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On the dynamics of solutions for autonomous reaction-diffusion equation in $\mathbb{R}^N$ with multivalued nonlinearity

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Abstract. We consider the dynamics of solutions for autonomous reaction-diffusion equation in $\mathbb{R}^n$ with multivalued nonlinearity. The a priory estimates for solutions are obtained. The existence of compact invariant global attractor for m-semiflow was justified.

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Introduction

The theory of global attractors for infinite-dimensional dynamical systems was founded in 70th of the last century in O. A. Ladyzhenskaya's works on studying the dynamics of two-dimensional system of Navier-Stokes's equations and in J. K. Hale's works, which concerned with investigation of high-quality behavior of functional-differential equations. However, swift development of this theory, which proceeds till today, came in the middle of 80th, when it turned out that at abstract level of those characteristic features which allowed to investigate the Navier–Stokes's equation and equations with delay from the point of global attractors theory, are peculiar to the wide class of evolution equations, which describe real natural phenomena: flow of viscid incompressible liquid, processes of chemical kinetics, various wave processes, physical processes assuming phase transitions, vibrations of shells in ultrafast gas streams, etc. An important contribution to foundation and development

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of classic theory of global attractors of the infinite-dimensional dynamic systems was made by M. I. Vishik, O. A. Ladyzhenskaya, V. S. Melnik, I. D. Chueshov, J. M. Ball, J. K. Hale, R. Temam, B. Wang, S. V. Zelik and their apprentices [1]– [18].

The results concerning with existence and properties of solutions of reaction-diffusion equation in the case of smooth by the phase variable nonlinear term as well as the results dealing with existence of global attractor under these conditions are classic and contain in [1, 16], for non-autonomous equations with almost periodic dependence on time variable — in [8], for inclusions in bounded domain in — [5,10], for equations in unbounded domain — in [14, 17]. The global attractor for inclusions in unbounded domain with continuous multivalued nonlinearity was considered in [19].

1. Problem definition

In the present paper it is investigated the asymptotic behavior of solutions of reaction-diffusion equation with multivalued nonlinearity of the next view:

$$y_t \in \triangle y - f(x,y), \quad x \in \mathbb{R}^N,\ t>0, \tag{1.1}$$

$$y(0) = y_0 \in L^2(\mathbb{R}^N), \tag{1.2}$$

where y is unknown function, $y_t = \partial y/\partial t$.

For numbers $a, b \in \mathbb{R}$ let $[a,b] = \{\alpha a + (1-\alpha)b \,|\, \alpha \in [0,1]\}$. Let us specify the conditions for parameters of the problem:

α_1) for almost each (a.e.) $x \in \mathbb{R}^N$, $\forall\, u \in \mathbb{R}$ $f(x,u) = [\underline{f}(x,u), \overline{f}(x,u)]$, where $\underline{f}, \overline{f} : \mathbb{R}^{N+1} \to \mathbb{R}$ are measurable functions such that for a.e. $x \in \mathbb{R}^N$ $\underline{f}(x,\cdot)$ is lower semi-continuous (l.s.c.), and $\overline{f}(x,\cdot)$ is upper semi-continuous (u.s.c.).

α_2) $\exists\, C_1 \in L^1(\mathbb{R}^N)$, $\alpha > 0$: for a.e. $x \in \mathbb{R}^N$, $\forall\, u \in \mathbb{R}$

$$\begin{aligned} \underline{f}(x,u)u &\geq \alpha|u|^2 - C_1(x), \quad u \geq 0, \\ \overline{f}(x,u)u &\geq \alpha|u|^2 - C_1(x), \quad u \leq 0. \end{aligned} \tag{1.3}$$

α_3) $\exists\, C_2 \in L^1(\mathbb{R}^N)$, $C_2 \geq 0$, $\exists\, \beta > 0$: for a.e. $x \in \mathbb{R}^N$, $\forall\, u \in \mathbb{R}$

$$\begin{gathered} |\underline{f}(x,u)|^2 \leq C_2(x) + \beta|u|^2, \\ |\overline{f}(x,u)|^2 \leq C_2(x) + \gamma|u|^2, \\ \underline{f}(x,u) \leq \overline{f}(x,u). \end{gathered} \tag{1.4}$$

Let us consider real spaces $H = L^2(\mathbb{R}^N)$, $V = H^1(\mathbb{R}^N)$ and $V^* = H^{-1}(\mathbb{R}^N)$ with corresponding norms $\|\cdot\|$, $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$. The norm in $\mathbb{R}^N$, inner product in H and in $\mathbb{R}^N$ we will denote by $|\cdot|$, $(\cdot,\cdot)_H$, $(\cdot,\cdot)$ respectively [14, p. 112].

The goal of this work is to prove the existence of compact global attractor for the solutions of the problem (1.1)–(1.2) in the phase space H.

2. Preliminaries

From (1.4) it follows, that for arbitrary $u, g \in L^2(0,T;H)$: $g(x,t) \in f(x,u(x,t))$ for a.e. $(x,t) \in \mathbb{R}^N \times (0,T)$

$$\int_0^T \int_{\mathbb{R}^N} |g(x,t)|^2 \, dx \, dt \le K_1 \left(T + \|u\|^2_{L^2(0,T;H)} \right). \tag{2.5}$$

Definition 2.1. *A function $y(x,t)$, $x \in \mathbb{R}^N$, $t \in [0,T]$ is called the weak solution of (1.1) on $[0,T]$, if $y \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and for some selector $d \in L^2(0,T;H)$: $d(x,t) \in f(x,y(x,t))$ for a.e. $(x,t) \in \mathbb{R}^N \times (0,T)$, the relation*

$$-\int_0^T (y, v_t) \, dt - \int_0^T (y, \triangle v) \, dt + \int_0^T \int_{\mathbb{R}^N} d(x,t) v(x,t) \, dx \, dt = 0, \tag{2.6}$$

holds true for every $v \in C_0^\infty(\mathbb{R}^N \times [0,T])$.

