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# Some results on fuzzy subsets in gamma-nearrings

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(*Presented by V. V. Sharko*)

**Abstract.** In this paper, we introduce the notions of bi-ideal, quasi-ideal and residual quotient sets in terms of fuzzy subsets and have studied their related properties. Also, we have characterized residual quotient fuzzy subsets in gamma-nearrings.

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## Introduction

Gamma-nearrings were defined by Satyanarayana [10] and Booth [2] studied the ideal theory in gamma-nearring. The notion of fuzzy subset in set was introduced by Zadeh [14], and this concept has been applied to various algebraic structures. Rosenfeld [9] defined the fuzzy subgroup and gave some of its properties. Since then, the study of fuzzy algebraic structure has been pursued in many directions such as groups, rings, modules, vector space and so on. Das [3] introduced the notion of fuzzy level set and explained the interrelationship between the fuzzy subgroup and their level subsets. Subsequently, many authors [1, 5, 7, 12] have studied several basic concepts pertaining to fuzzy ideals in rings. Jun et al. [6] introduced the notion of fuzzy ideals in gamma-nearrings and studied some of their results. Further, Satyanarayana [11] studied the fuzzy cosets in gamma-nearrings.

In [8] Liu, introduced residual quotient fuzzy subset $(\lambda : \mu)$ for any two fuzzy ideals in rings. Then, Dheena et al. [4] introduced the notion of residual quotient fuzzy subset $(\lambda : \mu)$ for any two fuzzy subset in near-

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ring, which is different from [8]. In this paper we introduce the notions of bi-ideals, quasi-ideals in terms of fuzzy sets and residual quotient fuzzy subset $(\lambda : \mu)$ for any two fuzzy subsets in gamma-nearring.

For convenience of readers, in Section 2, we list the basic definitions of gamma-nearrings, fuzzy set theory, ideal theory. In Section 3, we introduce the notions bi-ideal and quasi-ideal in terms of fuzzy subsets and studied their related properties. In Section 4, we introduce the notion of residual quotient fuzzy subsets and have studied some of their results.

## 1. Preliminary

In this section, we cite the fundamental definitions that will be used in the sequel:

**Definition 1.1.** *A near-ring $N$ is a system with two binary operations $+$ and $\cdot$ such that*

**N1.** $(N,+)$ *is a group, not necessarily abelian,*

**N2.** $(N,\cdot)$ *is a semigroup,*

**N3.** $(x+y)\cdot z = x\cdot z + y\cdot z$ *for all* $x,y,z \in N$.

**Definition 1.2.** *A gamma-nearring $M$ is a triple $(M,+,\Gamma)$ where*

**G1.** $(M,+)$ *is a group, not necessarily abelian,*

**G2.** $\Gamma$ *is a non-empty set of binary operators on $M$ such that for each* $\alpha \in \Gamma, (M,+,\alpha)$ *is a nearring,*

**G3.** $x\alpha(y\beta z) = (x\alpha y)\beta z$ , $x,y,z \in M$ *and* $\alpha,\beta \in \Gamma$.

**Definition 1.3.** *Let $(M,+,\Gamma)$ be a gamma-nearring. A subset $I$ of $M$ is said to be an ideal if*

**I1.** $(I,+)$ *is a normal divisor of* $(M,+)$,

**I2.** $x\alpha u \in I$, *for all* $x \in I, \alpha \in \Gamma, u \in M$,

**I3.** $u\alpha(v+x) - u\alpha v \in I$, *for all* $x \in I, \alpha \in \Gamma, u,v \in M$.

If $I$ satisfies **I1** and **I2**, then it is called a *right ideal* of $M$. If $I$ satisfies **I1** and **I3** , then it is called a *left ideal* of $M$. Let $M$ be a gamma-nearring. Given two subsets $A$ and $B$ of $M$, $A\Gamma B = \{a\gamma b | a \in M,\ b \in M,\ \gamma \in \Gamma\}$ as defined by [10] and $A\Gamma * B = \{a\gamma(a'+b) - a\gamma a' | a,\ a' \in A,\ b \in B, \gamma \in \Gamma\}$ as defined by [13].

From now on, throughout this paper $M$ will denote right distributive gamma-nearring, unless otherwise specified.

**Definition 1.4.** *A subgroup $Q$ of $(M,+)$ is said to be a quasi-ideal of $M$ if $(Q\Gamma M)\cap(M\Gamma Q)\cap(M\Gamma * Q)\subseteq Q$.*

**Definition 1.5.** *A subgroup $Q$ of $(M,+)$ is said to be a bi-ideal of $M$ if $(Q\Gamma M\Gamma Q)\cap(Q\Gamma M)\Gamma * Q\subseteq Q$.*

**Definition 1.6.** *A mapping $\mu : M\to[0,1]$ is called a fuzzy subset of $M$.*

A fuzzy subset $\mu$ is non-empty if $\mu$ is not the constant map which assumes the value 0, For any two fuzzy subsets $\lambda$ and $\mu$ of $M$, $\lambda\le\mu$ means that $\lambda(a)\le\mu(a)$ for all $a\in M$. The characteristic function of $M$ is denoted by $\mathbf{M}$ and, of its subset $A$ is denoted by $f_A$. The image of a fuzzy subset $\mu$ is denoted by $Im(\mu)=\{\mu(m)|m\in M\}$.

Hereafter, we consider only non-empty fuzzy subset of $M$.

**Definition 1.7.** *Let $\mu$ be a fuzzy subset of $M$. For any $t\in[0,1]$, the set*

$$\mu_t=\{x\in M|\mu(x)\ge t\}$$

*is called a level subset of $\mu$.*

**Definition 1.8.** *Let $f$ and $g$ be any fuzzy subset of $M$. Then*

$$f\cap g,\quad f\cup g,\quad f+g,\quad fg\ \ \text{and}\ \ f*g$$

*are fuzzy subsets of $M$ defined by*

$$(f\cap g)(x)=\min\{f(x),g(x)\},$$

$$(f\cup g)(x)=\max\{f(x),g(x)\},$$

$$(f+g)(x)=\begin{cases}\sup\limits_{x=y+z}\{\min\{f(y),g(z)\}\}, & \text{if } x \text{ is expressed as } x=y+z,\\ 0, & \text{otherwise.}\end{cases}$$

$$(f\Gamma g)(x)=\begin{cases}\sup\limits_{x=y\gamma z}\{\min\{f(y),g(z)\}\}, & \text{if } x \text{ is expressed as } x=y\gamma z,\\ 0, & \text{otherwise.}\end{cases}$$

$$(f * g)(x) = \begin{cases} \sup\limits_{x=a\gamma(b+c)-a\gamma b} \{\min\{f(a), g(c)\}\}, & \text{if } x = a\gamma(b+c) - a\gamma b, \\ 0, & \text{otherwise.} \end{cases}$$

*for all* $x, y, z, a, b, c \in M$ *and* $\gamma \in \Gamma$.

