Український математичний вісник
Том 5 **(2008)**, № 3, 305 – 326

General Beltrami equations and BMO

Bogdan V. Bojarski, Vladimir V. Gutlyanskiĭ,
Vladimir I. Ryazanov

Abstract. We study the Beltrami equations $\overline{\partial}f = \mu(z)\partial f + \nu(z)\overline{\partial f}$ under the assumption that the coefficients μ, ν satisfy the inequality $|\mu| + |\nu| < 1$ almost everywhere. Sufficient conditions for the existence of homeomorphic ACL solutions to the Beltrami equations are given in terms of the bounded mean oscillation by John and Nirenberg.

2000 MSC. 30C65, 30C75.

Key words and phrases. Degenerate Beltrami Equations, quasiconformal mappings, bounded mean oscillation.

1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$. We study the Beltrami equation

$$\overline{\partial}f = \mu(z)\partial f + \nu(z)\overline{\partial f} \text{ a.e. in } D \tag{1.1}$$

where $\overline{\partial}f = (f_x + if_y)/2$ and $\partial f = (f_x - if_y)/2$, $z = x + iy$, and μ and ν are measurable functions in D with $|\mu(z)| + |\nu(z)| < 1$ almost everywhere in D. Equation (1.1) arises, in particular, in the study of conformal mappings between two domains equipped with different measurable Riemannian structures, see [22]. Equation (1.1) and second order PDE's of divergent form are also closely related. For instance, given a domain D, let σ be the class of symmetric matrices with measurable entries, satisfying

$$\frac{1}{K(z)}|h|^2 \le \langle \sigma(z)h, h\rangle \le K(z)|h|^2, \quad h \in \mathbb{C}.$$

Assume that $u \in W^{1,2}_{\text{loc}}(D)$ is a week solution of the equation

$$\text{div}[\sigma(z)\nabla u(z)] = 0.$$

Received 25.09.2008

Consider the mapping $f = u + iv$ where $\nabla v(z) = J_f(z)\sigma(z)\nabla u(z)$ and $J_f(z)$ stands for the Jacobian determinant of f. It is easily to verify that f satisfies the Beltrami equation (1.1). On the other hand, the above second order partial differential equation naturally appears in a number of problems of mathematical physics, see, e.g., [3].

In the case $\nu(z) \equiv 0$ in (1.1) we recognize the classical Beltrami equation, which generates the quasiconformal mappings in the plane. Given an arbitrary measurable coefficient $\mu(z)$ with $\|\mu\|_\infty<1$ in $D \subseteq \mathbb{C}$, the well-known measurable Riemann mapping theorem for the Beltrami equation

$$\overline{\partial} f = \mu(z)\partial f \quad \text{a.e. in } D \subseteq \mathbb{C}, \tag{1.2}$$

see, e.g., [1, Chapter 5], [24, Chapter 5], guarantees the existence of a homeomorphic solution $f \in W^{1,2}_{\rm loc}(D)$ to the equation (1.2), which maps D onto an arbitrary conformally equivalent domain G. Moreover, the mapping f can be represented in the form $f = F\circ\omega$, where F stands for an arbitrary conformal mapping of D onto G and ω is a quasiconformal self-mapping of D with complex dilatation $\mu(z)$ a.e. in D. The corresponding measurable Riemann mapping theorem for the general Beltrami equation (1.1) was given in [5] and [6], Theorem 5.1 and Theorem 6.8, see also [38, Chapter 3, §17]. Bellow we give its statement in the form, convenient for our application.

Theorem 1.1. *Let B be the unit disk and G be a simply connected domain in $\mathbb{C}$. If μ and ν are measurable functions in B with $|\mu(z)|+|\nu(z)| \leq q < 1$ a.e. in B, then there exists a quasiconformal mapping $f : B \to G$, satisfying the equation* (1.1). *The mapping f has the representation $f = F \circ \omega$ where F stands for a conformal mapping of B onto G and the quasiconformal self-mapping ω of B can be normalized by $\omega(0) = 0$, $\omega(1) = 1$. If $F(\omega) \equiv \omega$, then the normalized solution is unique.*

Notice, that if f is a $W^{1,2}_{\rm loc}$ solution to (1.1), then f is also a solution to (1.2) where μ is replaced by $\tilde{\mu} = \mu + \nu\overline{\partial f}/\partial f$ if $\partial f \neq 0$ and $\tilde{\mu} = 0$ if $\partial f = 0$.