From (2.5), [14, p. 114] and from the Definition 2.1 of the weak solution y of the inclusion (1.1) it follows, that $y_t \in L^2(0,T;V^*)$, $y \in C(0,T;V^*)$ and $y \in C(0,T;H_w)$. Thus, for every $v \in L^2(0,T;V)$ the weak solution y satisfies such relation:

$$\int_0^T \langle y_t, v \rangle_V \, dt + \int_0^T (\nabla y, \nabla v) \, dt + \int_0^T \int_{\mathbb{R}^N} d(x,t) v(x,t) \, dx \, dt = 0. \tag{2.7}$$

where $\langle \cdot,\cdot \rangle_V$ is the pairing in the space V, that coincides on $H \times V$ with the inner product in H [2, p. 29].

The conditions α_1–α_3 don't provide the uniqueness of the solution of the problem (1.1)–(1.2) [10, p. 68], so let us introduce the definition of multivalued, in the general case, semiflow and its global attractor (see for example [10, p. 14]), that describe the dynamics of the solutions of initial problem as $t \to +\infty$.

Definition 2.2. *A map $G : \mathbb{R}_+ \times H \to P(H)$ is called the multivalued semiflow (m-semiflow) on H, if*

1) $G(0, \cdot) = I_H$ *is identical motion* H*;*

2) $G(t + s, x) \subset G(t, G(s, x))$ $\forall\, t, s \in \mathbb{R}_+$, $\forall\, x \in H$.

M-semiflow is called the strict, if $G(t + s, x) = G(t, G(s, x))$ $\forall\, t, s \in \mathbb{R}_+$, $\forall\, x \in H$.

Definition 2.3 ([14, p. 123]). *M-semiflow G is assiptotically compact, if for any bounded $B \in P(H)$ such, that $\gamma_T^+(B)$ is bounded for some $T = T(B) \geq 0$, an arbitrary sequence $\{\xi_n\}_{n\geq 1}$, $\xi_n \in G(t_n, B)$, $t_n \to +\infty$, is pre-compact in H.*

Definition 2.4. *A set $\mathcal{A} \subset H$, that satisfies the next properties:*

1) $\mathcal{A}$ *is absorbing set, i.e.,*

$$\operatorname{dist}(G(t, B), \mathcal{A}) \to 0, \quad \text{as} \quad t \to +\infty,$$

for any bounded set B, where $\operatorname{dist}(C, A) = \sup_{c\in C} \inf_{a\in A} \|c - a\|$.

2) $\mathcal{A}$ *is semi-invariant, i.e.,*

$$A \subset G(t, A), \quad \text{for every} \quad t \geq 0,$$

3) $\mathcal{A}$ *is minimal closed absorbing set (i.e. for any closed absorbing set C we have, that $A \subset C$)*

is called the global attractor $\mathcal{A}$ for m-semiflow G.

The global attractor is called invariant, if $A = G(t, A)$, for every $t \geq 0$.

Let now $\Omega \subset \mathbb{R}^N$ is bounded domain, $T > 0$, $Q = \Omega \times (0, T)$, $\mathcal{Y} = L^2(Q)$. The next lemma is necessary for proof of the main theorem.

Lemma 2.1. *Let f satisfies α_1, and $\{y_n, d_n\}_{n\geq 0} \subset \mathcal{Y}$ such, that*

1) *for a.e.* $(x, t) \in Q$ $y_n(x, t) \to y_0(x, t)$ *as* $n \to +\infty$,

2) $d_n \to d_0$ *weakly in* $\mathcal{Y}$ *as* $n \to +\infty$,

3) $\forall\, n \geq 1$ *for a.e.* $(x, t) \in Q$ $d_n(x, t) \in f(x, y_n(x, t))$.

Then for a.e. $(x, t) \in Q$ $d_0(x, t) \in f(x, y_0(x, t))$.

Proof. Let $\{y_n, d_n\}_{n\geq 0} \subset \mathcal{Y}$ satisfy the lemma conditions. Let us select the complete measure set $Q_1 \subset Q$ such, that

$$\forall\,(x,t) \in Q_1 \quad y_n(x,t) \to y_0(x,t) \quad \text{as} \quad n \to \infty. \tag{2.8}$$

As the space $L_2(Q)$ is a Hilbert space, then in view of [2, Remark I.6.2] it is uniformly convex (see for example [2, Definition I.5.9]). From the proof of [3, Theorem 1, p. 64–66] it follows that from weakly convergent to $\bar{0}$ in $\mathcal{Y}$ sequence $\{d_n - d_0\}_{n\geq 1}$ we can choose a subsequence $\{d_{n_k} - d_0\}_{k\geq 1} \subset \{d_n - d_0\}_{n\geq 1}$, for which the arithmetical means converge by norm to $\bar{0}$ in $L_2(Q)$ (in mentioned theorem from [3] it is proved stronger statement than the Banach–Saks property), i.e.

$$\left\| \frac{1}{k}\sum_{j=1}^{k}(d_{n_j} - d_0)\right\|_{\mathcal{Y}} \to 0 \quad \text{as} \quad k \to +\infty.$$

It means, that

$$\frac{1}{k}\sum_{j=1}^{k} d_{n_j} \to d_0 \quad \text{strongly in} \quad L_2(Q) \quad \text{as} \quad k \to +\infty. \tag{2.9}$$

