Український математичний вісник
Том 5 **(2008)**, № 3, 382 – 393

Optimal discretization for ill-posed integral equations with finitely smoothing operators

Sergey G. Solodky

(*Presented by A. M. Samoilenko*)

Abstract. A new approach to the approximate solution of Fredholm integral equations of the first kind with finitely smoothing operators is worked out. It is established that on wide classes of such equations this approach allows to achieve the given level of accuracy at the minimal expense of the discrete information.

2000 MSC. 65J20, 47A52.

Key words and phrases. Information complexity, ill-posed problems, optimization, regularization method, discretization, projection scheme.

1. Introduction

In a real Hilbert space $L_2 = L_2(0,1)$ with the usual inner product $(f,g) = \int_0^1 f(t)g(t)\,dt$ and the norm $\|f\| = \left(\int_0^1 f^2(t)\,dt\right)^{1/2}$ we shall consider a Fredholm integral equation of the first kind

$$Ax(t) := \int\limits_0^1 h(t,\tau)x(\tau)\,d\tau = f(t). \tag{1.1}$$

We assume that at some fixed $r = 1, 2, \ldots$ the integral operators A and A^* act from L_2 into the Sobolev space W_2^r of r times differentiable functions, where A^* is the adjoint operator of A, and the norm in W_2^r is defined as $\|f\|_{W_2^r} := \|f\| + \sum_{i=1}^r \|d^i f(t)/dt^i\|$. Besides let the kernel $h(t,\tau)$ of A be non-degenerate and for any $\|g\| \le 1$ it holds that

$$\left(\sum_{j=0}^{r} \left\| \int\limits_0^1 \frac{\partial^j h(\tau,t)}{\partial \tau^j} g(\tau)\,d\tau \right\|_{W_2^r}^2 \right)^{1/2} \le \gamma.$$

Received 30.09.2008

ISSN 1810 – 3200.

We denote the class of such operators A by $\mathcal{A}_\gamma^r$. Take an arbitrary real number ν_1, where $2 < \nu_1 < \infty$. Suppose that the interval $[2, \nu_1]$ contains such point ν (the value of ν is unknown) that

$$f \in AM_{\nu,\rho}(A) := \{g : \ g = Au, \ u \in M_{\nu,\rho}(A)\},$$

where $M_{\nu,\rho}(A) := \{u : \ u = |A|^\nu v, \ \|v\| \le \rho\}$, $|A| = (A^*A)^{1/2}$, $\rho > 0$ is known. Assume that instead of f we are given an approximation $f_\delta \in L_{2,\delta}$, where $L_{2,\delta}$ is a sphere of radius δ in L_2 with its centre at f. By Ψ_ν^δ we denote the class of Eqs. (1.1) with the operators $A \in \mathcal{A}_\gamma^r$, the right-hand sides $f \in AM_{\nu,\rho}(A)$ and with the perturbations f_δ filling the sphere $L_{2,\delta}$.

We study a problem of optimal recovery of Eqs. (1.1) solutions from the class

$$\Psi^\delta = \bigcup_{\nu \in [2,\nu_1]} \Psi_\nu^\delta.$$

In so doing we shall construct the approximations to the solutions $x^\dagger$ of (1.1) from $M_{\nu,\rho}(A)$ at all $\nu \in [2, \nu_1]$. Hereinafter, an optimal method for solving (1.1) is called a method that retains the given level of accuracy at minimal expenses of certain computational resources. By computational resources we shall understand a discrete information on (1.1) in the form of values of the functionals of a special kind (see (2.2)). At present similar studies are being intensively carried out in the framework of the Information Based Complexity Theory (see [1]) for a wide range of mathematical problems. In particular, for many classes of the 2-nd kind Fredholm integral equations their complexity is found and the corresponding optimal methods are constructed (see, for example, [2]). As to the 1-st kind Fredholm equations, the investigation of complexity of such equations was initiated in [3] for $f \in AM_{\nu,\rho}(A)$ in a case, when the value of ν is exactly known. The aim of the present article is to continue the indicated research on the case of an unknown ν.

