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Lévy approximation of processes with locally independent increments with semi-Markov switching

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Abstract. In this paper, the weak convergence of additive functionals of processes with locally independent increments and with semi-Markov switching in the scheme of Lévy approximation is investigated. Singular perturbation problem for the compensating operator of the extended Markov renewal process is used to prove the relative compactness.

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1. Introduction

Lévy approximation is still an active area of research in several theoretical and applied directions. Since Lévy processes are now standard, Lévy approximation is quite useful for analyzing complex systems (see, e.g. [1, 10]). Moreover they are involved in many applications, e.g., risk theory, finance, queueing, physics, etc. For a background on Lévy process see, e.g. [1, 4, 10].

In particular in [6, 7] it has been studied the following stochastic additive functional in the series scheme with the small parameter series $\varepsilon \to 0$ $(\varepsilon > 0)$

$$\xi^\varepsilon(t) = \xi_0 + \int_0^t \eta^\varepsilon(ds; x^\varepsilon(s/\varepsilon^2)), \quad t \geq 0, \tag{1.1}$$

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of a jump Markov process with locally independent increments (PLII) ([6]) $\eta(t;\cdot)$, $t \geq 0$, (also known as a Piecewise deterministic Markov process — PDMP, [2]), perturbed by the jump semi-Markov process $x(t)$, $t \geq 0$. The process (1.1) is studied in a (functional) Poisson approximation scheme, within an *ad hoc* time-scaling.

We propose to study functionals of PLII [6] using a combination of two methods. The one based on semimartingales theory, is combined with a solution of singular perturbation problem instead of ergodic theorem. So, the method includes two steps.

In the first step we prove the relative compactness of the semimartingales representation of the family ξ^ε, $\varepsilon > 0$, by proving the following two facts [3]:

$$\lim_{c\to\infty} \sup_{\varepsilon\leq\varepsilon_0} \mathbf{P}\big\{\sup_{t\leq T} |\xi^\varepsilon(t)| > c\big\} = 0,$$

known as the compact containment condition, and

$$\mathbf{E}|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq k|t-s|,$$

for some positive constant k.

In the second step we prove convergence of the process (1.1) by using singular perturbation technique as presented in [6].

The paper is organized as follows. In Section 2 we present the time-scaled additive functional (1.1), the PLII and the switching semi-Markov process. In the same section we present the main results of Lévy approximation. In Section 3 we present the proof of the theorem.

2. Main results

Let us consider the space $\mathbb{R}^d$ endowed with a norm $|\cdot|$ $(d \geq 1)$, and $(E, \mathcal{E})$, a *standard phase space*, (i.e., E is a Polish space and $\mathcal{E}$ its Borel σ-algebra). For a vector $v \in \mathbb{R}^d$ and a matrix $c \in \mathbb{R}^{d\times d}$, v^* and c^* denote their transpose respectively. Let $C_3(\mathbb{R}^d)$ be a measure-determining class of real-valued bounded functions, such that $g(u)/|u|^2 \to 0$, as $|u| \to 0$ for $g \in C_3(\mathbb{R}^d)$ (see [5,6]).

The additive functional $\xi^\varepsilon(t)$, $t \geq 0$, $\varepsilon > 0$ on $\mathbb{R}^d$ in the series scheme with small series parameter $\varepsilon \to 0$, $(\varepsilon > 0)$ are defined by the stochastic additive functional ([6, Section 3.3.1])

$$\xi^\varepsilon(t) = \xi_0^\varepsilon + \int_0^t \eta^\varepsilon(ds; x(s/\varepsilon^2)). \tag{2.1}$$

The family of the Markov jump processes with *locally independent increments* $\eta^\varepsilon(t;x),\ t\ge 0,\ x\in E$ on $\mathbb{R}^d$, is defined by the generators [6, Section 3.3.1] (see also [7])

$$\mathbf{\Gamma}^\varepsilon(x)\varphi(u)=\int_{\mathbb{R}^d}[\varphi(u+v)-\varphi(u)]\Gamma^\varepsilon(u,dv;x),\quad x\in E, \tag{2.2}$$