Further, $\exists\, Q_2 \subset Q_1$ such, that Q_2 is measurable, $\operatorname{mes}(Q_1 \backslash Q_2) = 0$ and $\forall\,(x,t) \in Q_2\ \forall\, k \geq 1$

$$\underline{f}(x, y_{n_k}(x,t)) \leq d_{n_k}(x,t) \leq \overline{f}(x, y_{n_k}(x,t)).$$

So, $\forall\, k \geq 1$, $\forall\,(x,t) \in Q_2$

$$\frac{1}{k}\sum_{j=1}^{k} \underline{f}(x, y_{n_j}(x,t)) \leq \frac{1}{k}\sum_{j=1}^{k} d_{n_j}(x,t) \leq \frac{1}{k}\sum_{j=1}^{k} \overline{f}(x, y_{n_j}(x,t)). \tag{2.10}$$

From (2.9) there exists a subsequence $\{\frac{1}{k_l}\sum_{j=1}^{k_l} d_{n_j}\}_{l\geq 1} \subset \{\frac{1}{k}\sum_{j=1}^{k} d_{n_j}\}_{k\geq 1}$ and a complete measure set $Q_3 \subset Q_2$:

$$\forall\,(x,t) \in Q_3 \quad \frac{1}{k_l}\sum_{j=1}^{k_l} d_{n_j}(x,t) \to d_0(x,t) \text{ as } l \to +\infty. \tag{2.11}$$

For a.e. $(x,t) \in Q_3$ let us set $a_k = \overline{f}(x, y_{n_k}(x,t))$, $k \geq 1$, $a_0 = \overline{f}(x, y_0(x,t))$. From α_1 and (2.8) it follows, that $\overline{\lim}_{k\to\infty} a_k \leq a_0$. Thus,

$$\overline{\lim_{k\to+\infty}}\, \frac{1}{k}\sum_{j=1}^{k} \overline{f}(x, y_{n_j}(x,t)) \leq \overline{f}(x, y_0(x,t)).$$

Similarly,

$$\varliminf_{k\to+\infty}\frac{1}{k}\sum_{j=1}^{k}\underline{f}(x,y_{n_j}(x,t))\geq\underline{f}(x,y_0(x,t)).$$

Taking into account (2.10)–(2.11), we obtain that for a.e. $(x,t)\in Q$ $d_0(x,t)\in f(x,y_0(x,t))$. □

3. Main results

Firstly we will obtain some a priory estimations of solutions.

Lemma 3.1. *For some weak solution y of the problem* (1.1)–(1.2) *we have:*

$$\|y\|_X\leq K_1(\|y_0\|,T),\tag{3.12}$$

$$\|y_t\|_U\leq K_2(\|y_0\|,T),\tag{3.13}$$

where K_i are nondecreasing by each variable functions, $X=L^2(0,T;V)\cap C([0,T],H)$ and $U=L^2(0,T;V^*)$.

Proof. From [14, Lemma 3] it follows, that

$$\frac{1}{2}\frac{d}{dt}\|y\|^2=-(\nabla y,\nabla y)-\int\limits_{\mathbb{R}^N}d\cdot y\,dx,\tag{3.14}$$

where $d\in L^2(0,T;H)$: $d(x,t)\in f(x,y(x,t))$ for a.e. $(x,t)\in\mathbb{R}^N\times(0,T)$. From α_1 and α_3 we obtain

$$\frac{1}{2}\frac{d}{dt}\|y\|^2+\|\nabla y\|^2\leq-\alpha\|y\|^2+\int\limits_{\mathbb{R}^N}C_1(x)\,dx.\tag{3.15}$$

So,

$$\forall\, t\in[0,T]\quad\|y(t)\|^2+2\int\limits_0^t\|\nabla y\|^2ds+2\alpha\int\limits_0^t\|y\|^2ds\leq\|y_0\|^2+2MT,\tag{3.16}$$

from here it follows (3.12).

On the other hand, as $-\triangle:V\to V^*$ is linear bounded operator, then from (3.12) it follows, that

$$\|-\triangle y\|_{L^2(0,T;V^*)}\leq K\|y\|_{L^2(0,T;V)}\leq KK_1.$$

Finally, from $y_t=\triangle y-d$ it follows (3.13). □

Theorem 3.1. *Let the suppositions α_1–α_3 hold true. Then for any $y_0 \in L^2(\mathbb{R}^N)$, $T > 0$, the problem* (1.1)–(1.2) *has at least one weak solution.*

Proof. In order to prove the solvability let us consider the homogeneous Dirichlet's problem in bounded domain

$$y_t \in \triangle y - f(x,y), \quad x \in \Omega_R,\, t > 0, \tag{3.17}$$

$$y_{|\partial\Omega} = 0, \quad t > 0, \tag{3.18}$$

$$y(0) = y_{0,R}, \quad x \in \Omega_R, \tag{3.19}$$

where $\Omega_R = B(0;R)$ is open ball of the radius $R \geq 1$ with the center in zero, $u_{0,R}(x) = u_0(x)\psi_R(|x|)$ and ψ_R is smooth function, which satisfies:

$$\psi_R(\xi) = \begin{cases} 1, & \text{if } 0 \leq \xi \leq R-1, \\ 0 \leq \psi_R(\xi) \leq 1, & \text{if } R-1 \leq \xi \leq R, \\ 0, & \text{if } \xi > R. \end{cases}$$