The case, when the assumption of strong ellipticity condition $|\mu(z)| + |\nu(z)| \leq q < 1$ is replaced by the assumption $|\mu(z)| + |\nu(z)| < 1$ a.e., the similar existence and uniqueness problem was not studied so far. Let us consider a couple of illustrative examples. Define the following Beltrami coefficients

$$\mu(z) = \left(\frac{|z|^2 - 1}{3|z|^2 + 1} + |z|\right) \cdot \frac{z}{\overline{z}}, \quad \nu(z) = -|z| \cdot \frac{z}{\overline{z}},$$

in the punctured unit disk $0 < |z| < 1$. Since $|\mu(z)| \to 1$ as $z \to 0$, we see that the equation $f_{\overline{z}} - \mu(z)\, f_z - \nu(z)\overline{f_z} = 0$ degenerates near the origin. It is easy to verify that the radial stretching

$$f(z) = (1+|z|^2)\frac{z}{|z|}, \quad 0 < |z| < 1.$$

satisfies the above equation and is a homeomorphic mapping of the punctured unit disk onto the annulus $1 < |w| < 2$. Thus, we observe the effect of cavitation. For the second example we choose

$$\mu(z) = \frac{i}{2}\frac{z}{\overline{z}}, \quad \nu(z) = \frac{i}{2}\frac{z}{\overline{z}}e^{2i\log|z|^2}.$$

In this case $|\mu(z)| + |\nu(z)| = 1$ holds for every $z \in \mathbb{C}$. In other words, we deal with "global" degeneration. However, the corresponding globally degenerate general Beltrami equation (1.1) admits the spiral mapping

$$f(z) = ze^{i\log|z|^2}$$

as a quasiconformal solution. The above observation shows, that in order to obtain existence or uniqueness results, some extra constraints must be imposed on μ and ν.

In this paper we give sufficient conditions for the existence of a homeomorphic ACL solution to the Beltrami equation (1.1), assuming that the degeneration of μ and ν is is controlled by a BMO function. More precisely we assume that the maximal dilatation function

$$K_{\mu,\nu}(z) = \frac{1+|\mu(z)|+|\nu(z)|}{1-|\mu(z)|-|\nu(z)|} \tag{1.3}$$

is dominated by a function $Q(z) \in$ BMO, where BMO stands for the class of functions with bounded mean oscillation in D, see [21].

Recall that, by John and Nirenberg in [21], a real-valued function u in a domain D in $\mathbb{C}$ is said to be of bounded mean oscillation in D, $u \in \mathrm{BMO}(D)$, if $u \in \mathrm{L}^1_{\mathrm{loc}}(D)$, and

$$\|u\|_* := \sup_B \frac{1}{|B|}\int\limits_B |u(z) - u_B|\, dx\, dy < \infty \tag{1.4}$$

where the supremum is taken over all discs B in D and

$$u_B = \frac{1}{|B|} \int\limits_B u(z)\, dx\, dy.$$

We also write $u \in \mathrm{BMO}$ if $D = \mathbb{C}$. If $u \in \mathrm{BMO}$ and c is a constant, then $u + c \in \mathrm{BMO}$ and $\|u\|_* = \|u + c\|_*$. The space of BMO functions modulo constants with the norm given by (1.4) is a Banach space. Note that $\mathrm{L}^\infty \subset BMO \subset L^p_{\mathrm{loc}}$ for all $p \in [1, \infty)$, see e.g. [21, 30]. Fefferman and Stein [13] showed that BMO can be characterized as the dual space of the Hardy space H^1. The space BMO has become an important concept in harmonic analysis, partial differential equations and related areas.

The case, when $\nu = 0$ and the degeneration of μ is expressed in terms of $|\mu(z)|$, has recently been extensively studied, see, e.g., [7–10, 16, 19, 20, 23, 25, 27, 32, 33, 36], and the references therein.

In this article, unless otherwise stated, by a solution to the Beltrami equation (1.1) in D we mean a sense-preserving homeomorphic mapping $f : D \to \mathbb{C}$ in the Sobolev space $W^{1,1}_{\mathrm{loc}}(D)$, whose partial derivatives satisfy (1.1) a.e. in D.