2. Statement of the problem

Let $E = \{e_1, e_2, \ldots, e_m, \ldots\}$ be an arbitrary orthonormal basis of L_2 and P_m an orthogonal projector on the linear span of $e_1, e_2, \ldots, e_m$, i.e.

$$P_m g = \sum_{i=1}^m (e_i, g) e_i.$$

It is known that an arbitrary linear continuous operator $A : L_2 \to L_2$ can be presented by the following infinite matrix

$$Ag = \sum_{i,j=1}^{\infty} (e_i, Ae_j)(e_j, g)e_i.$$

In the coordinate plane of E we take any bounded set $\Omega \subset [1,\infty)\times[1,\infty)$. Denote $\omega = \{i : (i,j) \in \Omega\}$. A projection discretization scheme (Ω, E) of Eq. (1.1) is called the passage from the input data A and f_δ of the initial problem to the elements

$$A_\Omega g = \sum_{(i,j)\in\Omega} (e_i, Ae_j)(e_j, g)e_i,$$

$$P_\omega f_\delta = \sum_{k\in\omega} (e_k, f_\delta)e_k.$$

Obviously, with various sets Ω and bases E it is possible to construct any possible projection discretization schemes (Ω, E), that use as a discrete information the inner products

$$(e_i, Ae_j), \quad (e_k, f_\delta), \qquad (i,j) \in \Omega, \ \ k \in \omega. \tag{2.2}$$

The set of numbers (2.2) is referred to as Galerkin information on the Eq. (1.1). As card(Ω) we denote the total number of inner products (2.2) involved in the scheme (Ω, E).

By a projection method for solving (1.1) we mean an arbitrary operator φ that assigns to the Galerkin information (2.2) an element $\varphi(\Omega, E, A, f_\delta) \in L_2$, which is taken as the approximate solution of (1.1). Furthermore, $\varphi(\Omega, E, A, f_\delta)$ is uniquely determinated by means of a finite number of numerical parameters. By $\Phi(\Omega, E)$ we understand a set of various possible projection methods φ, that apply the discretization scheme (Ω, E). Then $\Phi = \bigcup_{\Omega,E} \Phi(\Omega, E)$ means a set of all projection methods for solving (1.1). Here the union is executed over all orthonormal bases E in L_2 and bounded sets Ω of the corresponding coordinate planes.

The accuracy of the method $\varphi \in \Phi(\Omega, E)$ on the class Ψ_ν^δ is characterized by a maximal error

$$\mathcal{E}(\Psi_\nu^\delta, \varphi, \Omega, E) = \sup_{A\in\mathcal{A}_\gamma^r} \ \sup_{f\in AM_{\nu,\rho}(A)} \ \sup_{f_\delta : \|f-f_\delta\|\le\delta} \|x^\dagger - \varphi(\Omega, E, A, f_\delta)\|.$$

It is known [4] that on a class of Eqs. (1.1) with solutions from $M_{\nu,\rho}(A)$ and perturbed right-hand sides filling the sphere $L_{2,\delta}$ no approximate method (not necessarily projection method) can at best guarantee the accuracy of the recovery less than $\rho^{1/(\nu+1)}\delta^{\nu/(\nu+1)}$. Therefore, the value $O(\delta^{\nu/(\nu+1)})$ determines the optimal order of accuracy on the class Ψ_ν^δ.

Since the efficiency of an approximate method is characterized first of all by its accuracy on the class of problems under investigation, it is reasonable to separate from all sets of projection methods those, which attain the best order of accuracy on the class Ψ^δ. In other words, we shall study a subset $\Phi_{\text{opt}}(\Psi^\delta) \subset \Phi$ of such projection methods φ that at any $\nu \in [2, \nu_1]$ it holds that

$$\mathcal{E}(\Psi_\nu^\delta, \varphi, \Omega, E) \leq \xi \delta^{\nu/(\nu+1)}, \tag{2.3}$$

where the constant $\xi > 0$ does not depend on δ. Suppose that ξ is selected such that $\Phi_{\text{opt}}(\Psi^\delta)$ is not empty.