**Definition 1.9.** *Let* $\mu$ *be a non-empty fuzzy subset of* $M$*, if*

**FI1.** $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ ,

**FI2.** $\mu(y + x - y) \geq \mu(x)$ ,

**FI3.** $\mu(x\alpha y) \geq \mu(x)$,

**FI4.** $\mu(u\alpha(v + x) - u\alpha v) \geq \mu(x)$, *for all* $x, y, u, v \in M, \alpha \in \Gamma$.

If $\mu$ satisfies **FI1**, **FI2**, and **FI3**, then it is called a *fuzzy right ideal* of $M$. If $\mu$ satisfies **FI1**, **FI2**, and **FI4**, then it is called a *fuzzy left ideal* of $M$. If $\mu$ is both fuzzy right and left ideal of $M$, then $\mu$ is called a *fuzzy ideal* of $M$.

## 2. Bi-ideal and quasi-ideal in terms of fuzzy subsets

In this section, we introduce the notions of *fuzzy bi-ideal* and *fuzzy quasi-ideal* of $M$, and investigate related properties.

**Definition 2.1.** *A fuzzy subgroup* $\mu$ *of* $M$ *is said to be a fuzzy quasi-ideal of* $M$ *if* $(\mu\Gamma\mathbf{M}) \cap (\mathbf{M}\Gamma\mu) \cap (\mathbf{M}\Gamma * \mu) \leq \mu$.

**Example 2.1.** Let $M = \{0, a, b, c\}$ and $\Gamma$ be the non-empty set of binary operations such that $\alpha, \beta \in \Gamma$ are defined below:

| $+$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

| $\alpha$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $b$ | $0$ | $b$ |
| $b$ | $0$ | $0$ | $0$ | $0$ |
| $c$ | $0$ | $b$ | $0$ | $b$ |

| $\beta$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $0$ | $0$ |
| $b$ | $0$ | $0$ | $b$ | $0$ |
| $c$ | $0$ | $0$ | $0$ | $c$ |

Clearly $(M, +, \Gamma)$ is a $\Gamma$-nearring.

Define a fuzzy subset $\mu : M \to [0, 1]$ by $\mu(0) = 0.9$, $\mu(a) = 0.7$, $\mu(b) = 0.5$ and $\mu(c) = 0.3$. By usual calculations, it is clear that $\mu$ is a fuzzy subgroup of $M$ and hence $\mu$ is a fuzzy quasi-ideal of $M$.

**Definition 2.2.** *A fuzzy subgroup $\mu$ of $M$ is called a fuzzy bi-ideal of $M$ if* $(\mu\Gamma\mathbf{M}\Gamma\mu)\cap(\mu\Gamma\mathbf{M})\Gamma*\mu\leq\mu$.

Note that

$$\begin{aligned}((\mu\Gamma\mathbf{M})\Gamma*\mu)(w) &= \sup_{w=x\gamma(y+c)-x\gamma y}\{\min\{(\mu\gamma\mathbf{M})(x),\mu(c)\}\}\\ &= \sup_{w=x\gamma(y+c)-x\gamma y}\Big\{\min\{\sup_{x=x_1\gamma x_2}\mu(x_1),\mu(c)\}\Big\}\\ &= \sup_{w=(x_1\gamma x_2)\gamma(y+c)-(x_1\gamma x_2)\gamma y}\{\min\mu(x_1),\mu(c)\}\\ &= 0,\quad \text{otherwise.}\end{aligned}$$

It is very clear that if $M$ is a zero-symmetric gamma-nearring, then $\mu\Gamma\mathbf{M}\Gamma\mu\leq\mu$ for every fuzzy bi-ideal $\mu$.

**Example 2.2.** Let $M=\{0,a,b,c\}$ and $\Gamma$ be the non-empty set of binary operations such that $\alpha,\beta\in\Gamma$ are defined below:

| $+$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

| $\alpha$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $b$ | $0$ | $b$ |
| $b$ | $0$ | $0$ | $0$ | $0$ |
| $c$ | $0$ | $b$ | $0$ | $b$ |

| $\beta$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $0$ | $0$ |
| $b$ | $0$ | $0$ | $b$ | $0$ |
| $c$ | $0$ | $0$ | $0$ | $c$ |

Clearly $(M,+,\Gamma)$ is a $\Gamma$-near ring.

Define a fuzzy subset $\mu: M\to[0,1]$ by $\mu(0)=0.8$ and $\mu(x)=0.5$ for all $x\neq 0$. By usual calculations, it is clear that $\mu$ is a fuzzy subgroup of $M$.

Now

$$((\mu\Gamma\mathbf{M})\Gamma*\mu)(b) = \sup_{b=x\gamma(y+z)-x\gamma y}\{\min\{(\mu\gamma\mathbf{M})(x),\mu(z)\}\}$$

$$\begin{aligned}(\mu\Gamma\mathbf{M})(b) &= \sup_{b=x\gamma y}\{\min\{\mu(x),\mathbf{M}(y)\}\},\\ &= \sup\Big\{\min\{\mu(a),\mathbf{M}(a)\},\ \min\{\mu(a),\mathbf{M}(c)\},\\ &\quad \min\{\mu(c),\mathbf{M}(a)\},\ \min\{\mu(c),\mathbf{M}(c)\},\ \min\{\mu(b),\mathbf{M}(b)\}\Big\},\\ &\quad (\text{since } b=a\alpha a=a\alpha c=c\alpha a=c\alpha c=b\beta b)\\ &= \sup\Big\{\min\{0.5,1\},\ \min\{0.5,1\},\ \min\{0.5,1\},\ \min\{0.5,1\},\\ &\quad \min\{0.5,1\}\Big\},\\ &= \sup\{0.5,0.5,0.5,0.5,0.5\},\\ &= 0.5.\end{aligned}$$

Similarly

$$(\mu\Gamma\mathbf{M})(0) = (\mu\Gamma\mathbf{M})(a) = (\mu\Gamma\mathbf{M})(c) = 0.5$$

$$\begin{aligned}((\mu\Gamma\mathbf{M})\Gamma * \mu)(b) &= \sup_{b=x\gamma(y+z)-x\gamma y}\{\min\{(\mu\gamma\mathbf{M})(x), \mu(z)\}\} \\ &= 0.5.\end{aligned}$$

Thus

$$\begin{aligned}((\mu\Gamma\mathbf{M}\Gamma\mu) \cap (\mu\Gamma\mathbf{M})\Gamma * \mu)(b) &= \min\{(\mu\Gamma\mathbf{M}\Gamma\mu)(b), (\mu\Gamma\mathbf{M})\Gamma * \mu)(b)\} \\ &= \min\{0.5, 0.5\} \\ &= 0.5 \\ &\le \mu(b)\end{aligned}$$

Hence $\mu$ is a fuzzy bi-ideal of $M$.