Let us set $R \geq 1$, $Q_{R,T} = \Omega_R \times (0,T)$ $H_R = L^2(\Omega_R)$, $V_R = H^1(\Omega_R)$, $V_R^* = H^{-1}(\Omega_R)$, $\mathcal{X}_R = L^2(0,T;V_R)$, $\mathcal{X}_R^* = L^2(0,T;V_R^*)$, $\mathcal{Y}_R^* \equiv \mathcal{Y}_R = L^2(Q_R)$, $\langle\cdot,\cdot\rangle_R$ is pairing in $\mathcal{X}_R$, coinciding on $\mathcal{Y}_R \times X_R$ with the inner product in $\mathcal{Y}_R$ [2, p. 29], $W_R = \{y \in \mathcal{X}_R \,|\, y' \in \mathcal{X}_R^*\}$, where the derivative y' of an element $y \in \mathcal{X}_R$ is considered in the sense of distributions space $\mathcal{D}^*(0,T;V_R^*)$ [2, p. 168]. Let $A : \mathcal{X}_R \to \mathcal{X}_R^*$ is energetic extension $-\triangle$, $B : \mathcal{Y}_R \rightrightarrows \mathcal{Y}_R^*$ is the Nemitsky operator for f, i.e.

$$B(u) = \{v \in \mathcal{Y}_R^* \,|\, v(t,x) \in f(x,u(x,t)) \text{ for a.e. } (x,t) \in Q_{R,T}\},\ u \in \mathcal{Y}_R.$$

Let us denote, that A is linear, strongly monotone, bounded [2, 87–88]. From α_1–α_3 and Lemma 2.1 it follows, that B is bounded, and its graph is closed in $(\mathcal{Y}_R^*)_w \times \mathcal{Y}_R$ [6, p. 576]. Thus, there exist such $c_1, c_2, c_3 > 0$, that

$$\langle A(y), y\rangle_R = \|y\|^2_{\mathcal{X}_R},\ \|A(y)\|_{\mathcal{X}_R^*} = \|y\|_{\mathcal{X}_R}, \quad y \in \mathcal{X}_R;$$

$$\forall\, y \in \mathcal{Y}_R, \forall\, d \in B(y) \quad \langle d, y\rangle_R \geq c_1\|y\|_{\mathcal{Y}_R} - c_2,\ \|d\|_{\mathcal{Y}_R^*} \leq c_3(\|y\|_{\mathcal{Y}_R} + 1).$$

As the embedding $V_R \subset H_R$ is compact, then $W_R \subset \mathcal{Y}_R$ compactly [7, p. 70]. From [6, p. 557] the λ_0-pseudomonotony of $C = A + B : \mathcal{X}_R \rightrightarrows \mathcal{X}_R^*$ on W_R [6, p. 543] follows. So, from upper considered suppositions and from [4, Theorem 6.1.1] it follows, that the problem (3.17)–(3.19) has at least one weak (generalized) solution for any $y_{0,R} \in L^2(\Omega_R)$ (the definition of the weak solution is the same as in the definition 2.1, but there we considered Ω_R instead $\mathbb{R}^N$). So, there exists $d_R \in L^2(0,T;H_R)$ such, that $\forall\, v \in \mathcal{X}_R$

$$\int_0^T \left\langle \frac{\partial y_R}{\partial t}, v \right\rangle_V dt + \int_0^T (\nabla y_R, \nabla v)\, dt + \int_0^T \int_{\mathbb{R}^N} d_R(x,t) v(x,t)\, dx\, dt = 0, \tag{3.20}$$

$$d_R(x,t) \in f(x, y_R(x,t)) \text{ for a.e. } (x,t) \in \Omega_R \times (0,T). \tag{3.21}$$

Let y_{r_j}, $r_j \to +\infty$ be the sequence of solutions of (3.17)–(3.19) in the sense of (3.20)–(3.21). Let us denote, that

$$|y_0 - y_{0,r_j}|^2 = \int_{\mathbb{R}^N} (1 - \psi_{r_j}(|x|))^2 |y_0|^2\, dx$$

$$\leq \int_{|x| \geq r_j - 1} |y_0|^2\, dx \to 0, \; r_j \to +\infty. \tag{3.22}$$

Repeating the same suppositions as in the proof of Lemma 3.1, we will obtain

$$\|y(t)\|^2_{H_{r_j}} + 2\int_0^t \|\nabla y\|^2_{r_j}\, ds + 2\alpha \int_0^t \|y\|^2_{r_j}\, ds$$

$$\leq \|y_{0,r_j}\|^2 + 2T \int_{\mathbb{R}^N} C_1(x)\, dx, \quad \forall\, t \in [0,T].$$

So, from (3.22) it follows

$$\|y_{r_j}\|_{X_{r_j}} \leq K_1(\|y_{0,r_j}\|, T) \leq \tilde{K_1}(\|y_0\|, T), \tag{3.23}$$

where $X_{r_j} = L^2(0,T;V_{r_j}) \cap C([0,T], H_{r_j})$.

Let us continue solutions of (3.17)–(3.19) onto $\mathbb{R}^N$:

$$\hat{y}_{r_j}(x,t) = \begin{cases} y_{r_j}(x,t)\psi_{r_j}(|x|) & \text{in } B(0,r_j) \times (0,T), \\ 0, & \text{else,} \end{cases}$$

$$\hat{d}_{r_j}(x,t) = \begin{cases} d_{r_j}(x,t) & \text{in } B(0,r_j - 1) \times (0,T), \\ \overline{f}(x, \hat{y}_{r_j}(x,t)), & \text{else,} \end{cases}$$

$\hat{d}_{r_j}(x,t) \in f(x, \hat{y}_{r_j}(x,t))$ for a.e. $(x,t) \in \mathbb{R}^N \times (0,T)$. As y_{r_j} uniformly bounded in X_{r_j} by r_j, then $\{\hat{y}_{r_j}, \hat{d}_{r_j}\}$ is uniformly bounded sequence in

$X \times L^2(0,T;V^*)$. Thus, there exists a subsequence, which we denote again by $\{y_{r_j}, d_{r_j}\}$ such, that

$$\begin{aligned} &y_{r_j} \to y_\infty \text{ weakly in } L^2(0,T;V),\\ &y_{r_j} \to y_\infty \text{ weakly star in } L^\infty(0,T;H),\\ &d_{r_j} \to d_\infty \text{ weakly in } L^2(0,T;V^*). \end{aligned} \tag{3.24}$$

