

Proof. Step1. Let $\mathfrak{H}_b$ be the set of all functions $f \in \mathfrak{H}$ such that $f(t) = 0$ on some interval $(\eta, b) \subset \Delta$ (depending on f). First we show that

$$((A_0 - \lambda)^{-1} f)(x) = \int_0^b G_0(x,t,\lambda) f(t)\, dt, \quad f = f(\cdot) \in \mathfrak{H}_b, \quad \lambda \in \mathbb{C}_+, \tag{5.19}$$

where A_0 is the symmetric extension (3.14) and the operator kernel $G_0(x,t,\lambda)$ is

$$G_0(x,t,\lambda) = \begin{cases} -v_0(x,\lambda)\, c^*(t,\overline{\lambda}), & x > t \\ -c(x,\lambda)\, v_0^*(t,\overline{\lambda}), & x < t \end{cases}, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{5.20}$$

To prove (5.19) it is sufficient to show that for every $f = f(t) \in \mathfrak{H}_b$ the function

$$y = y(x,\lambda) := \int_0^b G_0(x,t,\lambda) f(t)\, dt, \quad \lambda \in \mathbb{C}_+ \tag{5.21}$$

belongs to $\mathcal{D}(A_0)$ and obeys the equality $l[y] - \lambda y = f$.

It follows from (5.21) and (5.20) that

$$y = y(x,\lambda) = c(x,\lambda) C_1(x) + v_0(x,\lambda) C_2(x) = Y_v(x,\lambda) C(x), \quad \lambda \in \mathbb{C}_+ \tag{5.22}$$

where

$$C_1(x) = -\int_x^b v_0^*(t,\overline{\lambda}) f(t)\, dt, \qquad C_2(x) = -\int_0^x c^*(t,\overline{\lambda}) f(t)\, dt, \tag{5.23}$$

$$Y_v(x,\lambda) = (c(x,\lambda) \;\; v_0(x,\lambda)) : H^n \oplus H^n \to H, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-, \tag{5.24}$$

$$C(x) = \{C_1(x), C_2(x)\} \; (\in H^n \oplus H^n). \tag{5.25}$$

Next we show that the functions (5.24), (5.25) satisfy hypothesis of Lemma 3.1.

In view of (3.21) and (3.22) $Y_v(\cdot,\lambda)$ is the solution of (3.4) with the initial data

$$\widetilde{Y}_v(0,\lambda) = \begin{pmatrix} c^{(1)}(0,\lambda) & v_0^{(1)}(0,\lambda) \\ c^{(2)}(0,\lambda) & v_0^{(2)}(0,\lambda) \end{pmatrix} = \begin{pmatrix} I_{H^n} & -m(\lambda) \\ 0 & I_{H^n} \end{pmatrix}, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{5.26}$$

Hence $0 \in \rho(\widetilde{Y}_v(0,\lambda))$ and, consequently, $0 \in \rho(\widetilde{Y}_v(x,\lambda))$ for all $x \in \Delta$. Next the direct calculation with taking the relation $m^*(\overline{\lambda}) = m(\lambda)$ into account gives $\widetilde{Y}_v^*(0,\overline{\lambda})J_{H^n}\widetilde{Y}_v(0,\lambda) = J_{H^n}$, where $J_{H^n} = \begin{pmatrix} 0 & -I_{H^n} \\ I_{H^n} & 0 \end{pmatrix}$. Moreover, the Lagrange's identity (3.8) implies

$$\widetilde{Y}_v^*(x,\overline{\lambda})J_{H^n}\widetilde{Y}_v(x,\lambda) = \widetilde{Y}_v^*(0,\overline{\lambda})J_{H^n}\widetilde{Y}_v(0,\lambda)\,(= J_{H^n}), \quad x \in \Delta.$$

Therefore in view of the invertibility of $\widetilde{Y}_v(x,\lambda)$ one has $\widetilde{Y}_v(x,\lambda)J_{H^n}\widetilde{Y}_v^*(x,\overline{\lambda}) = J_{H^n}$, which is equivalent to the relations

$$c^{(1)}(x,\lambda)v_0^{(1)*}(x,\overline{\lambda}) - v_0^{(1)}(x,\lambda)c^{(1)*}(x,\overline{\lambda}) = 0, \tag{5.27}$$

$$c^{(2)}(x,\lambda)v_0^{(2)*}(x,\overline{\lambda}) - v_0^{(2)}(x,\lambda)c^{(2)*}(x,\overline{\lambda}) = 0, \tag{5.28}$$

$$c^{(2)}(x,\lambda)v_0^{(1)*}(x,\overline{\lambda}) - v_0^{(2)}(x,\lambda)c^{(1)*}(x,\overline{\lambda}) = -I_{H^n}, \qquad x \in \Delta,\ \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \tag{5.29}$$