**Lemma 2.1.** *Let $\mu$ be a fuzzy subset of $M$. If $\mu$ is a fuzzy left ideal (fuzzy right ideal) of $M$, then $\mu$ is a fuzzy quasi-ideal of $M$.*

*Proof.* Let $\mu$ be a fuzzy left ideal of $M$. Let $x \in M$ and

$$x = u\gamma v = a\gamma(b + c) - a\gamma b,$$

where $u, v, a, b, c \in M$ and $\gamma \in \Gamma$. Consider

$$\begin{aligned}(\mu\Gamma\mathbf{M} \cap \mathbf{M}\Gamma\mu \cap \mathbf{M}\Gamma * \mu)(x) &= \min\big\{(\mu\Gamma\mathbf{M})(x),\ (\mathbf{M}\Gamma\mu)(x), \\ &\quad (\mathbf{M}\Gamma * \mu)(x)\big\} \\ &= \min\Big\{\sup_{x=u\gamma v}\{\mu(u)\},\ \sup_{x=v\gamma u}\{\mu(v)\}, \\ &\quad \sup_{x=a\gamma(b+c)-a\gamma b}\{\mu(c)\}\Big\} \\ &\le \min\Big\{1, 1, \sup_{x=a\gamma(b+c)-a\gamma b}\{\mu(a\gamma(b+c) \\ &\quad -a\gamma b)\}\Big\} \\ &\quad \text{as } \mu \text{ is a fuzzy left ideal,} \\ &\quad \{\mu(a\gamma(b+c) - a\gamma b)\} \ge \mu(c) \\ &\le \mu(x)\end{aligned}$$

Suppose that if $x$ is not expressed as $x = u\gamma v = a\gamma(b+c) - a\gamma b$, then $(\mu\Gamma\mathbf{M} \cap \mathbf{M}\Gamma\mu \cap \mathbf{M}\Gamma * \mu)(x) = 0 \le \mu(x)$. Thus $\mu\Gamma\mathbf{M} \cap \mathbf{M}\Gamma\mu \cap \mathbf{M}\Gamma * \mu \le \mu$. Hence $\mu$ is a fuzzy quasi-ideal of $M$. □

However, the following example shows that the converse of the above Lemma 2.1 is not true.

**Example 2.3.** From Example 2.1, it is clear that $\mu$ is not a fuzzy ideal of $M$. Since $\mu(a\alpha a) = \mu(b) = 0.5 < 0.7 = \mu(a)$.

**Lemma 2.2.** *For any non-empty subset $A$ and $B$ of $M$,*

1. $f_A \Gamma f_B = f_{A\Gamma B}$,
2. $f_A \cap f_B = f_{A\cap B}$,
3. $f_A * f_B = f_{A*B}$.

The proof is simple and straight forward.

**Theorem 2.1.** *Let $Q$ be a subgroup of $M$.*

*($i$) $Q$ is a quasi-ideal of $M$ if and only if $f_Q$ is a fuzzy quasi-ideal of $M$,*

*($ii$) $Q$ is a bi-ideal of $M$ if and only if $f_Q$ is a fuzzy bi-ideal of $M$.*

*Proof.* ($i$) Assume that $Q$ is a quasi-ideal of $M$. Then $f_Q$ is fuzzy subgroup of $M$.

$$f_Q \Gamma f_M \cap f_M \Gamma f_Q \cap f_M \Gamma * f_Q = f_{Q\Gamma M \cap M\Gamma Q \cap M\Gamma * Q} \leq f_Q.$$

Hence $f_Q$ is a fuzzy quasi-ideal of $M$.

Conversely, let us assume that $f_Q$ is a fuzzy quasi-ideal of $M$. Let $x \in Q\Gamma M \cap M\Gamma Q \cap M\Gamma * Q$. Then we have

$$f_Q(x) \geq (f_Q \Gamma f_M \cap f_M \Gamma f_Q \cap f_M \Gamma * f_Q)(x) = f_{Q\Gamma M \cap M\Gamma Q \cap M\Gamma * Q}(x) = 1.$$

Thus $x \in Q$ and $Q\Gamma M \cap M\Gamma Q \cap M\Gamma * Q \subseteq Q$. Hence $Q$ is a quasi-ideal of $M$.

($ii$) Assume that $Q$ is a bi-ideal of $M$. Then $f_Q$ is fuzzy subgroup of $M$.

$$f_Q \Gamma f_M \Gamma f_Q \cap f_Q \Gamma f_M \Gamma * f_Q = f_{Q\Gamma M\Gamma Q \cap Q\Gamma M\Gamma * Q} \leq f_Q.$$

Hence $f_Q$ is a fuzzy bi-ideal of $M$.

Conversely, let us assume that $f_Q$ is a fuzzy bi-ideal of $M$. Let $x \in Q\Gamma M\Gamma Q \cap M\Gamma Q\Gamma * M$. Then we have have

$$f_Q(x) \geq (f_Q \Gamma f_M \Gamma f_Q \cap f_Q \Gamma f_M \Gamma * f_Q)(x) = f_{Q\Gamma M\Gamma Q \cap Q\Gamma M\Gamma * Q}(x) = 1.$$

Thus $x \in Q$ and $Q\Gamma M\Gamma Q \cap Q\Gamma M\Gamma * Q \subseteq Q$. Hence $Q$ is a bi-ideal of $M$. □

**Theorem 2.2.** *Any fuzzy quasi-ideal of $M$ is a fuzzy bi-ideal of $M$.*

*Proof.* Let $\mu$ be any fuzzy quasi-ideal of $M$. Then, we have

$$\mu\Gamma\mathbf{M}\Gamma\mu \subseteq \mu\Gamma(\mathbf{M}\Gamma\mathbf{M}) \subseteq \mu\Gamma\mathbf{M}$$

$$\mu\Gamma\mathbf{M}\Gamma\mu \subseteq (\mathbf{M}\Gamma\mathbf{M})\Gamma\mu \subseteq \mathbf{M}\Gamma\mu$$

$$\mu\Gamma\mathbf{M}\Gamma * \mu \subseteq (\mathbf{M}\Gamma\mathbf{M})\Gamma * \mu \subseteq \mathbf{M}\Gamma * \mu$$

$$\mu\Gamma\mathbf{M}\Gamma\mu \cap \mu\Gamma\mathbf{M}\Gamma * \mu \subseteq \mu\Gamma\mathbf{M} \cap \mathbf{M}\Gamma\mu \cap \mathbf{M}\Gamma * \mu \subseteq \mu$$

Hence $\mu$ is a fuzzy bi-ideal of $M$. □

However, the following example shows that the converse of the above Theorem 2.2 is not true.