By $\Phi_{\text{opt}}(\Psi^\delta)_N$ we denote a set of all projection methods from $\Phi_{\text{opt}}(\Psi^\delta)$, satisfying the condition $\text{card}(\Omega) \leq N$. By information complexity of Eqs. (1.1) from Ψ^δ we understand the quantity

$$\text{Card}(\Psi^\delta) = \min \left\{ N : \ \Phi_{\text{opt}}(\Psi^\delta)_N \neq \emptyset \right\}.$$

This quantity determines the minimal volume of the discrete information (2.2), through which an optimal order of accuracy on the class Ψ^δ may be achieved. Our goal is to calculate exact orders of the quantity $\text{Card}(\Psi^\delta)$ at any $r = 1, 2, \ldots$.

The lower bound for $\text{Card}(\Psi^\delta)$ can be obtained from the previous results. With this purpose we shall consider the minimal radius of the Galerkin information (2.2) on the class Ψ_ν^δ, which is defined as

$$r_N(\Psi_\nu^\delta) = \inf_{E} \inf_{\substack{\Omega, \\ \text{card}(\Omega) \leq N}} \inf_{\varphi \in \Phi(\Omega, E)} \mathcal{E}(\Psi_\nu^\delta, \varphi, \Omega, E).$$

The value of $r_N(\Psi_\nu^\delta)$ characterizes a minimal error of the approximate solution of Eqs. (1.1) from Ψ_ν^δ, which can be guaranteed by using no more than N of Galerkin functionals (2.2). It should be noted that the quantity $r_N(\Psi_\nu^\delta)$ was first studied in [3] at the exactly known value of $\nu = 2$. The findings of the paper [3] were generalized in [5] on the case of an arbitrary known parameter $1 < \nu < \infty$. From these results it follows that at any $\nu_1 > 2$ and $N = O(\delta^{-\frac{\nu_1}{(\nu_1+1)r}})$ the bound

$$r_N(\Psi^\delta_{\nu_1}) = O(N^{-r}) = O(\delta^{\nu_1/(\nu_1+1)}) \tag{2.4}$$

is valid. With the help of this estimation it is easy to obtain the lower bound for the information complexity of the Eqs. (1.1). Namely, in the definitions of $\Phi_{\text{opt}}(\Psi^\delta)$, $\Phi_{\text{opt}}(\Psi^\delta)_N$, $\text{Card}(\Psi^\delta)$ we insert the class $\Psi^\delta_{\nu_1}$ instead of Ψ^δ. Thus, we obtain the quantity $\text{Card}(\Psi^\delta_{\nu_1})$ which is equal to the minimal volume of the Galerkin functionals (2.2) needed to attain the accuracy $\xi\delta^{\nu_1/(\nu_1+1)}$ on the class $\Psi^\delta_{\nu_1}$. Then by virtue of the definition of quantities $r_N(\Psi^\delta_\nu)$, $\text{Card}(\Psi^\delta)$ and $\text{Card}(\Psi^\delta_{\nu_1})$ from (2.4) it follows that

$$O(\delta^{-\frac{\nu_1}{(\nu_1+1)r}}) = \text{Card}(\Psi^\delta_{\nu_1}) \le \text{Card}(\Psi^\delta). \tag{2.5}$$

Thus, to obtain the exact order bound of $\text{Card}(\Psi^\delta)$ it is enough to construct at least one method from $\Phi_{\text{opt}}(\Psi^\delta)$ which supports (2.5) by means of the corresponding upper estimation of its accuracy.

3. Proposed approach to solve equations from Ψ^δ

The present section will introduce a set of projection methods which guarantee the attainment of the optimal order of accuracy on the class Ψ^δ of the Eqs. (1.1).