or, equivalently

$$\begin{aligned}\mathbf{\Gamma}^\varepsilon(x)\varphi(u)&=b_\varepsilon(u;x)\varphi'(u)+\frac{1}{2}c_\varepsilon(u;x)\varphi''(u)\\&\quad+\int_{\mathbb{R}^d}\Big[\varphi(u+v)-\varphi(u)-v\varphi'(u)-\frac{vv^*}{2}\varphi''(u)\Big]\Gamma^\varepsilon(u,dv;x),\end{aligned}$$

where $b_\varepsilon(u;x)=\int_{\mathbb{R}^d}v\Gamma^\varepsilon(u,dv;x)$, $c_\varepsilon(u;x)=\int_{\mathbb{R}^d}vv^*\Gamma^\varepsilon(u,dv;x)$, and $\Gamma^\varepsilon(u,dv;x)$ is the intensity kernel.

The switching semi-Markov process $x(t),t\ge 0$ on the standard phase space $(E,\mathcal{E})$, is defined by the semi-Markov kernel

$$Q(x,B,t)=P(x,B)F_x(t),\ x\in E,B\in\mathcal{E},\ t\ge 0,$$

which defines the associated Markov renewal process $x_n,\tau_n,n\ge 0$:

$$\begin{aligned}Q(x,B,t)&=P(x_{n+1}\in B,\ \theta_{n+1}\le t\mid x_n=x)\\&=P(x_{n+1}\in B\mid x_n=x)P(\theta_{n+1}\le t\mid x_n=x).\end{aligned}$$

Let the following conditions hold.

C1: The semi-Markov process $x(t),\ t\ge 0$ is uniformly ergodic with the stationary distribution

$$\pi(dx)q(x)=q\rho(dx),\qquad q(x):=1/m(x),\qquad q:=1/m,$$

$$m(x):=\mathbf{E}\theta_x=\int_0^\infty\overline{F}_x(t)\,dt,\qquad m:=\int_E\rho(dx)m(x),$$

$$\rho(B)=\int_E\rho(dx)P(x,B),\qquad \rho(E)=1.$$

C2: *Lévy approximation.* The family of processes with locally independent increments $\eta^\varepsilon(t;x),\ t\ge 0,\ x\in E$ satisfies the Lévy approximation conditions [6, Section 9.2].

L1: Approximation of the mean values:

$$b_\varepsilon(u;x) = \int_{\mathbb{R}^d} v\Gamma^\varepsilon(u,dv;x) = \varepsilon b_1(u;x) + \varepsilon^2[b(u;x) + \theta_b^\varepsilon(u;x)],$$

and

$$c_\varepsilon(u;x) = \int_{\mathbb{R}^d} vv^*\Gamma^\varepsilon(u,dv;x) = \varepsilon^2[c(u;x) + \theta_c^\varepsilon(u;x)].$$

L2: Poisson approximation condition for intensity kernel

$$\Gamma_g^\varepsilon(u;x) = \int_{\mathbb{R}^d} g(v)\Gamma^\varepsilon(u,dv;x) = \varepsilon^2[\Gamma_g(u;x) + \theta_g^\varepsilon(u;x)]$$

for all $g \in C_3(\mathbb{R}^d)$, and the kernel $\Gamma_g(u;x)$ is bounded for all $g \in C_3(\mathbb{R}^d)$, that is,

$$|\Gamma_g(u;x)| \le \Gamma_g \quad \text{(a constant depending on } g\text{)},$$

where the kernel $\Gamma(u,dv;x)$ is defined on the class $C_3(\mathbb{R}^d)$ by the relation

$$\Gamma_g(u;x) = \int_{\mathbb{R}^d} g(v)\Gamma(u,dv;x), \quad g \in C_3(\mathbb{R}^d).$$

The above negligible terms $\theta_g^\varepsilon, \theta_b^\varepsilon, \theta_c^\varepsilon$ satisfy the condition

$$\sup_{x\in E} |\theta_\cdot^\varepsilon(u;x)| \to 0, \quad \varepsilon \to 0.$$

L3: *Balance condition.*

$$\int_E \rho(dx) b_1(u;x) = 0.$$

L4: Initial value condition

$$\sup_{\varepsilon>0} E|\xi_0^\varepsilon| \le C < \infty$$

and

$$\xi_0^\varepsilon \Rightarrow \xi_0.$$

In addition the following conditions are used:

C3: *Uniform square-integrability*:

$$\lim_{c\to\infty} \sup_{x\in E} \int\limits_{|v|>c} vv^*\Gamma(u,dv;x) = 0.$$

C4: *Linear growth*: there exists a positive constant L such that

$$|b(u;x)| \le L(1+|u|), \quad \text{and} \quad |c(u;x)| \le L(1+|u|^2),$$

and for any real-valued non-negative function $f(v), v \in \mathbb{R}^d$, such that $\int_{\mathbb{R}^d\setminus\{0\}}(1+f(v))|v|^2\,dv < \infty$, we have

$$|\Lambda(u,v;x)| \le Lf(v)(1+|u|),$$

where $\Lambda(u,v;x)$ is the Radon–Nikodym derivative of $\Gamma(u,B;x)$ with respect to Lebesgue measure dv in $\mathbb{R}^d$, that is,

$$\Gamma(u,dv;x) = \Lambda(u,v;x)\,dv.$$

The main result of our work is the following.

Theorem 2.1. *Under conditions* **C1–C4** *the weak convergence*

$$\xi^\varepsilon(t) \Rightarrow \xi^0(t), \quad \varepsilon \to 0$$

takes place.

The limit process $\xi^0(t),\ t \ge 0$ is a Lévy process defined by the generator **L** *as follows*

$$\mathbf{L}\varphi(u) = (\widehat{b}(u) - \widehat{b}_0(u))\varphi'(u) + \frac{1}{2}\sigma^2(u)\varphi''(u) + \lambda(u)\int\limits_{\mathbb{R}^d}[\varphi(u+v)-\varphi(u)]\Gamma^0(u,dv), \quad (2.3)$$

where:

$$\widehat{b}(u) = q\int\limits_E \rho(dx)b(u;x), \qquad \widehat{b}_0(u) = \int\limits_E v\Gamma(u,dv),$$

$$\Gamma(u,dv) = q\int\limits_E \rho(dx)\Gamma(u,dv;x),$$

$$\widehat{b_1^2}(u) = q\int\limits_E \rho(dx)b_1^2(u;x), \qquad \widetilde{b}_1(u;x) := q(x)\int\limits_E P(x,dy)b_1(u;x),$$

$$c_0(u;x) = \int_E vv^* \Gamma(u, dv; x),$$

$$\sigma^2(u) = 2 \int_E \pi(dx) \{\widetilde{b}_1(u;x) \widetilde{R}_0 \widetilde{b}_1^*(u;x) + \frac{1}{2}[c(u;x) - c_0(u;x)]\} - \widehat{b}_1^2(u),$$

$$\sigma^2(u) \geq 0$$

$$\lambda(u) = \Gamma(u, \mathbb{R}^d), \qquad \Gamma^0(u, dv) = \Gamma(u, dv)/\lambda(u),$$

here $\widetilde{R}_0$ *is the potential operator of embedded Markov chain* (*see* (3.9)).

Remark 2.1. The limit Lévy process consists of three parts: deterministic drift, diffusion part and Poisson part.

There are some possible cases:

1. If the condition $\varepsilon^{-2} \int_{|v|>\delta} y^2 \Gamma^\varepsilon(x, dy) \to 0$, $\varepsilon \to 0$, $\forall \delta > 0$ (see Theorem 4.21 in [5, p. 558]), then the limit process does not have Poisson part.

2. If $\widehat{b}(u) - \widehat{b}_0(u) = 0$ then the limit process does not have deterministic drift.

3. If $\sigma^2(u) = 0$ then the limit process does not have diffusion part. As a variant of this case we note that if $c(u;x) = c_0(u;x)$ then also $b_1(u;x) = 0$ and we obtain the conditions of Poisson approximation after re-normation $\varepsilon^2 = \widetilde{\varepsilon}$ (see, for example Chapter 7 in [6]).

Remark 2.2. In the work [6, Theorem 9.3] an analogical result was obtained for impulsive process with Markov switching. If we study an ordinary impulsive process without switching, we should obtain $\sigma^2 = E(\alpha_k^\varepsilon)^2 - (E(\alpha_k^\varepsilon))^2 = (c - c_0) - b_1^2$. This result correlates with the similar results from [5]. In case of our Theorem this may be easily shown, but in [6, Theorem 9.3] it is not obvious.

The difference is that we used $\widetilde{R}_0$ — the potential operator of embedded Markov chain instead of R_0 — the potential operator of Markov process. Due to this, our result obviously correlates with other well-known result.