**Example 2.4.** Let $M = \{0, a, b, c\}$ and $\Gamma$ be the non-empty set of binary operations such that $\alpha, \beta \in \Gamma$ is defined below:

| $+$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

| $\alpha$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $b$ | $0$ | $b$ |
| $b$ | $0$ | $0$ | $0$ | $0$ |
| $c$ | $0$ | $b$ | $0$ | $b$ |

| $\beta$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $0$ | $0$ |
| $b$ | $0$ | $0$ | $b$ | $0$ |
| $c$ | $0$ | $0$ | $0$ | $c$ |

Clearly $(M, +, \Gamma)$ is a $\Gamma$-nearring.

Define a fuzzy subset $\mu : M \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0, a \\ 0, & \text{otherwise} \end{cases}$$

For any $t \in [0, 1]$, $\mu_t = \{0, a\}$ or $\{0, a, b, c\}$. Since $\{0, a\}$ and $\{0, a, b, c\}$ are bi-ideal in $M$, $\mu_t$ is the bi-ideal in $M$ for all $t$. Hence $\mu$ is a fuzzy bi-ideal of $M$. Now

$$\begin{aligned}
(\mu\Gamma\mathbf{M})(b) &= \sup_{b=x\gamma y}\big\{\min\{\mu(x), \mathbf{M}(y)\}\big\}, \\
&= \sup\big\{\min\{\mu(a), \mathbf{M}(a)\},\ \min\{\mu(a), \mathbf{M}(c)\}, \\
&\quad \min\{\mu(c), \mathbf{M}(a)\},\ \min\{\mu(c), \mathbf{M}(c)\},\ \min\{\mu(b), \mathbf{M}(b)\}\big\}, \\
&\quad (\text{since } b = a\alpha a = a\alpha c = c\alpha a = c\alpha c = b\beta b) \\
&= \sup\big\{\min\{1, 1\},\ \min\{1, 1\},\ \min\{0, 1\},\ \min\{0, 1\}, \\
&\quad \min\{0, 1\}\big\}, \\
&= \sup\big\{1, 1, 0, 0, 0\big\}, \\
&= 1.
\end{aligned}$$

Similarly, we can get $(\mathbf{M}\Gamma\mu)(b) = 1$ and this implies

$$(\mu\Gamma\mathbf{M} \cap \mathbf{M}\Gamma\mu)(b) = \min\{(\mu\Gamma\mathbf{M})(b), (\mathbf{M}\Gamma\mu)(b)\} = \min\{1,1\} = 1.$$

But, we have $\mu(b) = 0$ and so $(\mu\Gamma\mathbf{M} \cap \mathbf{M}\Gamma\mu)(b) = 1 > \mu(b) = 0$. Hence $\mu$ is not a fuzzy quasi-ideal of $M$.

**Lemma 2.3.** *Let $\mu$ be a fuzzy subset of $M$. If $\mu$ is a fuzzy left ideal (right ideal) of $M$, then $\mu$ is a fuzzy bi-ideal of $M$.*

*Proof.* If $\mu$ is a fuzzy left ideal of $M$. Then by Lemma 3.5, we have $\mu$ is a fuzzy quasi-ideal of $M$. Hence, by Theorem 2.2, $\mu$ is a fuzzy bi-ideal of $M$. □

However, the following example shows that the converse of the above Lemma 2.3 is not true.

**Example 2.5.** Let $M = \{0, a, b, c\}$ and $\Gamma$ be the non-empty set of binary operations such that $\alpha, \beta \in \Gamma$ are defined below:

| $+$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

| $-$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $0$ | $a$ | $b$ | $c$ |
| $c$ | $a$ | $0$ | $c$ | $b$ |

| $\alpha$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $b$ | $0$ | $b$ |
| $b$ | $0$ | $0$ | $0$ | $0$ |
| $c$ | $0$ | $b$ | $0$ | $b$ |

| $\beta$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $0$ | $0$ |
| $b$ | $0$ | $0$ | $b$ | $0$ |
| $c$ | $0$ | $0$ | $0$ | $c$ |

Clearly $(M, +, \Gamma)$ is a $\Gamma$-near ring.

Define a fuzzy subset $\mu : M \to [0, 1]$ by $\mu(0) = 0.8$ and $\mu(x) = 0.5$ for all $x \neq 0$. From Example 2.2, it is clear that $\mu$ is a fuzzy bi-ideal of $M$. But $\mu$ is not a fuzzy ideal of $M$. Since $\mu(a\alpha(a + 0) - a\alpha a) = \mu(b) = 0.5 < 0.8 = \mu(0)$.

**Theorem 2.3.** *Let $\mu$ be a fuzzy subset of $M$. Then $\mu$ is a fuzzy quasi-ideal of $M$ if and only if $\mu_t$ is a quasi-ideal of $M$, for all $t \in Im(\mu)$*

*Proof.* Let $\mu$ be a fuzzy quasi-ideal of $M$. Let $t \in Im(\mu)$. Suppose $x, y \in M$ such that $x, y \in \mu_t$. Then, $\mu(x) \geq t$, $\mu(y) \geq t$ and $\min\{\mu(x), \mu(y)\} \geq t$. As $\mu$ is a fuzzy quasi-ideal, $\mu(x - y) \geq t$ and this implies $x - y \in \mu_t$. Let $x \in M$. Suppose $x \in \mu_t\Gamma M \cap M\Gamma\mu_t \cap M\Gamma * \mu_t$. Then there exist

$a, b, c \in \mu_t$, $m_1, m_2, m_3, m_4 \in M$ and $\gamma \in \Gamma$ such that $x = a\gamma m_1 = m_2\gamma b = m_3\gamma(m_4 + c) - m_3\gamma m_4$. Then

$$\begin{aligned}(\mu\Gamma\mathbf{M} \cap \mathbf{M}\Gamma\mu \cap \mathbf{M}\Gamma * \mu)(x) &= \min\big\{(\mu\Gamma\mathbf{M})(x),\ (\mathbf{M}\Gamma\mu)(x),\\ &\quad (\mathbf{M}\Gamma * \mu)(x)\big\}\\ &= \min\Big\{\sup_{x=a\gamma m_1} \mu(a), \sup_{x=m_2\gamma b} \mu(m_2),\\ &\quad \sup_{\substack{x=m_3\gamma(m_4+c)\\ -m_3\gamma m_4}} \mu(c)\Big\}\\ &\geq t.\end{aligned}$$

As $\mu$ is the fuzzy quasi-ideal of $M$, $\mu(x) \geq t$ implies $x \in \mu_t$. Hence $\mu_t$ is a quasi-ideal of $M$.