By virtue of the given assumptions about the operator A, made in Section 1, it is true that $\text{Range}(A) \neq \overline{\text{Range}(A)}$. It is known ([4]) that in this case the problem (1.1) is ill-posed and to ensure stable approximations it is required to apply special regularization methods (see [6]). In view of the problem formulated above we restrict ourselves by the study of such regularization methods with the help of which it is possible to achieve the best accuracy of approximations on the class of equations under investigation. Following [7], as a regularization method we use an operator $R_\alpha = R_\alpha(A) : L_2 \to L_2$ such that as an approximate solution of (1.1) one takes the element $R_\alpha(A)f_\delta$, where the number $\alpha > 0$ is referred to as regularization parameter and R_α is of the form

$$R_\alpha(A) = g_\alpha(A^*A)A^*. \tag{3.6}$$

Here $g_\alpha(\lambda)$ is a Borel measurable function on $[0,\infty)$ which satisfies the following conditions

$$\sup_{0\le\lambda<\infty} \lambda^\nu |1 - \lambda g_\alpha(\lambda)| \le \chi_\nu \alpha^\nu, \qquad 0 \le \nu \le \nu_*, \tag{3.7}$$

$$\sup_{0\le\lambda<\infty} \lambda^{1/2} |g_\alpha(\lambda)| \le \chi_* \alpha^{-1/2}, \tag{3.8}$$

where

$$\nu_1 \leq 2\nu_* - 1 \tag{3.9}$$

and ν_* is called a qualification of the method R_α and χ_ν, χ_* are some positive constants ($\chi_0 = 1$) independent of α. The qualification ν_* characterizes the greatest value of ν, at which the method R_α guarantees attainment of the optimal order of accuracy. Note that many known regularization methods satisfy (3.6)–(3.9) at various ν_*; for example, the Showalter method and the nonstationary iterated Tikhonov method (with any $\nu_* < \infty$), the iterative Landweber and Fakeev–Lardy methods (with $\nu_* = \infty$), etc. The set of all methods R_α satisfying (3.6)–(3.9) is denoted by $\mathcal{R}$.

Let us pass to the description of a projection discretization scheme that will be used in solving (1.1). As Ω we take a set of the coordinate plane of the following form

$$\Gamma_n := \{1\} \times [1, 2^{2n}] \bigcup_{k=1}^{n} (2^{k-1}, 2^k] \times [1, 2^{2n-ak}],$$

where $a = (3\nu_1 - 2)/(2\nu_1)$, $1 < a < 3/2$. Then the proposed projection discretization scheme consists in replacing the coefficients $A \in \mathcal{A}_\gamma^r$ and f_δ of the initial problem by their finite-dimensional analogues

$$\begin{aligned} A_n = A_{\Gamma_n} &:= \sum_{k=1}^{n} (P_{2^k} - P_{2^{k-1}}) A P_{2^{2n-ak}} + P_1 A P_{2^{2n}}, \\ P_{2^n} f_\delta &= \sum_{k=1}^{2^n} (f_\delta, \hat{e_k}) \hat{e_k}, \end{aligned} \tag{3.10}$$

where $\hat{E} = \{\hat{e_1}, \hat{e_2}, \ldots, \hat{e_m}, \ldots\}$ is an orthonormal basis of the space L_2, ensuring the order-optimal approximation to all functions from the space W_2^r by their partial Fourier sums. By virtue of the definition of $\mathcal{A}_\gamma^r$ the last condition on $\hat{E}$ means that at any $m = 1, 2, \ldots$ and some $\beta_r > 0$ it holds that

$$\|(I - P_m)A\| \leq \beta_r m^{-r}, \qquad \|A(I - P_m)\| \leq \beta_r m^{-r}, \tag{3.11}$$

where $A \in \mathcal{A}_\gamma^r$. As an example of the basis satisfying (3.11) we can recall the orthonormal system of Legendre's polynomials considered on the interval $[0, 1]$. If $2n - ak$ is not an integer, we will write $P_{2^{2n-ak}}$ but mean $P_{[2^{2n-ak}]}$ where $[g]$ is the integer part of g.