Remark 2.3. Asymptotic of the second moment in the condition **L1** contains second modified characteristics $c(u;x)$ (see correlation 4.2 in [5, p. 555]). This characteristics in limit contains both second moment of Poisson part and dispersion of diffusion part, namely $c = c_0 + \sigma^2$.

3. Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the semimartingale representation of the additive functional process (2.1). We split the proof in the following two steps.
Step 1. In this step we establish the relative compactness of the family of processes $\xi^\varepsilon(t),\ t \geq 0,\ \varepsilon > 0$ by using the approach developed in [8]. Let us remind that the space of all probability measures defined on the standard space $(E, \mathcal{E})$ is also a Polish space; so the relative compactness and tightness are equivalent.

First we need the following lemma.

Lemma 3.1. *Under assumption* **C4** *there exists a constant $k > 0$, independent of ε and dependent on T, such that*

$$\mathbf{E} \sup_{t \leq T} |\xi^\varepsilon(t)|^2 \leq k_T.$$

Corollary 3.1. *Under assumption* ***C4****, the following compact containment condition (CCC) holds:*

$$\lim_{c\to\infty} \sup_{\varepsilon \leq \varepsilon_0} \mathbf{P}\{\sup_{t \leq T} |\xi^\varepsilon(t)| > c\} = 0.$$

Proof. The proof of this corollary follows from Kolmogorov's inequality by using the estimation of Lemma 3.1. □

Proof of Lemma 3.1 (*following* [8]). The semimartingale (2.1) has the following representation

$$\xi^\varepsilon(t) = u + A_t^\varepsilon + M_t^\varepsilon, \tag{3.1}$$

where $u = \xi^\varepsilon(0)$; A_t^ε is the predictable drift (see [4]):

$$\begin{aligned} A_t^\varepsilon &= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} v\Gamma(\xi^\varepsilon(s), dv; x_s^\varepsilon)\, ds + \theta_b^\varepsilon(t) \\ &= \varepsilon \int_0^t b_1(\xi^\varepsilon(s), x_s^\varepsilon)\, ds + \varepsilon^2 \int_0^t b(\xi^\varepsilon(s), x_s^\varepsilon)\, ds + \theta_b^\varepsilon(t) \\ &=: B_1^\varepsilon(t) + B^\varepsilon(t) + \theta_b^\varepsilon(t), \end{aligned}$$

and M_t^ε is the locally square integrable martingale

$$M_t^\varepsilon = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} v[\mu(ds, dv; x_s^\varepsilon) - \Gamma(\xi^\varepsilon(s), dv; x_s^\varepsilon)\, ds] + \theta_c^\varepsilon(t),$$

and for every finite $T > 0$

$$\sup_{0\le t\le T} |\theta^\varepsilon_\cdot(t)| \to 0, \quad \varepsilon \to 0.$$

To verify compactness of the process $\xi^\varepsilon(t)$ we split it at two parts.

The first part of order ε

$$B_1^\varepsilon(t) = \varepsilon \int_0^t b_1(\xi^\varepsilon(s), x_s^\varepsilon)\, ds,$$

can be characterized by the compensating operator

$$\mathbf{L}^\varepsilon \varphi(u;x) = \varepsilon^{-2} q(x)[\mathbf{B}_1^\varepsilon(x)P - I]\varphi(u;x),$$

where $\mathbf{B}_1^\varepsilon(x)\varphi(u) = \varphi(u + \varepsilon b_1(u;x)) = \varepsilon b_1(u;x)\varphi'(u) + \varepsilon\theta^\varepsilon(x)\varphi(u)$.

After simple calculations we may rewrite the operator:

$$\mathbf{L}^\varepsilon = \varepsilon^{-2}\mathbf{Q} + \varepsilon^{-1}\mathbf{B}_1(x)P + \theta^\varepsilon,$$

here $\mathbf{B}_1(x)\varphi(u) = \varepsilon b_1(u;x)\varphi'(u)$.

Corresponding martingale characterization is the following

$$\widetilde{\mu}_t^\varepsilon = \varphi^\varepsilon(B_1^\varepsilon(t), x_t^\varepsilon) + \varphi^\varepsilon(B_1^\varepsilon(0), x_0^\varepsilon) - \int_0^t \mathbf{L}^\varepsilon \varphi^\varepsilon(B_1^\varepsilon(s), x_s^\varepsilon)\, ds,$$

where $x_t^\varepsilon := x(t/\varepsilon^2)$.