Conversely, let us assume that $\mu_t$ is a quasi-ideal of $M$, for all $t \in Im(\mu)$.

Let $x \in M$. Consider

$$\begin{aligned}(\mu\Gamma\mathbf{M} \cap \mathbf{M}\Gamma\mu \cap \mathbf{M}\Gamma * \mu)(x) &= \min\big\{(\mu\Gamma\mathbf{M})(x),\ (\mathbf{M}\Gamma\mu)(x),\\ &\quad (\mathbf{M}\Gamma * \mu)(x)\big\}\\ &= \min\Big\{\sup_{x=a\gamma b}\{\min\{\mu(a), \mathbf{M}(b)\}\},\\ &\quad \sup_{x=a\gamma b}\{\min\{\mathbf{M}(a), \mu(b)\}\},\\ &\quad \sup_{\substack{x=m_1\gamma(m_2+c)\\ -m_1\gamma m_2}}\{\min\{\mathbf{M}(m_1), \mu(c)\}\}\Big\}\\ &= \min\Big\{\sup_{x=a\gamma b} \mu(a), \sup_{x=a\gamma b} \mu(b),\\ &\quad \sup_{\substack{x=m_1\gamma(m_2+c)\\ -m_1\gamma m_2}} \mu(c)\Big\}\end{aligned}$$

Let

$$\sup_{x=a\gamma b}\{\mu(a)\} = t_1, \quad \sup_{x=a\gamma b}\{\mu(b)\} = t_2, \quad \sup_{x=m_1\gamma(m_2+c)-m_1\gamma m_2}\{\mu(c)\} = t_3,$$

for any $a, b, c, m_1, m_2 \in M$ and $\gamma \in \Gamma$.

Assume that $\min\{t_1, t_2, t_3\} = t_1$. Then $a, b, c \in \mu_{t_1}$. Since $\mu_{t_1}$ is a quasi-ideal of $M$, then

$$x = a\gamma b \in M\Gamma\mu_{t_1},$$

$$x = a\gamma b \in \mu_{t_1}\Gamma M,$$
$$x = m_1\gamma(m_2 + c) - m_1\gamma m_2 \in M\Gamma * \mu_{t_1}.$$

This implies

$$x \in \mu\Gamma M \cap M\Gamma\mu \cap M\Gamma * \mu \subseteq \mu_{t_1}.$$

Thus

$$\mu(x) \geq t_1 = \min\{t_1, t_2, t_3\}.$$

Hence

$$(\mu\Gamma\mathbf{M} \cap \mathbf{M}\Gamma\mu \cap \mathbf{M}\Gamma * \mu)(x) \leq t_1 \leq \mu(x).$$

Similarly, if we take $\min\{t_1, t_2, t_3\} = t_2$ or $t_3$, we can prove that

$$(\mu\Gamma\mathbf{M} \cap \mathbf{M}\Gamma\mu \cap \mathbf{M}\Gamma * \mu)(x) \leq t_2 \quad \text{or} \quad t_3 \leq \mu(x).$$

Thus

$$(\mu\Gamma\mathbf{M} \cap \mathbf{M}\Gamma\mu \cap \mathbf{M}\Gamma * \mu)(x) \leq \mu(x).$$

for all $x \in M$. Hence $\mu$ is a fuzzy quasi-ideal of $M$. □

**Theorem 2.4.** *Let $\mu$ be a fuzzy subset of $M$. Then $\mu$ is a fuzzy bi-ideal of $M$ if and only if $\mu_t$ is a bi-ideal of $M$, for all $t \in Im(\mu)$*

*Proof.* Let $\mu$ be a fuzzy bi-ideal of $M$. Let $t \in Im(\mu)$. Suppose $x, y \in M$ such that $x, y \in \mu_t$. Then, $\mu(x) \geq t$, $\mu(y) \geq t$ and $\min\{\mu(x), \mu(y)\} \geq t$. As $\mu$ is a fuzzy bi-ideal, $\mu(x - y) \geq t$ and this implies $x - y \in \mu_t$. Let $z \in M$. Suppose $z \in \mu_t\Gamma M\Gamma\mu_t \cap \mu_t\Gamma M\Gamma * \mu_t$. Then there exist $x, y, a, b, c \in \mu_t$, $m_1, m_2, m_3 \in M$ and $\gamma \in \Gamma$ such that $z = x\gamma m_1\gamma y = a\gamma m_2\gamma(b\gamma m_3 + c) - a\gamma m_2\gamma b\gamma m_3$. Then

$$(\mu\Gamma\mathbf{M}\Gamma\mu \cap \mu\Gamma\mathbf{M}\Gamma * \mu)(z) \quad = \quad \min\{(\mu\Gamma\mathbf{M}\Gamma\mu)(z), (\mu\Gamma\mathbf{M}\Gamma * \mu)(z)\}$$

Also, we have

$$(\mu\Gamma\mathbf{M}\Gamma\mu)(z) = \sup_{z = x\gamma y} \{\min\{\mu(x), \mu(y)\}\} \geq t$$
$$(\mu\Gamma\mathbf{M}\Gamma * \mu)(z) = \sup_{\substack{z = a\gamma m_2\gamma(b\gamma m_3 + c) \\ -a\gamma m_2\gamma b\gamma m_3}} \{\min\{\mu(a), \mu(c)\}\} \geq t.$$

Thus, we obtain

$$\min\{(\mu\Gamma\mathbf{M}\Gamma\mu)(z), (\mu\Gamma\mathbf{M}\Gamma * \mu)(z)\} \geq t$$

and

$$(\mu\Gamma\mathbf{M}\Gamma\mu \cap \mu\Gamma\mathbf{M}\Gamma * \mu)(z) \geq t.$$

As $\mu$ is bi-ideal of $M$, $\mu(z) \geq t$ implies $z \in \mu_t$. Hence $\mu_t$ is a bi-ideal of $M$. Conversely, let us assume that $\mu_t$ is a bi-ideal of $M$, $t \in Im(\mu)$.