It follows from (5.27), (5.29) that

$$\begin{gathered} c^{(1)}(x,\lambda)v_0^*(x,\overline{\lambda}) - v_0^{(1)}(x,\lambda)c^*(x,\overline{\lambda}) = 0, \\ c^{(2)}(x,\lambda)v_0^*(x,\overline{\lambda}) - v_0^{(2)}(x,\lambda)c^*(x,\overline{\lambda}) = (-I_H\ \ 0\ \ \ldots\ \ 0)^\top, \\ x \in \Delta, \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-. \end{gathered}$$

Moreover the equalities (5.23), (5.25) give

$$C'(x) = (v_0^*(x,\overline{\lambda}) \quad - c^*(x,\overline{\lambda}))^\top f(x) \quad (\bmod \mu).$$

Hence nearly everywhere on Δ one has

$$\begin{aligned} \widetilde{Y}_v(x,\lambda)C'(x) &= \begin{pmatrix} c^{(1)}(x,\lambda) & v_0^{(1)}(x,\lambda) \\ c^{(2)}(x,\lambda) & v_0^{(2)}(x,\lambda) \end{pmatrix} \begin{pmatrix} v_0^*(x,\overline{\lambda}) \\ -c^*(x,\overline{\lambda}) \end{pmatrix} f(x) \\ &= \{\underbrace{0,\ldots,0,}_{n} -f(x), \underbrace{0,\ldots,0}_{n-1}\}, \end{aligned}$$

which coincides with (3.6). Therefore according to Lemma 3.1 the function (5.22) belongs to $\mathcal{D}(l)$ and obeys the relations

$$l[y] - \lambda y = f, \qquad \widetilde{y}(x,\lambda) = \widetilde{Y}_v(x,\lambda)C(x). \tag{5.30}$$

Since $f \in \mathfrak{H}_b$, it follows from (5.23) that $C_1(x) \equiv 0$ and $C_2(x) \equiv C_2\,(\in H^n)$ on some interval $(\eta, b) \subset \Delta$. Hence by (5.22) $y = v_0(x,\lambda)C_2$, $x \in (\eta, b)$, which yields the inclusion $y \in \mathcal{D}$. Moreover, according to [16] (see proof of Lemma 3.4) $\Gamma_0' y = \Gamma_0'(v_0(x,\lambda)C_2)$ and by (3.16) $\Gamma_0' y = 0$.

Finally combining the second equality in (5.30) with (5.26) and (5.25), we obtain $y^{(2)}(0,\lambda)=C_2(0)=0$.

Thus the function (5.21) satisfies the boundary conditions in the right hand part of (3.14) and, consequently, it belongs to $\mathcal{D}(A_0)$.

Step2. Next by using formula for resolvents (2.28) we prove (5.18) for all $f=f(t)\in\mathfrak{H}_b$ and $\lambda\in\rho(\widetilde{A})\cap\mathbb{C}_+$. Applying Lemma 4.1, 3) to the solution $Z_-(\cdot,\overline{\lambda})$ and taking (3.24) into account one obtains

$$\gamma_-^*(\overline{\lambda})f=\int_0^b Z_-^*(t,\overline{\lambda})f(t)\,dt,\quad \lambda\in\mathbb{C}_+,\quad f\in\mathfrak{H}_b.$$

This and (3.23) yield

$$\begin{aligned}\big(\gamma_+(\lambda)(C_0+C_1M_+(\lambda))^{-1}C_1\gamma_-^*(\overline{\lambda})f\big)(x)\\ =\int_0^b Z_+(x,\lambda)(C_0+C_1M_+(\lambda))^{-1}C_1Z_-^*(t,\overline{\lambda})f(t)\,dt\\ =\int_0^b G_1(x,t,\lambda)f(t)\,dt,\end{aligned}$$

where

$$G_1(x,t,\lambda)=Z_+(x,\lambda)(C_0+C_1M_+(\lambda))^{-1}C_1Z_-^*(t,\overline{\lambda}),\quad \lambda\in\rho(\widetilde{A})\cap\mathbb{C}_+. \tag{5.31}$$

Moreover the resolvent $(A_0-\lambda)^{-1}$ is defined by (5.19). Hence formula (2.28) for the decomposing D-triplet Π takes the form

$$((\widetilde{A}-\lambda)^{-1}f)(x)=\int_0^b G(x,t,\lambda)f(t)\,dt,\quad f=f(\cdot)\in\mathfrak{H}_b,\quad \lambda\in\rho(\widetilde{A})\cap\mathbb{C}_+, \tag{5.32}$$

where $G(x,t,\lambda)=G_0(x,t,\lambda)-G_1(x,t,\lambda)$, $\lambda\in\rho(\widetilde{A})\cap\mathbb{C}_+$. Now it remains to show that $G(x,t,\lambda)$ coincides with the Green function $G_\theta(x,t,\lambda)$ (see (5.4)).