The proposed approach to solving (1.1) is as follows. In the framework of a projection method φ we take an arbitrary regularization method

$R_\alpha \in \mathcal{R}$ and perform the discretization according to (3.10), (3.11), where the parameter n is calculated in accordance with the condition

$$2^{-2(\nu_1+1)rn/\nu_1} \le \frac{\delta}{c_1} < 2^{-2(\nu_1+1)r(n-1)/\nu_1} \tag{3.12}$$

with $c_1 = \frac{\gamma\bar{\gamma}2^{2ar+1}}{2^{(2a-1)r}-1}\beta_r^3\rho$, $\bar{\gamma} = \max\{1, \gamma^{\nu_1}\}$. By an approximate solution $\varphi_\alpha = \varphi_\alpha(\Gamma_n, \hat{E}, A, f_\delta)$ we understand the element

$$\varphi_\alpha = R_\alpha(A_n)P_{2^n}f_\delta, \tag{3.13}$$

where the regularization parameter α is chosen according to the discrepancy principle [8], i.e. the computational procedure (3.13) terminates as soon as is executed

$$b_1\delta \le \|P_{2^n}f_\delta - A_n\varphi_\alpha\| \le b_2\delta, \quad 2 < b_1 \le b_2. \tag{3.14}$$

By $\Phi(\mathcal{R}, \Gamma_n, \hat{E}, b_1, b_2)$ we denote the set of all methods described above, i.e. of all projection methods satisfying (3.6)–(3.14).

4. Information complexity

In what follows we shall need some approximate properties of the operator A_n.

Lemma 4.1. *Let $A \in \mathcal{A}_\gamma^r$, $x^\dagger \in M_{\nu,\rho}(A)$, $\nu \in [2, \nu_1]$. If the discretization parameter n is chosen according to* (3.12)*, then*

$$\|A_n x^\dagger - P_{2^n}f_\delta\| \le 2\delta,$$

$$\| A^*A - A_n^*A_n\| \le c_2 2^{-2rn} \le c_3\delta^{\nu_1/(\nu_1+1)},$$

where $c_2 = \gamma^2\big(\beta_r^2 + \frac{\beta_r^3 2^{2r+1}}{1-2^{(a-2)r}}\big)$, $c_3 = c_2/c_1^{\nu_1/(\nu_1+1)}$.

Proof. Using Lemma 2 [9] and the relation (3.12), we obtain

$$\begin{aligned}\|A_n x^\dagger - P_{2^n}f_\delta\| &\\ &\le \|A_n x^\dagger - P_{2^n}f\| + \|P_{2^n}(f - f_\delta)\| \\ &\le c_1 2^{-2(\nu_1+1)rn/\nu_1} + \delta \le 2\delta.\end{aligned}$$

The bound of the norm of $A^*A - A_n^*A_n$ is given by Lemma 1 [9]. □

Theorem 4.1. *Within the framework of any method from $\Phi(\mathcal{R}, \Gamma_n, \hat{E}, b_1, b_2)$ the optimal order of the accuracy $O(\delta^{\nu/(\nu+1)})$ is achieved on the class Ψ^δ.*

Proof. To establish the present statement we use the technique applied earlier in the proof of Theorem 3.3 [10] and Theorem 4.1 [11]. One can write a representation for the error of the arbitrary method $\varphi \in \Phi(\mathcal{R}, \Gamma_n, \hat{E}, b_1, b_2)$ as

$$x^\dagger - \varphi_\alpha = R_{\alpha,n}(A_n x^\dagger - P_{2^n} f_\delta) + S_{\alpha,n} x^\dagger,$$

where $R_{\alpha,n} = g_\alpha(A_n^* A_n)A_n^*$, $S_{\alpha,n} = I - g_\alpha(A_n^* A_n)A_n^* A_n$. From (3.8) one obtains

$$\| R_{\alpha,n}\| = \|g_\alpha(A_n^* A_n)A_n\| \le \chi_* \alpha^{-1/2}.$$

Then follows the error estimation

$$\| x^\dagger - \varphi_\alpha\| \le \|S_{\alpha,n} x^\dagger\| + 2\chi_* \alpha^{-1/2}\delta. \tag{4.15}$$

Consider the following element

$$\begin{aligned} A_n S_{\alpha,n} x^\dagger &= (A_n - A_n R_{\alpha,n} A_n)x^\dagger = A_n x^\dagger - A_n R_{\alpha,n} A_n x^\dagger \\ &= (P_{2^n} f_\delta - A_n \varphi_\alpha) + (I - A_n R_{\alpha,n})(A_n x^\dagger - P_{2^n} f_\delta). \end{aligned} \tag{4.16}$$