Let $u \in M$. Consider

$$\begin{aligned}(\mu\Gamma\mathbf{M}\Gamma\mu \cap \mu\Gamma\mathbf{M}\Gamma * \mu)(u) &= \min\left\{(\mu\Gamma\mathbf{M}\Gamma\mu)(u),\ (\mu\Gamma\mathbf{M}\Gamma * \mu)(u)\right\} \\ &= \min\Bigg\{\sup_{u=x\gamma y}\{\min\{\mu(x),\mu(y)\}\}, \\ &\qquad \sup_{\substack{u=a\gamma m_1\gamma(b+c)\\ -a\gamma m_1\gamma b}}\{\min\{\mu(a),\mu(c)\}\}\Bigg\} \\ &= \sup_{\substack{u=x\gamma y=a\gamma m_1\gamma(b+c)\\ -a\gamma m_1\gamma c}}\{\min\{\mu(x),\mu(y),\mu(a), \\ &\qquad \mu(c)\}\}\end{aligned}$$

Let

$$\mu(x) = t_1 < \mu(y) = t_2 < \mu(a) = t_3 < \mu(c) = t_4.$$

Then

$$\mu_{t_1} \supseteq \mu_{t_2} \supseteq \mu_{t_3} \supseteq \mu_{t_4}.$$

If $x, y, a, c \in \mu_{t_1}$, $u = x\gamma y \in \mu_{t_1}\Gamma\mathbf{M}\Gamma\mu_{t_1}$, and $u = a\gamma n\gamma(b+c) - a\gamma n\gamma b \in \mu_{t_1}\Gamma\mathbf{M}\Gamma * \mu_{t_1}$. Thus $u \in \mu_{t_1}$ and $(\mu_{t_1}\Gamma\mathbf{M}\Gamma\mu_{t_1} \cap \mu_{t_1}\Gamma\mathbf{M}\Gamma * \mu_{t_1}) \subseteq \mu_{t_1}$.

This implies $\mu(u) \geq t_1$ and hence $(\mu\Gamma\mathbf{M}\Gamma\mu \cap \mu\Gamma\mathbf{M}\Gamma * \mu) \leq \mu$. Hence $\mu$ is a fuzzy bi-ideal of $M$. □

## 3. Residual quotient fuzzy subset

In this section, we introduce the notion of *residual quotient fuzzy subset* in $M$ and we have characterized residual quotient fuzzy subsets in $M$.

**Definition 3.1.** *Let $\lambda$ and $\mu$ be any two fuzzy subsets of $M$. The residual quotient fuzzy subset $(\lambda : \mu)$ of $M$ is defined as*

$$(\lambda : \mu)(x) = \begin{cases} \bigvee_{t \in Im(\lambda)} \{t : x \in (\lambda_t : \mu_\alpha),\ \textit{where}\ \alpha = \sup\{Im(\mu)\}\}, \\ 0, \quad \textit{otherwise.}\end{cases}$$

*where* $(\lambda_t : \mu_\alpha) = \{x \in M\ \textit{and}\ \gamma \in \Gamma |\ x\gamma\mu_\alpha \subseteq \alpha_t\}$.

It is clear that, for any $t_1, t_2 \in Im(\lambda)$ and $\alpha = \sup\{Im(\mu)\}$ with $t_1 < t_2$, we have$(\lambda_{t_2} : \mu_\alpha) \subseteq (\lambda_{t_1} : \mu_\alpha)$. For $x \in (\lambda_{t_2} : \mu_\alpha)$. Then, $x\gamma\mu_\alpha \subseteq \lambda_{t_2} \subseteq \lambda_{t_1}$. Thus, $x \in (\lambda_{t_1} : \mu_\alpha)$ and therefore $(\lambda_{t_2} : \mu_\alpha) \subseteq (\lambda_{t_1} : \mu_\alpha)$.

**Lemma 3.1.** *If $\mu$ is a fuzzy left ideal of $M$, then $\mu(m_0\gamma x) \geq \mu(x)$ for all $x \in M$ and $m_0 \in M_0$*

*Proof.* If $\mu$ is a fuzzy left ideal of $M$, then we have $\mu(a\gamma(b+c) - a\gamma b \geq \mu(c)$, for all $a, b, c \in M$. Taking $a = m_0 \in M_0$ and $b = 0$, we have $\mu(m_0\gamma c) \geq \mu(c)$. □

**Lemma 3.2.** *$M$ is a zero-symmetric gamma-nearring if and only if each fuzzy left ideal of $M$ is a fuzzy $M$-subgroup of $M$.*

*Proof.* Assume that $M = M_0$. Let $\mu$ be a fuzzy left ideal of $M$. As $\mu$ is a fuzzy left ideal of $M$, by Lemma 3.1, $\mu(m_0\gamma x) \geq \mu(x)$, for all $x \in M$ and $m_0 \in M_0 = M$. Hence $\mu$ is a fuzzy $M$-subgroup of $M$.

Conversely, let us assume that each fuzzy ideal of $M$ is a fuzzy $M$-subgroup of $M$. Let $L$ be a left ideal of $M$. Then, $f_L$ is a fuzzy $M$-subgroup of $M$. This implies $f_L(m\gamma x) \geq f_L(x)$, for all $m, x \in M$. In particular, $x \in L$ and $m \in M$, then $M\Gamma L \subseteq L$. Taking $L$ as $\{0\}$, we have $M\Gamma\{0\} \subseteq \{0\}$. This implies $M\gamma 0 = \{0\}$ and hence $M = M_0$. □

**Theorem 3.1.** *Let $\lambda$ and $\mu$ be any two fuzzy subsets of $M$. If $\lambda$ is a fuzzy left ideal of $M$ if and only if $(\lambda : \mu)$ is a fuzzy left ideal of $M$.*