Thus (see, for example Theorem 1.2 in [6]), it has quadratic characteristic

$$\langle \widetilde{\mu}^\varepsilon \rangle_t = \int_0^t \left[\mathbf{L}^\varepsilon(\varphi^\varepsilon(B_1^\varepsilon(s), x_t^\varepsilon))^2 - 2\varphi^\varepsilon(B_1^\varepsilon(s), x_s^\varepsilon)\mathbf{L}^\varepsilon\varphi^\varepsilon(B_1^\varepsilon(s), x_s^\varepsilon)\right] ds.$$

Applying the operator $\mathbf{L}^\varepsilon = \varepsilon^{-2}\mathbf{Q} + \varepsilon^{-1}\mathbf{B}_1(x)P + \theta^\varepsilon$ to test-function $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1$ we obtain the integrand of the view

$$Q\varphi_1^2 - 2\varphi_1 Q\varphi_1 + \theta^\varepsilon\varphi^\varepsilon.$$

Thus the integrand is limited. The boundedness of the quadratic characteristic provides $\widetilde{\mu}_t^\varepsilon$ is compact. Thus, $\varphi(B_1^\varepsilon(t))$ is compact too and bounded uniformly by ε. By the results from [3] we obtain compactness of $B_1^\varepsilon(t)$, because the test-function $\varphi(u)$ belongs to the measure-determining class.

Now we should study the second part of order ε^2.

For a process $y(t),\ t\ge 0$, let us define the process $y_t^\dagger=\sup_{s\le t}|y(s)|$, then from (3.1) we have

$$((\xi_t^\varepsilon)^\dagger)^2\le 3[u^2+((A_t^\varepsilon)^\dagger)^2+((M_t^\varepsilon)^\dagger)^2]. \tag{3.2}$$

Condition **C4** implies that

$$(B_t^\varepsilon)^\dagger=\varepsilon^2\int_0^t b(\xi^\varepsilon(s),x(s/\varepsilon^2))\,ds$$
$$=\varepsilon^2\int_0^{t/\varepsilon^2} b(\xi^\varepsilon(s),x(s))\,ds\le L\varepsilon^2\int_0^{t/\varepsilon^2}(1+(\xi_s^\varepsilon)^\dagger)\,ds. \tag{3.3}$$

Now, by Doob's inequality (see, e.g., [9, Theorem 1.9.2]),

$$\mathbf{E}((M_t^\varepsilon)^\dagger)^2\le 4|\mathbf{E}\langle M^\varepsilon\rangle_t|,$$

and condition **C4** we obtain

$$|\langle M^\varepsilon\rangle_t|=\left|\int_0^t\int_{\mathbb{R}^d\setminus\{0\}} vv^*\Gamma(\xi^\varepsilon(s),dv;x_s^\varepsilon)\,ds\right|$$
$$=\left|\varepsilon^2\int_0^t c(\xi^\varepsilon(s);x_s^\varepsilon)\,ds\right|\le L\varepsilon^2\int_0^{t/\varepsilon^2}[1+((\xi_s^\varepsilon)^\dagger)^2]\,ds. \tag{3.4}$$

Inequalities (3.2)–(3.4) and Cauchy–Bunyakovsky–Schwarz inequality, $([\int_0^t\varphi(s)\,ds]^2\le t\int_0^t\varphi^2(s)\,ds)$, imply

$$\mathbf{E}((\xi_t^\varepsilon)^\dagger)^2\le k_1+k_2\varepsilon^2\int_0^{t/\varepsilon^2}\mathbf{E}((\xi_s^\varepsilon)^\dagger)^2\,ds=k_1+k_2\int_0^t\mathbf{E}((\xi_s^\varepsilon)^\dagger)^2\,ds,$$

where k_1 and k_2 are positive constants independent of ε.