Using (3.7), we find

$$\| I - A_n R_{\alpha,n}\| = \|I - g_\alpha(A_n A_n^*)A_n A_n^*\| \le \sup_{0\le\lambda<\infty} |1 - \lambda g_\alpha(\lambda)| \le 1.$$

Hence with the help of (3.14) and (4.16) we have

$$b_1\delta - \| A_n x^\dagger - P_{2^n} f_\delta\| \le \| A_n S_{\alpha,n} x^\dagger\| \le b_2\delta + \| A_n x^\dagger - P_{2^n} f_\delta\|. \tag{4.17}$$

By virtue of Lemma 4.1 it follows from the left-hand side of (4.17)

$$\begin{gathered} b_1\delta - 2\delta \le \| A_n S_{\alpha,n} x^\dagger\|, \\ \alpha^{-1/2}\delta \le \alpha^{-1/2}(b_1 - 2)^{-1}\| A_n S_{\alpha,n} x^\dagger\|. \end{gathered} \tag{4.18}$$

To estimate the norm of $A_n S_{\alpha,n}$ we apply the polar decomposition $A_n = U(A_n^* A_n)^{1/2}$, $\|U\| = 1$, then

$$A_n S_{\alpha,n} = U(I - |A_n| g_\alpha(|A_n|^2)|A_n|)|A_n| = U S_{\alpha,n}|A_n|.$$

Further,

$$\begin{aligned} \alpha^{-1/2}\| A_n S_{\alpha,n} x^\dagger\| &= \alpha^{-1/2}\| A_n S_{\alpha,n}|A|^\nu v\| \\ &\le \rho\alpha^{-1/2}\| A_n S_{\alpha,n}|A_n|^\nu\| + \| A_n S_{\alpha,n}(|A|^\nu - |A_n|^\nu)\| \\ &\le \rho\alpha^{-1/2}\| U S_{\alpha,n}|A_n|^{\nu+1}\| + \|S_{\alpha,n}|A_n|\,\|\|(A^*A)^{\nu/2} - (A_n^* A_n)^{\nu/2}\|. \end{aligned}$$

The application of Lemmas 4.1 [10], 4.1 and (3.7) gives us

$$\begin{aligned}\alpha^{-1/2}\| A_n S_{\alpha,n} x^\dagger \| &\le \rho\alpha^{-1/2}\|S_{\alpha,n}|A_n|^{\nu+1}\| + z(\nu)\|S_{\alpha,n}|A_n|\|\|A^*A - A_n^*A_n\| \\ &\le \rho\chi_{\frac{\nu+1}{2}}\alpha^{\nu/2} + \rho\chi_{1/2}c_3 z(\nu)\delta^{\nu/(\nu+1)}, \quad (4.19)\end{aligned}$$

where $z(\nu)$ is a function bounded on $(0,\infty)$.

Similarly we estimate

$$\begin{aligned}\| S_{\alpha,n}x^\dagger\| = \| S_{\alpha,n}|A|^\nu v\| &\le \rho\|S_{\alpha,n}|A_n|^\nu\| + \rho\|S_{\alpha,n}\|\||A|^\nu - |A_n|^\nu\| \\ &\le \rho\chi_{\nu/2}\alpha^{\nu/2} + \rho z(\nu)c_3\delta^{\nu/(\nu+1)}. \quad (4.20)\end{aligned}$$

Substituting (4.19) into (4.18) we find

$$\alpha^{-1/2}\delta \le (b_1-2)^{-1}(\rho\chi_{\frac{\nu+1}{2}}\alpha^{\nu/2} + \rho\chi_{1/2}c_3 z(\nu)\delta^{\nu/(\nu+1)}).$$