*Proof.* Let $x, y \in M$ and $\alpha = \sup\{Im(\mu)\}$. Suppose $(\lambda : \mu)(x) = t_1$ and $(\lambda : \mu)(y) = t_2$ where $t_1, t_2 \neq 0 \in Im(\lambda)$. Assume that $t_1 < t_2$. Then, $(\lambda_{t_2} : \mu_\alpha) \subseteq (\lambda_{t_1} : \mu_\alpha)$. For all $x, y \in (\lambda_{t_1} : \mu_\alpha)$ and $b \in \mu_\alpha$, we have $(x-y)\gamma b = x\gamma b - y\gamma b \in \lambda_{t_1}$. This implies $(x-y)\gamma\mu_\alpha \subseteq \lambda_{t_1}$. Hence $(x - y) \in (\lambda_{t_1} : \mu_\alpha)$ which implies that $(\lambda : \mu)(x - y) \geq t_1$ and $(\lambda : \mu)(x - y) \geq t_1 = \min\{t_1, t_2\}$. Similarly, If $t_1 > t_2$ then $(\lambda : \mu)(x - y) \geq t_2 \geq \min\{t_1, t_2\}$. This implies $(\lambda : \mu)(x - y) \geq \min\{(\lambda : \mu)(x), (\lambda : \mu)(y)\}$. For other choices of $t_1$ and $t_2$, it can be verified that $(\lambda : \mu)(x - y) \geq \min\{(\lambda : \mu)(x), (\lambda : \mu)(y)\}$. Suppose $(\lambda : \mu)(x + y) = t$. Then $x + y \in (\lambda_t : \mu_\alpha)$. As $\lambda_t$ is normal subgroup of $M$, $(\lambda_t : \mu_\alpha)$ is also normal subgroup of $M$. This implies $y + x \in (\lambda_t : \mu_\alpha)$ and so $(\lambda : \mu)(y + x) \geq t = (\lambda : \mu)(x+y)$. Similarly, $(\lambda : \mu)(x+y) \geq t = (\lambda : \mu)(y+x)$ Hence $(\lambda : \mu)(y + x) = (\lambda : \mu)(x + y)$. Let $(\lambda : \mu)(x) = t$. Now, $x \in (\lambda_t : \mu_\alpha)$. Thus, for any $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma(b + x) - a\gamma b \in (\lambda_t : \mu_\alpha)$. This implies $(\lambda : \mu)(a\gamma(b+x) - a\gamma b) \geq t = (\lambda : \mu)(x)$. Hence $(\lambda : \mu)$ is a fuzzy left ideal of $M$. □

The following example shows that, $\lambda$ and $\mu$ be any two fuzzy subsets of $M$. If $\lambda$ is a fuzzy left ideal of $M$, then $(\lambda : \mu)$ is not necessarily fuzzy ideal of $M$.

**Example 3.1.** Let $M = \{0, a, b, c\}$ and $\Gamma$ be the non-empty set of binary operations such that $\alpha, \beta \in \Gamma$ is defined below:

| $+$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

| $\alpha$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $0$ | $a$ | $b$ | $c$ |
| $c$ | $a$ | $0$ | $c$ | $b$ |

| $\beta$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $0$ | $a$ |
| $b$ | $0$ | $b$ | $0$ | $c$ |
| $c$ | $0$ | $0$ | $0$ | $b$ |

Clearly $(M, +, \Gamma)$ is a gamma-nearring.

Define a fuzzy subset $\lambda : M \to [0, 1]$ by

$$\lambda(0) = \lambda(b) = 1 \quad \text{and} \quad \lambda(a) = \lambda(c) = 0.$$

For any $t \in [0, 1]$, $\mu_t = \{0, b\}$ or $\{0, a, b, c\}$. Clearly, $\lambda$ is a fuzzy left ideal on $M$.

Define a fuzzy subset $\mu : M \to [0, 1]$ by

$$\mu(c) = 1 \quad \text{and} \quad \mu(0) = \lambda(a) = \mu(b) = 0.$$

Now, we have $(\lambda : \mu)(0) = 1, (\lambda : \mu)(a) = 0, (\lambda : \mu)(b) = 0, (\lambda : \mu)(c) = 1$.

Thus

$$(\lambda : \mu)(c\alpha c) = (\lambda : \mu)(b) = 0 < (\lambda : \mu)(c) = 1.$$

Hence, $(\lambda : \mu)$ is not a fuzzy ideal of $M$.

Now we find the conditions under which $(\lambda : \mu)$ is a fuzzy ideal of $M$.

**Theorem 3.2.** *Let $\lambda$ and $\mu$ be any two fuzzy subsets of $M$. If $\lambda$ is a fuzzy left ideal of $M$ and $\mu$ is a fuzzy $M$-subgroup of $M$, then $(\lambda : \mu)$ is a fuzzy ideal of $M$.*

*Proof.* By Theorem 3.1, it is clear that $(\lambda : \mu)$ is a fuzzy left ideal of $M$. Let $(\lambda : \mu)(x) = t$. Then $x \in (\lambda_t : \mu_\alpha)$, where $\alpha = \sup\{Im(\mu)\}$. Now, $x\gamma\mu_\alpha \subseteq \lambda_t$. Let $m \in M$. Consider $x\gamma m\gamma b = x\gamma(m\gamma b) \in x\gamma\mu_\alpha \subseteq \lambda_t$. Since $\mu_\alpha$ is a $M$-subgroup of $M$, then $m\gamma b \in \mu_\alpha$. Thus, for all $m \in M, \gamma \in \Gamma$ and $b \in \mu_\alpha$, we have $x\gamma m\gamma b \subseteq \lambda_t$. This implies $x\gamma m \in (\lambda_t : \mu_\alpha)$ and so $(\lambda : \mu)(x\gamma m) \geq t = (\lambda : \mu)(x)$. This shows that $(\lambda : \mu)$ is a fuzzy right ideal of $M$. Hence $(\lambda : \mu)$ is a fuzzy ideal of $M$. □

The following example shows that, $\lambda$ and $\mu$ are any two fuzzy subsets of $M$. If $\lambda$ is not a fuzzy right ideal of $M$ and $\mu$ is a fuzzy $M$-subgroup of $M$, then $(\lambda : \mu)$ is a fuzzy ideal of $M$.

**Example 3.2.** Let $M = \{0, a, b, c\}$ and $\Gamma$ be the non-empty set of binary operations such that $\alpha, \beta \in \Gamma$ is defined below:

| $+$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

| $\alpha$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $b$ | $c$ |
| $b$ | $0$ | $0$ | $0$ | $0$ |
| $c$ | $0$ | $a$ | $b$ | $c$ |

| $\beta$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $0$ | $0$ | $0$ | $b$ |
| $c$ | $a$ | $a$ | $a$ | $c$ |

Clearly $(M, +, \Gamma)$ is a gamma-nearring.

Define a fuzzy subset $\lambda : M \to [0,1]$ by

$$\lambda(x) = \begin{cases} 1, & \text{if } x = 0, a, \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $\lambda$ is a fuzzy left ideal of $M$. But $\lambda$ is not a fuzzy right ideal of $M$, since $0 = \lambda(a\alpha c) < \lambda(a) = 1$.

Define a fuzzy subset $\mu : M \to [0,1]$ by

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0, c, \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $\mu$ is a fuzzy $M$-subgroup of $M$. Now for any $x \in M$,

$$\begin{aligned} (\lambda : \mu)(x) = 1 \quad &\Leftrightarrow \quad x \in \ (\lambda_1 : \mu_1) \\ &\Leftrightarrow \quad x \in (\{0, a\} : \{0, c\}) \\ &\Leftrightarrow \quad x \in \{0, b\} \end{aligned}$$

Thus

$$(\lambda : \mu)(0) = (\lambda : \mu)(b) = 1 \quad \text{and} \quad (\lambda : \mu)(a) = (\lambda : \mu)(c) = 0.$$

Hence $(\lambda : \mu)$ is a fuzzy ideal of $M$.