Denote by $G_+(x,t,\lambda)$ and $G_-(x,t,\lambda)$ restrictions of $G(x,t,\lambda)$ onto the sets $\{(x,t):x>t\}$ and $\{(x,t):x<t\}$ respectively. It follows from (5.20) and (5.31) that

$$G_+(x,t,\lambda)=-v_0(x,\lambda)c^*(t,\overline{\lambda})-Z_+(x,\lambda)(C_0+C_1M_+(\lambda))^{-1}C_1Z_-^*(t,\overline{\lambda}), \tag{5.33}$$

$$G_-^*(x,t,\lambda) = -v_0(t,\overline{\lambda})c^*(x,\lambda) - Z_-(t,\overline{\lambda})C_1^*(C_0^* + M_-(\overline{\lambda})C_1^*)^{-1}Z_+^*(x,\lambda). \tag{5.34}$$

Next assume that $\theta^\times = \{(C_{0\times}, C_{1\times}); \mathcal{K}_\times\} \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ is the $\times$-adjoint operator pair (4.2) and let $\theta^* = \{(C_{1*}, C_{0*}); \mathcal{K}_\times\} \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$ be the adjoint pair with C_{1*} and C_{0*} given by (2.3). It follows from (4.2) that the block-matrix representations

$$\begin{aligned} C_{1*} &= (\hat{C}_{2\times} \;\; C'_{1*}) : H^n \oplus \mathcal{H}'_1 \to \mathcal{K}_\times, \\ C_{0*} &= (-\hat{C}_{1\times} \;\; C'_{0*}) : H^n \oplus \mathcal{H}'_0 \to \mathcal{K}_\times. \end{aligned} \tag{5.35}$$

are valid with $C'_{1*} = C'_{0\times} \restriction \mathcal{H}'_1$ and $C'_{0*} = C'_{1\times}P'_1 - iC'_{0\times}P'_2$. Now by using Lemma 2.1 we can rewrite (5.34) as

$$G_-^*(x,t,\lambda) = -v_0(t,\overline{\lambda})c^*(x,\lambda) - Z_-(t,\overline{\lambda}) \times (C_{1*} + C_{0*}M_-(\overline{\lambda}))^{-1}C_{0*}Z_+^*(x,\lambda). \tag{5.36}$$

Let

$$\begin{aligned} Y_1(t,\overline{\lambda}) &:= (-c(t,\overline{\lambda}) \;\; 0) : H^n \oplus \mathcal{H}'_0 \to H, \\ Y_2(t,\lambda) &:= (-c(t,\lambda) \;\; 0) : H^n \oplus \mathcal{H}'_1 \to H \end{aligned} \tag{5.37}$$

and let $Y_-(\cdot,\overline{\lambda}) : \Delta \to [\mathcal{H}_0, H]$, $Y_+(\cdot,\lambda) : \Delta \to [\mathcal{H}_1, H]$ be operator solutions given by

$$Y_-(t,\overline{\lambda}) := Y_1(t,\overline{\lambda}) - Z_-(t,\overline{\lambda})C_1^*(C_0^* + M_-(\overline{\lambda})C_1^*)^{-1} \tag{5.38}$$

$$Y_+(t,\lambda) := Y_2(t,\lambda) - Z_+(t,\lambda)C_{0*}^*(C_{1*}^* + M_+(\lambda)C_{0*}^*)^{-1} \tag{5.39}$$

Combining (5.37) with (3.19) and (3.20) one obtains

$$-v_0(x,\lambda)c^*(t,\overline{\lambda}) = Z_+(x,\lambda)Y_1^*(t,\overline{\lambda}),$$

$$-v_0(t,\overline{\lambda})c^*(x,\lambda) = Z_-(t,\overline{\lambda})Y_2^*(x,\lambda).$$

Hence the equalities (5.33) and (5.36) can be represented as

$$G_+(x,t,\lambda) = Z_+(x,\lambda)Y_-^*(t,\overline{\lambda}), \quad G_-^*(x,t,\lambda) = Z_-(t,\overline{\lambda})Y_+^*(x,\lambda) \tag{5.40}$$