Using (4.15) and the above bounds for values $\alpha^{-1/2}\delta$ and $\|S_{\alpha,n}x^\dagger\|$, we obtain

$$\begin{aligned}\| x^\dagger - \varphi_\alpha\| &\le \rho(\chi_{\nu/2}\alpha^{\nu/2} + z(\nu)c_3\delta^{\nu/(\nu+1)} \\ &\quad + 2(b_1-2)^{-1}\chi_*(\chi_{\frac{\nu+1}{2}}\alpha^{\nu/2} + \chi_{1/2}c_3 z(\nu)\delta^{\nu/(\nu+1)}) \\ &= c_4\alpha^{\nu/2} + c_5\delta^{\nu/(\nu+1)},\end{aligned}$$

where $c_4 = \rho(\chi_{\nu/2} + 2\chi_*\chi_{\frac{\nu+1}{2}}(b_1-2)^{-1})$, $c_5 = \rho z(\nu)c_3(2\chi_*\chi_{1/2}(b_1-2)^{-1} + 1)$. Obviously, at $\alpha \le \delta^{2/(\nu+1)}$ it is true that

$$\|x^\dagger - \varphi_\alpha\| \le (c_4 + c_5)\delta^{\nu/(\nu+1)}.$$

Thus in this case Theorem 4.1 is proved.

For arbitrary $\alpha_1 > 0$ we take any function g_α satisfying (3.7)–(3.9). It is known (see Lemma 3.2 [10]) that there is a constant $c_* > 0$ such that for all $0 \le \lambda < \infty$ and $\alpha \ge \alpha_1$

$$(1-\lambda g_\alpha(\lambda))^2 \le c_*(1-\lambda g_{\alpha_1}(\lambda))^2 + \alpha_1^{-1}(\lambda(1-\lambda g_\alpha(\lambda)))^2.$$

Then

$$\| S_{\alpha,n}x^\dagger\|^2 \le c_*(\|S_{\alpha_1,n}x^\dagger\|^2 + \alpha_1^{-1}\|A_nS_{\alpha,n}x^\dagger\|^2). \quad (4.21)$$

Suppose $\alpha_1 = \delta^{2/(\nu+1)}$. Keeping in mind Lemma 4.1, from the right-hand side of (4.17) we get

$$\alpha_1^{-1}\|A_n S_{\alpha,n} x^\dagger\|^2 \le (b_2+2)^2 \delta^{2\nu/(\nu+1)}.$$

From (4.20) we obtain

$$\|S_{\alpha_1,n}x^\dagger\|^2 \le (\rho\chi_{\nu/2}\delta^{\nu/(\nu+1)} + \rho z(\nu) c_3 \delta^{\nu/(\nu+1)})^2 = \delta^{2\nu/(\nu+1)}(\rho\chi_{\nu/2} + \rho z(\nu)c_3)^2.$$

Substituting the above bounds for values $\alpha_1^{-1}\|A_n S_{\alpha,n}x^\dagger\|^2$ and $\|S_{\alpha_1,n}x^\dagger\|^2$ into (4.21) we find

$$\|S_{\alpha,n}x^\dagger\|^2 \le c_*\big((\rho\chi_{\nu/2} + \rho z(\nu)c_3)^2\delta^{2\nu/(\nu+1)} + (b_2+2)^2\delta^{2\nu/(\nu+1)}\big) = c_*\big((\rho\chi_{\nu/2} + \rho z(\nu)c_3)^2 + (b_2+2)^2\big)\delta^{2\nu/(\nu+1)}.$$

By virtue of $\alpha \ge \alpha_1 = \delta^{2/(\nu+1)}$ it is easy to see that

$$\alpha^{-1/2}\delta \le \alpha_1^{-1/2}\delta = \delta^{\nu/(\nu+1)}.$$

Then, we substitute into (4.15) the bounds for values $\alpha^{-1/2}\delta$ and $\|S_{\alpha,n}x^\dagger\|$

$$\|x^\dagger - \varphi_\alpha\| \le c_6 \delta^{\nu/(\nu+1)},$$

where $c_6 = \big(c_*(\rho\chi_{\nu/2} + \rho z(\nu)c_3)^2 + (b_2+2)^2\big)^{1/2} + 2\chi_*$.