By Gronwall inequality (see, e.g., [3, p. 498]), we obtain

$$\mathbf{E}((\xi_t^\varepsilon)^\dagger)^2\le k_1\exp(k_2 t).$$

Hence the lemma is proved. □

Lemma 3.2. *Under assumption* **C4** *there exists a constant $k>0$, independent of ε such that*

$$\mathbf{E}|\xi^\varepsilon(t)-\xi^\varepsilon(s)|^2\le k|t-s|.$$

Proof. In the same manner with (3.2), we may write

$$|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \le 2|A_t^\varepsilon - A_s^\varepsilon|^2 + 2|M_t^\varepsilon - M_s^\varepsilon|^2.$$

By using Doob's inequality, we obtain

$$\mathbf{E}|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \le 2\mathbf{E}\{|A_t^\varepsilon - A_s^\varepsilon|^2 + 8|\langle M^\varepsilon\rangle_t - \langle M^\varepsilon\rangle_s|\}.$$

Now (3.3), (3.4), and assumption **C5** imply

$$|A_t^\varepsilon - A_s^\varepsilon|^2 + 8|\langle M^\varepsilon\rangle_t - \langle M^\varepsilon\rangle_s| \le k_3[1 + ((\xi_T^\varepsilon)^\dagger)^2]|t-s|,$$

where k_3 is a positive constant independent of ε:

From the last inequality and Lemma 3.1 the desired conclusion is obtained. □

The conditions proved in Corollary 3.1 and Lemma 3.2 are necessary and sufficient for the compactness of the family of processes $\xi^\varepsilon(t)$, $t \ge 0,\ \varepsilon > 0$.

Step 2. At the next step of proof we apply the problem of singular perturbation to the generator of the process $\xi^\varepsilon(t)$. To do this, we mention the following theorem.

$C_0^2(\mathbb{R}^d \times E)$ is the space of real-valued twice continuously differentiable functions on the first argument, defined on $\mathbb{R}^d \times E$ and vanishing at infinity, and $C(\mathbb{R}^d \times E)$ is the space of real-valued continuous bounded functions defined on $\mathbb{R}^d \times E$.

Theorem 3.1 ([6, Theorem 6.3]). *Let the following conditions hold for a family of coupled Markov processes* $\xi^\varepsilon(t),\ x^\varepsilon(t),\ t \ge 0,\ \varepsilon > 0$*:*

CD1: *There exists a family of test functions* $\varphi^\varepsilon(u,x)$ *in* $C_0^2(\mathbb{R}^d \times E)$*, such that*

$$\lim_{\varepsilon\to 0} \varphi^\varepsilon(u,x) = \varphi(u),$$

uniformly on u, x.

CD2: *The following convergence holds*

$$\lim_{\varepsilon\to 0} \mathbf{L}^\varepsilon \varphi^\varepsilon(u,x) = \mathbf{L}\varphi(u),$$

uniformly on u, x*. The family of functions* $\mathbf{L}^\varepsilon\varphi^\varepsilon$*,* $\varepsilon > 0$ *is uniformly bounded, and* $\mathbf{L}\varphi(u)$ *and* $\mathbf{L}^\varepsilon\varphi^\varepsilon$ *belong to* $C(\mathbb{R}^d \times E)$.

CD3: *The quadratic characteristics of the martingales that characterize a coupled Markov process $\xi^\varepsilon(t)$, $x^\varepsilon(t)$, $t \geq 0$, $\varepsilon > 0$ have the representation $\langle \mu^\varepsilon \rangle_t = \int_0^t \zeta^\varepsilon(s)\, ds$, where the random functions ζ^ε, $\varepsilon > 0$, satisfy the condition*

$$\sup_{0 \leq s \leq T} \mathbf{E}|\zeta^\varepsilon(s)| \leq c < +\infty.$$

CD4: *The convergence of the initial values holds and*

$$\sup_{\varepsilon > 0} \mathbf{E}|\zeta^\varepsilon(0)| \leq C < +\infty.$$

Then the weak convergence

$$\xi^\varepsilon(t) \Rightarrow \xi(t), \quad \varepsilon \to 0,$$

takes place.