Let us further prove, that y_∞ is the weak solution of (1.1)–(1.2). Let us fix an arbitrary r_k. As $r_j \to +\infty$, then we can suppose, that $r_k \le r_j - 1$. The projection y_{r_j} on $B(0, r_k)$ we will denote by:

$$y_{k,j} = L_k y_{r_j}.$$

From (3.23) it is obviously follows, that $\{y_{k,j}\}_j$ is bounded in X_{r_k}. So, up to the subsequence (let us denote it again by y_{r_j}), $y_{k,j} = L_k y_{r_j} \to y_{k,\infty}$ weakly in $L^2(0,T;V_{r_k})$ and weakly star in $L^\infty(0,T;H_{r_k})$. Following by [14, p. 118], we obtain, that $L_k y_\infty = y_{k,\infty}$. In order to show, that y_∞ is the weak solution of the problem (1.1)–(1.2), it is sufficiently to check, that $L_k y_\infty$ is the weak solution on $\Omega_{r_k} \times (0,T)$ [14, p. 118].

Let $v \in C_0^\infty(\Omega_{r_k} \times [0,T])$. As $B(0,r_k) \subset B(0,r_j)$, then $v \in C_0^\infty(\Omega_{r_j} \times [0,T])$. So,

$$\begin{aligned} &\int_0^T \int_{\Omega_{r_k}} \big(-L_k y_{r_j} \cdot v_t - L_k y_{r_j} \cdot \triangle v + L_k d_{r_j} \cdot v\big)\, dx\, dt \\ &\qquad = \int_0^T \int_{\Omega_{r_j}} \big(-L_k y_{r_j} \cdot v_t - L_k y_{r_j} \cdot \triangle v + L_k d_{r_j} \cdot v\big)\, dx\, dt = 0. \end{aligned} \tag{3.25}$$

Following by the proof of the Lemma 3.1 and [14, Theorem 5], we will obtain:

$$L_k \frac{\partial y_{r_j}}{\partial t} = \frac{\partial L_k y_{r_j}}{\partial t} \to \frac{\partial y_{k,\infty}}{\partial t} \text{ weakly in } L^2(0,T;V_{r_k}^*). \tag{3.26}$$

As the embedding $H_0^1(\Omega_{r_k}) \subset L^2(\Omega_{r_k})$ is compact, and the embedding $L^2(\Omega_{r_k}) \subset H^{-1}(\Omega_{r_k})$ is continuous, using the compactness lemma [7, p. 70] and Lemma 2.1, from convergences (3.12), (3.26) and from conditions α_1, α_3, similarly to [14, p. 119] we will obtain, that $d_\infty(x,t) \in f(x, y_\infty(x,t))$ for a.e. $(x,t) \in \Omega_{r_k} \times (0,T)$. In order to complete the proof it remains to pass to the limit in (3.25) as $r_j \to +\infty$. □

Since in the Theorem 3.1 $T > 0$ is arbitrary, as the concatenation of solutions is the solution (see the proof of 1° of the Theorem 3.2), then by analogical suppositions to [14, p. 119], each solution can be continued to the global, that is defined on $[0, +\infty)$.

Let us denote the family of all global solutions of the problem (1.1)–(1.2), corresponding to the initial condition y_0 by $\mathcal{D}(y_0)$. It is obviously, that $\mathcal{D}(y_0) \subset L^2_{loc}(0, +\infty; V) \cap C([0, +\infty), H)$. Let us show, that $\mathcal{D}(y_0) \subset L^\infty(0, +\infty; H)\ \forall\, y_0 \in L^2(\mathbb{R}^N)$.

Lemma 3.2. *If y is the weak solution of the problem* (1.1)–(1.2), *then*

$$\forall\, t \geq 0 \quad \|y(t)\|^2 + 2\int\limits_0^t e^{-\alpha(t-s)}\|\nabla y\|^2\, ds \leq \|y(0)\|^2 e^{-2\alpha t} + D, \quad (3.27)$$

where $D = \|C_1\|_{L^1(\mathbb{R}^N)}/\alpha$.

Proof. The proof follows from (3.15) and from the Gronwall–Bellman lemma. □

Let $y_0 \in H$, $P(H) = 2^H \setminus \{\emptyset\}$. We define (in general the multivalued) map $G : \mathbb{R}_+ \times H \to P(H)$:

$$G(t, y_0) = \{z \in H \mid \exists u \in \mathcal{D}(y_0) :\ y(0) = y_0,\, y(t) = z\}.$$

Theorem 3.2. *Under the conditions α_1–α_3, the problem* (1.1)–(1.2) *defines m-semiflow in the phase space H, that possesses the invariant global attractor.*

Proof. 1° Firstly we will show, that G is the strict m-semiflow. The proof of $G(t+s, x) \subset G(t, G(s, x))$ repeats the proof of similar inclusion from [14, Lemma 7]. Let us check $G(t, G(s, x)) \subset G(t+s, x)$. Let $u \in G(t, G(s, x))$. Then there exist z_1, $y_1(\cdot) \in \mathcal{D}(x)$, $y_2(\cdot) \in \mathcal{D}(z_1)$, d_1, d_2 such, that

$$y_1(0) = x, \quad y_1(s) = z_1,$$

$$y_2(0) = z_1, \quad y_2(t) = u,$$

$$d_1 = \triangle y_1 - \frac{\partial y_1}{\partial t}, \quad d_1(\xi, \zeta) \in f(\xi, y_1(\xi, \zeta)) \text{ for a.e. } (\xi, \zeta) \in \mathbb{R}^N \times \mathbb{R}_+,$$

$$d_2 = \triangle y_2 - \frac{\partial y_2}{\partial t}, \quad d_2(\xi, \zeta) \in f(\xi, y_2(\xi, \zeta)) \text{ for a.e. } (\xi, \zeta) \in \mathbb{R}^N \times \mathbb{R}_+.$$