We finally obtain that generally the following holds

$$\|x^\dagger - \varphi_\alpha\| \le c_7 \delta^{\nu/(\nu+1)}$$

with $c_7 = \max\{c_6, c_4 + c_5\}$. Thus, Theorem 4.1 is proved. □

Theorem 4.2. *At any* $r = 1, 2, \dots$

$$\mathrm{Card}(\Psi^\delta) \le O(\delta^{-\frac{\nu_1}{(\nu_1+1)r}}).$$

The exact order of $\mathrm{Card}(\Psi^\delta)$ *is retained within any method from the set* $\Phi(\mathcal{R}, \Gamma_n, \hat{E}, b_1, b_2)$.

Proof. We calculate the volume of Galerkin information (2.2) involved by the projection scheme $(\Gamma_n, \hat{E})$. Thus,

$$\mathrm{card}(\Gamma_n) = 2^n + 2^{2n}\left(1 + \frac{1}{2}\sum_{k=1}^{n}\left[2^{k(1-a)}\right]\right) = O(2^{2n}).$$

Having chosen the parameter n (3.12) we get

$$\operatorname{card}(\Gamma_n) = O(2^{2n}) = O(\delta^{-\frac{\nu_1}{(\nu_1+1)r}}). \qquad (4.22)$$

Since at $c_7 \le \xi$ any method from the set $\varphi \in \Phi(\mathcal{R}, \Gamma_n, \hat{E}, b_1, b_2)$ satisfies (2.3), then $\Phi(\mathcal{R}, \Gamma_n, \hat{E}, b_1, b_2) \subset \Phi_{\text{opt}}(\Psi^\delta)_N$ at $N = O(2^{2n})$ and

$$\operatorname{Card}(\Psi^\delta) \le \operatorname{card}(\Gamma_n).$$

The last inequality together with (4.22) gives us the upper bound for the quantity $\operatorname{Card}(\Psi^\delta)$. The corresponding lower bound is established by the relation (2.5). □

References

[1] J. F. Traub and H. Wozniakowski, *Theory of Optimal Algorithms*, Academic Press, New York, 1980.

[2] S. V. Pereverzev, *Optimization of Methods for Approximate Solution of Operator Equations*, Nova Science Publishers, New York, 1996.

[3] S. V. Pereverzev and S. G. Solodky, *The Minimal Radius of Galerkin Information for the Fredholm Problem of the First Kind* // J. Complexity **12** (1996), 176–202.

[4] H. W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Kluver Ac. Publ., Dordrecht-Boston-London, 1996.

[5] S. G. Solodky, *Information complexity of projection algorithms for the solution of Fredholm equations of the first kind. II* // Ukrainian Math. J. **50** (1998), 952–959.

[6] A. N. Tikhonov and V. A. Arsenin, *Solutions of ill-posed problems*, Wiley, New York, 1977.

[7] A. B. Bakushinski, *A general method of constructing regularizing algorithms for a linear ill-posed equation in Hilbert Space* // Zh. Vychisl. Mat. Mat. Fiz. **7** (1967), 672–677.

[8] V. A. Morozov, *On the solution of functional equations by the method of regularization* // Soviet Math. Dokl. **7** (1966), 414–417.

[9] S. G. Solodky, *A Generalized projection scheme for solving ill-posed problems* // J. Inverse Ill-Posed Probl. **7** (1999), 185–200.

[10] R. Plato and G. Vainikko, *On the Regularization of Projection Methods for solving Ill-posed Problems* // Numer. Math. **57** (1990), 63–79.

[11] S. V. Pereverzev and S. G. Solodky, *An Efficient Discretization for Solving Ill-Posed Problems*, in: The mathematics of numerical analysis (Park City, UT, 1995), Lectures in Appl. Math., **32**, Amer. Math. Soc., Providence, RI, (1996) pp. 643–649.

Contact information

Sergey G. Solodky
Institute of Mathematics,
National Academy of Sciences of Ukraine,
Tereshscenkivska Str. 3,
01601 Kiev,
Ukraine
E-Mail: solodky@imath.kiev.ua