We consider the the extended Markov renewal process

$$\xi_n^\varepsilon,\ x_n^\varepsilon,\ \tau_n^\varepsilon, \quad n \geq 0, \tag{3.5}$$

where $x_n^\varepsilon = x^\varepsilon(\tau_n^\varepsilon)$, $x^\varepsilon(t) := x(t/\varepsilon^2)$, $\xi_n^\varepsilon = \xi^\varepsilon(\tau_n^\varepsilon)$ and $\tau_{n+1}^\varepsilon = \tau_n^\varepsilon + \varepsilon^2 \theta_n^\varepsilon$, $n \geq 0$, and

$$P(\theta_{n+1}^\varepsilon \leq t | x_n^\varepsilon = x) = F_x(t) = P(\theta_x \leq t).$$

Definition 3.1 ([11]). *The compensating operator $\mathbf{L}^\varepsilon$ of the Markov renewal process* (3.5) *is defined by the following relation*

$$\mathbf{L}^\varepsilon \varphi(\xi_0^\varepsilon, x_0, \tau_0) = q(x_0)\mathbf{E}[\varphi(\xi_1^\varepsilon, x_1, \tau_1) - \varphi(\xi_0^\varepsilon, x_0, \tau_0)|\mathcal{F}_0],$$

where

$$\mathcal{F}_t := \sigma(\xi^\varepsilon(s),\ x^\varepsilon(s),\ \tau^\varepsilon(s);\ 0 \leq s \leq t).$$

Using Lemma 9.1 from [6] we obtain that the compensating operator of the extended Markov renewal process from Definition 3.1 can be defined by the relation (see also Section 2.8 in [6])

$$\mathbf{L}^\varepsilon \varphi(u, v; x) = \varepsilon^{-2} q(x) \left[\int_E P(x, dy) \int_{\mathbb{R}^d} \Gamma^\varepsilon(u, dz; x) \varphi(u + z, v; y) - \varphi(u, v; x) \right]. \tag{3.6}$$

By analogy with [6, Lemma 9.2] we may prove the following result:

Lemma 3.3. *The main part in the asymptotic representation of the compensating operator* (3.6) *is as follows*

$$\begin{aligned}\mathbf{L}^{\varepsilon}\varphi(u,v,x) &= \varepsilon^{-2}\mathbf{Q}\varphi(\cdot,\cdot,x)\\ &+\varepsilon^{-1}b_1(u;x)\mathbf{Q}_0\varphi'_u(u,\cdot,\cdot)+[b(u;x)-b_0(u;x)]\mathbf{Q}_0\varphi'_u(u,\cdot,\cdot)\\ &+\frac{1}{2}[c(u;x)-c_0(u;x)]\mathbf{Q}_0\varphi''_{uu}(u,\cdot,\cdot)+\mathbf{\Gamma}_{u,x}\mathbf{Q}_0\varphi(u,\cdot,\cdot)\end{aligned} \tag{3.7}$$

where

$$\mathbf{Q}_0\varphi(x) := q(x)\int_E P(x,dy)\varphi(y), \mathbf{\Gamma}_{u,x}\varphi(u) := \int_{\mathbb{R}^d}[\varphi(u+z)-\varphi(u)]\Gamma(u,dz;x),$$

$$b_0(u;x)=\int_E v\Gamma(u,dv;x), \qquad c_0(u;x)=\int_E vv^*\Gamma(u,dv;x).$$

Proof. We may rewrite (3.6) in the view:

$$\begin{aligned}&\mathbf{L}^{\varepsilon}\varphi(u,v;x)=\varepsilon^{-2}q(x)\varphi(\cdot,\cdot;x)\\ &+\varepsilon^{-2}q(x)\int_E P(x,dy)\int_{\mathbb{R}^d}[\varphi(u+z,v;y)-\varphi(u,v;x)]\Gamma^{\varepsilon}(u,dz;x).\end{aligned} \tag{3.8}$$

We have for the operator

$$\begin{aligned}&\int_{\mathbb{R}^d}[\varphi(u+z,v;y)-\varphi(u,v;x)]\Gamma^{\varepsilon}(u,dz;x)\\ &=\int_{\mathbb{R}^d}\Big\{\Big[\varphi(u+z)-\varphi(u)-z\varphi'(u)-zz^*\frac{\varphi''(u)}{2}\Big]\Gamma^{\varepsilon}(u,dz;x)\\ &\qquad+\Big[z\varphi'(u)+zz^*\frac{\varphi''(u)}{2}\Big]\Gamma^{\varepsilon}(u,dz;x)\Big\}\\ &\qquad= (\text{due to conditions } \mathbf{L1}, \mathbf{L2} \text{ and as soon as}\\ &\qquad\varphi(u+z)-\varphi(u)-z\varphi'(u)-zz^*\tfrac{\varphi''(u)}{2}\in C_3(\mathbb{R}^d))\\ &=\varepsilon^2\Bigg(\int_{\mathbb{R}^d}\Big[\varphi(u+z)-\varphi(u)-z\varphi'(u)-zz^*\frac{\varphi''(u)}{2}\Big]\Gamma(u,dz;x)\\ &\qquad+[\varepsilon^{-1}b_1(u;x)+b(u;x)]\varphi'(u)+\frac{1}{2}c(u;x)\varphi''(u)\Bigg)\end{aligned}$$