It follows from (3.23) and (3.24) that the corresponding operators (4.28) and (5.2) for $Z_+(\cdot,\lambda)$ and $Z_-(\cdot,\overline{\lambda})$ are

$$T = (C_0\Gamma_0 + C_1\Gamma_1)\gamma_+(\lambda), \qquad T_\times = (C_{0\times}\Gamma_0 + C_{1\times}\Gamma_1)\gamma_-(\overline{\lambda}).$$

Therefore by Lemma 2.3 one has

$$T = C_0 + C_1M_+(\lambda), \qquad T_\times = C_{1*} + C_{0*}M_-(\overline{\lambda}). \tag{5.41}$$

Next by (5.38) $Y_-(\cdot,\overline{\lambda})$ is the operator solution if the equation $l[y]-\overline{\lambda}y=0$ and

$$\widetilde{Y}_-(0,\overline{\lambda})=\widetilde{Y}_1(0,\overline{\lambda})-\widetilde{Z}_-(0,\overline{\lambda})C_1^*(C_0^*+M_-(\overline{\lambda})C_1^*)^{-1}$$
$$=X(C_0^*+M_-(\overline{\lambda})C_1^*)^{-1},$$

where

$$X=\widetilde{Y}_1(0,\overline{\lambda})(C_0^*+M_-(\overline{\lambda})C_1^*)-\widetilde{Z}_-(0,\overline{\lambda})C_1^*$$
$$=\begin{pmatrix}-I & 0\\ 0 & 0\end{pmatrix}\left[\begin{pmatrix}\hat{C}_2^*\\ C_0'^*\end{pmatrix}+\begin{pmatrix}m(\overline{\lambda}) & M_{2-}(\overline{\lambda})\\ M_{3-}(\overline{\lambda}) & M_{4-}(\overline{\lambda})\end{pmatrix}\begin{pmatrix}-\hat{C}_1^*\\ C_1'^*\end{pmatrix}\right]$$
$$-\begin{pmatrix}-m(\overline{\lambda}) & -M_{2-}(\overline{\lambda})\\ I_{H^n} & 0\end{pmatrix}\begin{pmatrix}-\hat{C}_1^*\\ C_1'^*\end{pmatrix}=\begin{pmatrix}-\hat{C}_2^*+m(\overline{\lambda})\hat{C}_1^*-M_{2-}(\overline{\lambda})C_1'^*\\ 0\end{pmatrix}$$
$$-\begin{pmatrix}m(\overline{\lambda})\hat{C}_1^*-M_{2-}(\overline{\lambda})C_1'^*\\ -\hat{C}_1^*\end{pmatrix}=\begin{pmatrix}-\hat{C}_2^*\\ \hat{C}_1^*\end{pmatrix}$$

(here we made use of the block-matrix representations (4.1), (3.13) and (3.22)). This implies that the solution $Y_-(\cdot,\overline{\lambda})$ satisfies the initial data

$$\widetilde{Y}_-(0,\overline{\lambda})=(-\hat{C}_2^*\quad \hat{C}_1^*)^\top(C_0^*+M_-(\overline{\lambda})C_1^*)^{-1}=(-\hat{C}_2^*\quad \hat{C}_1^*)^\top T^{-1*} \tag{5.42}$$

Similar calculations for $Y_+(t,\lambda)$ (with taking (5.35) into account) gives

$$\widetilde{Y}_+(0,\lambda)=(-\hat{C}_{2\times}^*\ \ \hat{C}_{1\times}^*)^\top(C_{1*}^*+M_+(\lambda)C_{0*}^*)^{-1}=(-\hat{C}_{2\times}^*\ \ \hat{C}_{1\times}^*)^\top T_\times^{-1*}. \tag{5.43}$$

Comparing (5.42) and (5.43) with (5.1) and (5.3), one obtains $Y_-(t,\overline{\lambda})=Y_\theta(t,\overline{\lambda})$ and $Y_+(t,\lambda)=Y_{\theta^\times}(t,\lambda)$. Now the equality $G(x,t,\lambda)=G_\theta(x,t,\lambda)$ is implied by (5.40).

Step 3. To complete the proof it is necessary to extend the above result to all $f\in\mathfrak{H}$ and $\lambda\in\rho(\widetilde{A})$.