**Theorem 3.3.** *Let $\lambda$ and $\mu$ be any two fuzzy subsets of $M$. If $\lambda$ and $\mu$ are fuzzy ideals of zero-symmetric gamma-nearring $M$, then $(\lambda : \mu)$ is a fuzzy ideal of $M$.*

*Proof.* By Theorem 3.1 , it is clear that $(\lambda : \mu)$ is a fuzzy left ideal of $M$. Also, we have $\mu$ is a fuzzy ideal of $M$, then by Lemma 3.2, $\mu$ is a fuzzy $M$-subgroup of $M$. Hence, by the Theorem 3.2, $(\lambda : \mu)$ is a fuzzy ideal of $M$. □

**Theorem 3.4.** *Let $\lambda$ be a fuzzy bi-ideal of $M$ and let $\mu$ is a fuzzy $M$-subgroup of zero-symmetric gamma-nearring $M$, then $(\lambda : \mu)$ is a fuzzy bi-ideal of $M$.*

*Proof.* Let $\lambda$ be a fuzzy bi-ideal of $M$ and $\mu$ be a fuzzy $M$-subgroup of zero-symmetric gamma-nearring $M$. Clearly $(\lambda : \mu)$ is a fuzzy subgroup of $M$. Next we need to prove that $(\lambda : \mu)$ is a fuzzy bi-ideal of $M$. Let $a, b, t, m \in M$ and $\gamma \in \Gamma$ such that $t = a\gamma m\gamma b$. Consider

$$\begin{aligned}((\lambda : \mu)\Gamma\mathbf{M}\Gamma(\lambda : \mu))(t) &= \sup_{t=a\gamma m\gamma b}\{\min\{(\lambda : \mu)(a), \mathbf{M}(m), (\lambda : \mu)(b)\}\} \\ &= \sup_{t=a\gamma m\gamma b}\{\min\{(\lambda : \mu)(a), (\lambda : \mu)(b)\}\}\end{aligned}$$

Let $\min\{(\lambda : \mu)(a), (\lambda : \mu)(b)\} = t$. This implies that $(\lambda : \mu)(a) \geq t$ and $(\lambda : \mu)(b) \geq t$. Thus $a, b \in (\lambda_t : \mu_\alpha)$. As $\lambda$ is the fuzzy bi-ideal and $\mu$ is the fuzzy $M$-subgroup, $(\lambda_t : \mu_\alpha)$ is a bi-ideal of $M$. Hence $a\gamma m\gamma b \in (\lambda_t : \mu_\alpha)$. This implies $(\lambda : \mu)(a\gamma m\gamma b) \geq t = \min\{(\lambda : \mu)(a), (\lambda : \mu)(b)\}$. Thus

$$\min\{(\lambda : \mu)(a), (\lambda : \mu)(b)\} \leq (\lambda : \mu)(a\gamma m\gamma b\}.$$

This shows that

$$\sup_{t=a\gamma m\gamma b}\{\min\{(\lambda : \mu)(a), (\lambda : \mu)(b)\} \leq (\lambda : \mu)(a\gamma m\gamma b)\}.$$

Thus, we have

$$((\lambda : \mu)\Gamma\mathbf{M}\Gamma(\lambda : \mu))(t) \leq (\lambda : \mu)(t).$$

Hence $((\lambda : \mu)\Gamma\mathbf{M}\Gamma(\lambda : \mu)) \leq (\lambda : \mu)$ and $(\lambda : \mu)$ is a fuzzy bi-ideal of $M$. □

**Theorem 3.5.** *Let $\lambda$ and $\mu$ be a fuzzy subsets of $M$. If $\lambda$ is a fuzzy ideal of $M$ and $\mu$ is a fuzzy $M$-subgroup of $M$, then $(\lambda : \mu)$ is a fuzzy quasi-ideal of $M$.*

*Proof.* Let $\lambda$ and $\mu$ be a fuzzy subsets of $M$. If $\lambda$ is a fuzzy ideal of $M$ and $\mu$ is a fuzzy $M$-subgroup of $M$ then by Theorem 3.2, we have $(\lambda : \mu)$ is a fuzzy ideal of $M$. Let $t \in M$, we have

$$((\lambda : \mu)\Gamma\mathbf{M})(t) = \sup_{t=a\gamma m}\{\min\{(\lambda : \mu)(a), \mathbf{M}(m)\}\} = \sup_{t=a\gamma m}\{(\lambda : \mu)(a)\}.$$

$$(\mathbf{M}\Gamma(\lambda : \mu))(t) = \sup_{t=a\gamma m}\{\min\{\mathbf{M}(m), (\lambda : \mu)(a)\}\} = \sup_{t=a\gamma m}\{(\lambda : \mu)(a)\}.$$

$$\begin{aligned}(\mathbf{M}\Gamma * (\lambda : \mu))(t) &= \sup_{t=x\gamma(m+a)-x\gamma m}\{\min\{\mathbf{M}(m), (\lambda : \mu)(a)\}\} \\ &= \sup_{t=x\gamma(m+a)-x\gamma m}\{(\lambda : \mu)(a)\}.\end{aligned}$$

for all $a, m, x \in M$ and $\gamma \in \Gamma$

Thus

$$
\begin{aligned}
&(((\lambda:\mu)\Gamma\mathbf{M}) \cap (\mathbf{M}\Gamma(\lambda:\mu)) \cap (\mathbf{M}\Gamma * (\lambda:\mu)))(t) \\
&= \min\Big\{ \sup_{t=a\gamma m}\{(\lambda:\mu)(a)\}, \sup_{t=a\gamma m}\{(\lambda:\mu)(a)\}, \sup_{\substack{t=x\gamma(m+a)\\ -x\gamma m}}\{(\lambda:\mu)(a)\}\Big\} \\
&\qquad \le \min\Big\{1, 1, \sup_{\substack{t=x\gamma(m+a)\\ -x\gamma m}}\{(\lambda:\mu)(x\gamma(m+a)-x\gamma m)\}\Big\} \\
&\qquad\qquad \le \text{(since } (\lambda:\mu) \text{is a fuzzy ideal of } M\text{ )} \le (\lambda:\mu)(t).
\end{aligned}
$$

Hence,

$$(((\lambda:\mu)\Gamma\mathbf{M}) \cap (\mathbf{M}\Gamma(\lambda:\mu)) \cap (\mathbf{M}\Gamma * (\lambda:\mu))) \le (\lambda:\mu).$$

This completes the proof. □

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