Let us show, that there exists $y(\cdot) \in \mathcal{D}(y_0)$: $y(0) = x$, $y(t+s) = u$. Let us define y by:

$$y(r) = \begin{cases} y_1(r), & 0 \leq r \leq s, \\ y_2(r-s), & s \leq r. \end{cases}$$

For a.e. $(\xi,\zeta)\in\mathbb{R}^N\times\mathbb{R}_+$ let us set

$$d(\xi,\zeta)=\begin{cases} d_1(\xi,\zeta), & 0\le\zeta\le s,\\ d_2(\xi,\zeta-s), & s\le\zeta.\end{cases}$$

We remark, that

$$d(\xi,\zeta)\in f(\xi,y(\xi,\zeta)) \text{ for a.e. } (\xi,\zeta)\in\mathbb{R}^N\times\mathbb{R}_+.$$

The next suppositions complete the proof

$$\begin{aligned}
&\int_0^T\left\langle\frac{\partial y}{\partial r},v\right\rangle_V dr+\int_0^T\left[(\nabla y,\nabla v)\,dr+\int_{\mathbb{R}^N} d(x,r)v(x,r)\,dx\right]dr\\
&=\int_0^s\left\langle\frac{\partial y_1}{\partial r},v\right\rangle_V dr+\int_0^s\left[(\nabla y_1,\nabla v)\,dr+\int_{\mathbb{R}^N} d_1(x,r)v(x,r)\,dx\right]dr\\
&\quad+\int_s^T\left\langle\frac{\partial y_2(r-s)}{\partial r},v\right\rangle_V dr+\int_s^T\Bigg[(\nabla y_2(r-s),\nabla v)\,dr\\
&\quad+\int_{\mathbb{R}^N} d_2(x,r-s)v(x,r-s)\,dx\Bigg]dr=0+\int_0^{T-s}\left\langle\frac{\partial y_2}{\partial r},v\right\rangle_V dr\\
&\quad+\int_0^{T-s}\left[(\nabla y_2,\nabla v)\,dr+\int_{\mathbb{R}^N} d_2(x,r)v(x,r)\,dx\right]dr=0
\end{aligned}$$

$\forall\, T>s+t$, $\forall\, v\in C_0^\infty([0,T]\times\mathbb{R}^N)$.

For fixed $k>0$ the ball of the radius k with the center in 0 we denote by Ω_k.

2° Let us prove, that for an arbitrary nonempty bounded set $B\subset H$, an arbitrary $y_0\in B$, an arbitrary weak solution $y\in\mathcal{D}(y_0)$, an arbitrary $\varepsilon>0$ there exist $T(\varepsilon,B)$, $K(\varepsilon,B)$ such, that

$$\forall\, t\ge T,\ k\ge K\quad \int_{|x|\ge\sqrt{2}k}|y(x,t)|^2\,dx\le\varepsilon.$$

Indeed, let $s\in\mathbb{R}_+$. Let us define the smooth function

$$\theta(s)=\begin{cases}0, & 0\le s\le 1,\\ 0\le\theta(s)\le 1, & 1\le s\le 2,\\ 1, & s\ge 2\end{cases}$$

such, that $|\theta'(s)| \le C \ \forall s \in \mathbb{R}_+$. Moreover, let us suppose, that $\sqrt{\theta}$ is smooth too.

Let us apply [14, Lemma 3] to $\rho(x) = \sqrt{\theta(\frac{|x|^2}{k^2})}$. From the definition of the weak solution of the equation (1.1) it follows, that

$$\text{for a.e. } t \ge 0 \qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big)|y|^2\,dx = \langle y_t, \rho^2 y\rangle_V$$
$$= \langle \triangle y, \rho^2 y\rangle_V - \int\limits_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big) d(x,t)y(x,t)\,dx, \quad (3.28)$$

where

$$d = \triangle y - \frac{\partial y}{\partial t}, \quad d(\xi,\zeta) \in f(\xi, y(\xi,\zeta)) \text{ for a.e. } (\xi,\zeta) \in \mathbb{R}^N \times \mathbb{R}_+,$$

Similarly to [14, p. 122–123], the first term in the right part of the last relation is estimated by the next way:

$$\langle \triangle y, \rho^2 y\rangle_V \le \varepsilon'(1 + \|\nabla y\|^2) \quad (3.29)$$

for an arbitrary $k \ge K_1(\varepsilon')$, where $\varepsilon' > 0$ is an arbitrary and rather small.

For the second term from (3.28) with the help of α_2 and α_3 we will obtain:

$$-\int\limits_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big) d(x,t)y(x,t)\,dx$$
$$\le -\alpha \int\limits_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big)|y(x,t)|^2\,dx + \int\limits_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big) C_1(x)\,dx$$
$$\le -\alpha \int\limits_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big)|y(x,t)|^2\,dx + 2\varepsilon', \quad (3.30)$$

as soon as $k \ge K_2(\varepsilon')$. Let us set

$$Y(t) = \int\limits_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big)|y(x,t)|^2\,dx.$$

Then from (3.28)–(3.30) it follows, that

$$\frac{1}{2}\frac{d}{dt}Y(t) + \alpha Y(t) \le 3\varepsilon' + \varepsilon\|\nabla y\|^2,$$

as soon as $k \geq \max\{K_1, K_2\}$. By using the Gronwall–Bellman lemma and Lemma 3.2, we obtain

$$Y(t) \leq Y(0)e^{-2\alpha t} + \frac{3}{\alpha}\varepsilon' + \frac{\varepsilon'}{2}(\|y_0\|^2 + D).$$