$$= \varepsilon^2 \Bigg(\int_{\mathbb{R}^d} [\varphi(u+z) - \varphi(u)]\Gamma(u, dz; x) + \varepsilon^{-1} b_1(u;x)\varphi'(u)$$
$$+ [b(u;x) - b_0(u;x)]\varphi'(u) + \frac{1}{2}[c(u;x) - c_0(u;x)]\varphi''(u) \Bigg).$$

Putting this representation into (3.8), we obtain (3.7). □

The solution of the singular perturbation problem at the test functions $\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x)$ in the form

$$\mathbf{L}^\varepsilon \varphi^\varepsilon = \mathbf{L}\varphi + \theta^\varepsilon \varphi$$

can be found in the same manner with Lemma 9.3 in [6].

To simplify the formula, we refer to the embedded Markov chain. Corresponding generator $\widetilde{\mathbf{Q}} := P - I$, and the potential operator satisfies the correlation

$$\widetilde{R}_0(P - I) = \widetilde{\Pi} - I. \tag{3.9}$$

From (3.7) we obtain

$$\widetilde{\mathbf{Q}}\varphi = 0,$$
$$\widetilde{\mathbf{Q}}\varphi_1 + \mathbf{B}_1(x)P\varphi = 0,$$
$$\widetilde{\mathbf{Q}}\varphi_2 + \mathbf{B}_1(x)P\varphi_1 + (\mathbf{B}(x) + \mathbf{C}(x) + \mathbf{G}_{u,x})P\varphi = m(x)\mathbf{L}\varphi,$$

where

$$\mathbf{B}(x)\varphi(u) := [b(u;x) - b_0(u;x)]\varphi'(u),$$
$$\mathbf{B}_1(x)\varphi(u) := b_1(u;x)\varphi'(u),$$
$$\mathbf{C}(x) := \frac{1}{2}[c(u;x) - c_0(u;x)]\varphi''_{uu}(u).$$

From the second equation we obtain $\varphi_1 = \widetilde{R}_0\mathbf{B}_1(x)\varphi$, and substituting it into the last equation we have:

$$\widetilde{\mathbf{Q}}\varphi_2 + \mathbf{B}_1(x)P\widetilde{R}_0\mathbf{B}_1(x)\varphi + (\mathbf{B}(x) + \mathbf{C}(x) + \mathbf{G}_{u,x})\varphi = m(x)\mathbf{L}\varphi.$$

As soon as $P\widetilde{R}_0 = \widetilde{R}_0 + \widetilde{\Pi} - I$ we finally obtain

$$q^{-1}\mathbf{L} = \widetilde{\Pi}[(\mathbf{B}(x) + \mathbf{C}(x) + \mathbf{G}_{u,x}) + \mathbf{B}_1(x)\widetilde{R}_0\mathbf{B}_1(x) - \mathbf{B}_1^2(x)]\widetilde{\Pi}. \tag{3.10}$$

Simple calculations give us (2.3) from (3.10).

Now Theorem 3.1 can be applied.

We see from (3.6) and (3.10) that the solution of singular perturbation problem for $\mathbf{L}^\varepsilon\varphi^\varepsilon(u,v;x)$ satisfies the conditions **CD1, CD2**. Condition **CD3** of this theorem implies that the quadratic characteristics of the martingale, corresponding to a coupled Markov process, is relatively compact. The same result follows from the CCC (see Corollary 3.1 and Lemma 3.2) by [5]. Thus, the condition **CD3** follows from the Corollary 3.1 and Lemma 3.2. Due to **L4** we see that the condition **CD4** is also satisfied. Thus, all the conditions of above Theorem 3.1 are satisfied, so the weak convergence $\xi^\varepsilon(t) \Rightarrow \xi^0(t)$ takes place.

Theorem 2.1 is proved.

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