If $\lambda\in\rho(\widetilde{A})\cap\mathbb{C}_-$, then $\overline{\lambda}\in\rho(\widetilde{A}^*)\cap\mathbb{C}_+$ and, consequently,

$$((\widetilde{A}^*-\overline{\lambda})^{-1}f)(x)=\int_0^b G_{\theta^\times}(x,t,\overline{\lambda})f(t)\,dt,\quad f=f(\cdot)\in\mathfrak{H}_b. \tag{5.44}$$

Since $(\widetilde{A}-\lambda)^{-1}=((\widetilde{A}^*-\overline{\lambda})^{-1})^*$, it follows from (5.44) that $(\widetilde{A}-\lambda)^{-1}|\mathfrak{H}_b$ is the integral operator with the kernel $G'(x,t,\lambda)=(G_{\theta^\times}(t,x,\overline{\lambda}))^*=G_\theta(x,t,\lambda)$ (here the second equality is immediate from (5.4)). Therefore (5.18) holds for all $f\in\mathfrak{H}_b$ and $\lambda\in\rho(\widetilde{A})\cap\mathbb{C}_-$.

Next assume that $\lambda_0 \in \rho(\widetilde{A}) \cap \mathbb{R}$. Then there exists a disk $U(\lambda_0) = \{\lambda \in \mathbb{C} : |\lambda\lambda_0| < \varepsilon\}$ $(\varepsilon > 0)$ such that $U(\lambda_0) \subset \rho(\widetilde{A})$. Let $\mathcal{Z}(\cdot)$ be an operator function (4.26) and let $\mathcal{Z}(\cdot, \lambda)$ $(\lambda \in \rho(\widetilde{A}))$ be a family of fundamental solutions (4.27) (see proof of Theorem 4.3). Since by Proposition 2.2 $\Gamma_j \in [\mathcal{D}_+, \mathcal{H}_j]$, $j \in \{0, 1\}$, the operator functions

$$T(\lambda) = (C_0\Gamma_0 + C_1\Gamma_1)\mathcal{Z}(\lambda), \quad T_\times(\lambda) = (C_{0\times}\Gamma_0 + C_{1\times}\Gamma_1)\mathcal{Z}(\lambda), \ \lambda \in U(\lambda_0)$$

are holomorphic on $U(\lambda_0)$. Let now $\mathcal{Y}(\cdot, \lambda) : \Delta \to [\mathcal{K}', H]$ and $\mathcal{Y}_\times(\cdot, \lambda) : \Delta \to [\mathcal{K}', H]$ be operator solutions of (3.4) with

$$\widetilde{\mathcal{Y}}(0, \lambda) = (-\hat{C}_2^* \quad \hat{C}_1^*)^\top (T(\overline{\lambda}))^{-1*},$$
$$\widetilde{\mathcal{Y}}_\times(0, \lambda) = (-\hat{C}_{2\times}^* \quad \hat{C}_{1\times}^*)^\top (T_\times(\overline{\lambda}))^{-1*}, \quad \lambda \in U(\lambda_0)$$

Then for every $\lambda \in U(\lambda_0)$ the corresponding Green function (5.4) can be written as

$$G_\theta(x, t, \lambda) = \begin{cases} \mathcal{Z}(x, \lambda)\, \mathcal{Y}^*(t, \overline{\lambda}), & x > t \\ \mathcal{Y}_\times(x, \lambda)\, \mathcal{Z}^*(t, \overline{\lambda}), & x < t \end{cases}. \tag{5.45}$$

Since all initial data $\widetilde{\mathcal{Z}}(0, \lambda), \widetilde{\mathcal{Y}}(0, \lambda)$ and $\widetilde{\mathcal{Y}}_\times(0, \lambda)$ are continuous (and even holomorphic) at the point λ_0, it follows from (5.45) that

$$\lim_{\lambda \to \lambda_0} \sup_{(x,t) \in R} \| G_\theta(x, t, \lambda) - G_\theta(x, t, \lambda_0) \| = 0$$

for each closed rectangle $R \subset \Delta \times \Delta$. Therefore for every $f = f(\cdot) \in \mathfrak{H}_b$ we can pass to the limit in the equality (5.18) as $\mathbb{C}_+ \ni \lambda \to \lambda_0$, which gives the same equality for $\lambda = \lambda_0$. Hence (5.18) holds for all $\lambda \in \rho(\widetilde{A})$ and $f \in \mathfrak{H}_b$.

Finally (4.23) implies existence of the limit

$$\lim_{\eta \uparrow b} \int_0^\eta G_\theta(x, t, \lambda) f(t)\, dt$$

for all $f(\cdot) \in \mathfrak{H}$, which completely proves (5.18). □

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Contact information

Vadim Mogilevskii
Department of Calculus
Lugans'k National University
2 Oboronna Str.,
Lugans'k, 91011
Ukraine
E-Mail: vim@mail.dsip.net