Choosing ε', $T(\varepsilon, B)$ such, that

$$\frac{3}{\alpha}\varepsilon' + \frac{\varepsilon'}{2}(\|y_0\|^2 + D) \leq \frac{\varepsilon}{2}, \quad Y(0)e^{-2\alpha t} \leq \frac{\varepsilon}{2}, \quad \forall\, y_0 \in B,\, t \geq T,$$

we will obtain $Y(t) \leq \varepsilon$ and

$$\int\limits_{|x| \geq \sqrt{2}k} |y(x,t)|^2\, dx \leq \int\limits_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big)|y(x,t)|^2\, dx \leq \varepsilon.$$

3° For bounded set $B \subset H$ and $T \in \mathbb{R}_+$ let us consider

$$\gamma_T^+(B) = \bigcup_{t \geq T} G(t, B).$$

Following by the proof of [14, Lemma 8] and the proof of the Theorem 3.1 we obtain the next result, that is necessity for the proof of asymptotic compactness of m-semiflow G, namely, the graph of $G(t,\cdot)$ is weakly closed, i.e. if $\xi_n \to \xi_\infty$, $\beta_n \to \beta_\infty$ weakly in H, where $\xi_n \in G(t, \beta_n)$ $\forall\, n \geq 1$, then $\xi_\infty \in G(t, \beta_\infty)$.

4° Let us show, that m-semiflow G is assiptotically compact. Let $\xi_n \in G(t_n, v_n)$ $v_n \in B$, B be bounded set in H. Since $\gamma_{T(B)}^+(B)$ is bounded and $\xi_n \in G(t_n, v_n) \subset \gamma_{T(B)}^+(B)$ for $n \geq n_0$, then there exists the weakly convergent in H subsequence (let us denote it by ξ_n again) to some ξ. Let $T_0 > 0$ be an arbitrary number. Using 1°, we have, that $\xi_n \in G(t_n, v_n) = G(T_0, G(t_n - T_0, v_n))$. Then there exists such $\beta_n \in G(t_n - T_0, v_n)$, that $\xi_n \in G(T_0, \beta_n)$. Let us choose $N(B, T_0)$ such, that $\forall\, n \geq N(B, T_0)$ $t_n - T_0 \geq T(B)$ and such, that $G(t_n - T_0, v_n) \subset \gamma_{T(B)}^+(B)$ is bounded, $\beta_n \to \xi_{T_0}$ weakly in H. From 3° it follows, that the graph $G(T_0, \cdot)$ is weakly closed. So, $\xi \in G(T_0, \xi_{T_0})$ and $\varliminf_{n\to\infty} \|\xi_n\| \geq \|\xi\|$. If we show, that up to subsequence, $\varlimsup_{n\to\infty} \|\xi_n\| \leq \|\xi\|$, then, up to subsequence, $\xi_n \to \xi$ strongly in H, that we need to show.

From (3.14) it follows, that any weak solution y satisfies

$$\frac{1}{2}\frac{d}{dt}\|y\|^2 + \frac{1}{2}\|y\|^2 + \|\nabla y\|^2 = -\int\limits_{\mathbb{R}^N} d \cdot y\, dx + \frac{1}{2}\|y\|^2, \quad \text{a.e. on } [0, T],$$

where $d \in L^2(0,T;H)$: $d(x,t) \in f(x,y(x,t))$ for a.e. $(x,t) \in \mathbb{R}^N \times (0,T)$. Let $y_n(\cdot)$ is the sequence of weak solutions, for which $y_n(T_0) = \xi_n$ and $y_n(0) = \beta_n$. In view of the Gronwall–Bellman lemma,

$$\|\xi_n\|^2 = e^{-T_0}\|\beta_n\|^2 - 2\int_0^{T_0} e^{-(T_0-s)}\|\nabla y_n\|^2\,ds \\ - 2\int_0^{T_0}\int_{\mathbb{R}^N} e^{-(T_0-s)} d_n \cdot y_n\,dx\,ds + \int_0^{T_0} e^{-(T_0-s)}\|y_n\|^2\,ds, \quad (3.31)$$

where $d_n \in L^2(0,T;H)$: $d_n(x,t) \in f(x,y_n(x,t))$ for a.e. $(x,t) \in \mathbb{R}^N \times (0,T)$. From the Lemma 3.1 and the Banach–Alaoglu theorem, up to subsequence (we denote it again by $\{y_n, d_n\}$), y_n converges to some weak solution y by the next way:

$$\begin{aligned} y_n &\to y \quad \text{weakly in } L^2(0,T;V), \\ y_n &\to y \quad \text{weakly star in } L^\infty(0,T;H), \\ d_n &\to d \quad \text{weakly in } L^2(0,T;V^*), \\ \frac{\partial y_n}{\partial t} &\to \frac{\partial y}{\partial t} \quad \text{weakly in } L^2(0,T;V^*). \end{aligned} \quad (3.32)$$

From 3°, $y(0) = \xi_{T_0}$, $y(T_0) = \xi$.

Since the sequence $\{\beta_n\}$ is bounded, then

$$\forall\, n \quad e^{-T_0}\|\beta_n\|^2 \le e^{-T_0} M. \quad (3.33)$$

Further,

$$\varlimsup_{n\to\infty}\left(-2\int_0^{T_0} e^{-(T_0-s)}\|\nabla y_n\|^2\,ds\right) \le -2\int_0^{T_0} e^{-(T_0-s)}\|\nabla y\|^2\,ds. \quad (3.34)$$

On the other hand,

$$\int_0^{T_0} e^{-(T_0-s)}\|y_n\|^2\,ds = \int_0^{T_0}\int_{\Omega_k} e^{-(T_0-s)}|y_n|^2\,dx\,ds \\ + \int_0^{T_0} e^{-(T_0-s)}\int_{|x|\ge k}|y_n|^2\,dx\,ds.$$

From 1° it follows, that $y_n(s) \in G(s, G(t_n - T_0, v_n)) = G(s + t_n - T_0, v_n)$. From 2°, for any $\varepsilon > 0$ there exist such $T(\varepsilon, B)$, $K_1(\varepsilon, B) > 0$, that

$$\int\limits_{|x|\geq k} |y_n(s)|^2\, dx \leq \varepsilon,$$

as soon as $k \geq K_1$, $t_n - T_0 \geq T$. Repeating corresponding suppositions from the proof of the Theorem 3.1, we have, that (up to subsequence) $L_k y_n \to L_k y$ strongly in $L^2(0, T; H_k)$. So,

$$\overline{\lim_{n\to\infty}} \int\limits_0^{T_0} e^{-(T_0 - s)} \|y_n\|^2\, ds \leq \int\limits_0^{T_0} e^{-(T_0 - s)} \|y\|^2\, dx\, ds + \varepsilon. \tag{3.35}$$

Let us consider the "nonlinear term" from (3.25). At first we remark, that from α_3 it follows that

$$-2 \int\limits_0^{T_0} \int\limits_{|x|\geq k} e^{-(T_0 - s)} d_n \cdot y_n\, dx\, ds \leq 4\varepsilon \int\limits_0^{T_0} e^{-(T_0 - s)}\, ds \leq 4\varepsilon,$$

as soon as $k \geq K_2(\varepsilon)$. Since $y_n \to y$ strongly in $L^2(0, T; H_k)$, then up to the subsequence, $y_n(t, x) \to y(t, x)$ for a.e. $(t, x) \in (0, T_0) \times \Omega_k$. From the Lemma 2.1 and (3.32) we have

$$\lim_{n\to\infty} \left(-2 \int\limits_0^{T_0} \int\limits_{\Omega_k} e^{-(T_0 - s)} d_n \cdot y_n\, dx\, ds \right) = -2 \int\limits_0^{T_0} \int\limits_{\Omega_k} e^{-(T_0 - s)} d \cdot y\, dx\, ds.$$

Thus,

$$\overline{\lim_{n\to\infty}} \left(-2 \int\limits_0^{T_0} \int\limits_{\Omega_k} e^{-(T_0 - s)} d_n \cdot y_n\, dx\, ds \right) \leq -2 \int\limits_0^{T_0} \int\limits_{\Omega_k} e^{-(T_0 - s)} d \cdot y\, dx\, ds + 4\varepsilon. \tag{3.36}$$

Passing to the limit as $k \to \infty$ in (3.36) and using (3.31) and (3.33)–(3.36) we find, that

$$\overline{\lim_{n\to\infty}} \|\xi_n\|^2 \leq e^{-T_0} M - 2 \int\limits_0^{T_0} \int\limits_{\mathbb{R}^N} e^{-(T_0 - s)} |\nabla y|^2\, dx\, ds$$
$$+ \int\limits_0^{T_0} \int\limits_{\mathbb{R}^N} e^{-(T_0 - s)} |y|^2\, dx\, ds - 2 \int\limits_0^{T_0} \int\limits_{\mathbb{R}^N} e^{-(T_0 - s)} d \cdot y\, dx\, ds + 5\varepsilon$$

$$= \|\xi\|^2 + e^{-T_0}M - e^{-T_0}\|\xi_{T_0}\|^2 + 5\varepsilon. \quad (3.37)$$

Passing to the limit as $T_0 \to +\infty$, and then directing $\varepsilon \to 0$, we will obtain the next inequality

$$\overline{\lim_{n\to\infty}} \|\xi_n\|^2 \le \|\xi\|^2.$$

5° Let us prove the semi-continuity of m-semiflow G [14, p. 126], namely, let us prove, that the map $G(t,\cdot)$ is upper semi-continuous and has compact values for any $t \ge 0$. Indeed, let $\xi_n \in G(t,x_n)$ and $x_n \to x_0$. Let us prove, that the sequence ξ_n is pre-compact in H. From the Lemma 3.2, the sequence ξ_n is bounded, so, up to the subsequence it is weakly convergent to some ξ. Supposing analogically to the proof of 4°, there exist weak solutions $y_n(\cdot)$, $y(\cdot)$ such, that $y_n(t) = \xi_n$, $y_n(0) = x_n$, $y(t) = \xi$, $y(0) = x_0$ and y_n converges to y in the sense of (3.32). Repeating the suppositions from 4° we will obtain, that $\overline{\lim}_{n\to\infty} \|\xi_n\|^2 \le \|\xi\|^2$. Thus, $\xi_n \to \xi$ strongly in H. So, taking into account 3°, $G(t,x_0)$ is compact.

Now, if $G(t,\cdot)$ is not upper semi-continuous, then there exists the point x_0, the neighborhood $\mathcal{O}$ of the set $G(t,x_0)$ and the sequence $\xi_n \in G(t,x_n)$ such, that $\|x_n - x_0\| \to 0$ as $n \to +\infty$ and $\xi_n \notin \mathcal{O}\ \forall n$. Passing to the subsequences we have, that $\xi_{n_k} \to \xi$, $x_{n_k} \to x_0$ strongly in H. From 3° it follows, that $\xi \in G(t,x_0)$. We obtained the contradiction.

From the properties 1°–5°, there exists the global compact invariant attractor for G (see [13, Theorem 3, Remark 8]), that is minimal closed absorbing set. Thus, the Theorem 3.2 is proved. ☐

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