Theorem 1.2. *Let μ, ν be measurable functions in $D \subset \mathbb{C}$, such that $|\mu| + |\nu| < 1$ a.e. in D and*

$$\frac{1 + |\mu(z)| + |\nu(z)|}{1 - |\mu(z)| - |\nu(z)|} \le Q(z) \tag{1.5}$$

a.e. in D for some function $Q(z) \in BMO(\mathbb{C})$. Then the Beltrami equation (1.1) has a homeomorphic solution $f : D \to \mathbb{C}$ which belongs to the space $W^{1,s}_{\mathrm{loc}}(D)$ for all $s \in [1, 2)$. Moreover, this solution admits a homeomorphic extension to $\overline{\mathbb{C}}$ such that f is conformal in $\overline{\mathbb{C}} \setminus \overline{D}$ and $f(\infty) = \infty$. For the extended mapping $f^{-1} \in W^{1,2}_{\mathrm{loc}}$, and for every compact set $E \subset \mathbb{C}$ there are positive constants C, C', a and b such that

$$C \exp\left(-\frac{a}{|z' - z''|^2}\right) \le |f(z') - f(z'')| \le C' \left|\log \frac{1}{|z' - z''|}\right|^{-b} \tag{1.6}$$

for every pair of points $z', z'' \in E$ provided that $|z' - z''|$ is sufficiently small.

Remark 1.1. Note that C' is an absolute constant, b depends only on E and Q.

Remark 1.2. Prototypes of Theorem 1.2 when $\nu(z) \equiv 0$ can be found in the pioneering papers on the degenerate Beltrami equation [27] and [10], see also [32] and [36].

In [2] it was shown that a necessary and sufficient condition for a measurable function $K(z) \geq 1$ to be majorized in $D \subset \mathbb{C}$ by a function $Q \in \mathrm{BMO}$ is that

$$\iint\limits_D e^{\eta K(z)} \frac{dx\,dy}{1+|z|^3} < \infty \tag{1.7}$$

for some positive number η. Thus, the inequality (1.7) can be viewed as a test for $K_{\mu,\nu}(z)$ to satisfy the hypothesis of Theorem 1.2.

2. Auxiliary lemmas

For the proof of Theorem 1.2 we need the following lemmas.

Lemma 2.1. *Let $f_n : D \to \mathbb{C}$ be a sequence of homeomorphic ACL solutions to the equation* (1.1) *converging locally uniformly in D to a homeomorphic limit function f. If*

$$K_{\mu_n,\nu_n}(z) \leq Q(z) \in L^p_{\mathrm{loc}}(D) \tag{2.1}$$

a.e. in D for some $p > 1$, then the limit function f belongs to $W^{1,s}_{\mathrm{loc}}$ where $s = 2p/(1+p)$ and $\overline{\partial} f_n$ and ∂f_n converge weakly in $L^s_{\mathrm{loc}}(D)$ to the corresponding generalized derivatives of f.

Proof. First, let us show that the partial derivatives of the sequence f_n are bounded by the norm in L^s over every disk B with $\overline{B} \subset D$. Indeed,

$$|\overline{\partial} f_n| \leq |\partial f_n| \leq |\partial f_n| + |\overline{\partial} f_n| \leq Q^{1/2}(z) \cdot J_n^{1/2}(z)$$

a.e. in B and by the Hölder inequality and Lemma 3.3 of Chapter III in [24]

$$\|\partial f_n\|_s \leq \|Q\|_p^{1/2} \cdot |f_n(B)|^{1/2}$$

where $s = 2p/(1+p)$, J_n is the Jacobian of f_n and $\|\cdot\|_p$ denotes the L^p-norm in B.

By the uniform convergence of f_n to f in $\overline{B}$, for some $\lambda > 1$ and large n, $|f_n(B)| \leq |f(\lambda B)|$ and, consequently,

$$\|\partial f_n\|_s \leq \|Q\|_p^{1/2} \cdot |f(\lambda B)|^{1/2}.$$

Hence $f_n \in W^{1,s}_{\mathrm{loc}}$, see e.g. Theorem 2.7.1 and Theorem 2.7.2 in [26]. On the other hand, by the known criterion of the weak compactness in the space L^s, $s \in (1,\infty)$, see [12, Corollary IV.8.4], $\partial f_n \to \partial f$ and $\overline{\partial} f_n \to \overline{\partial} f$ weakly in L^s_{loc} for such s. Thus, f belongs to $W^{1,s}_{\mathrm{loc}}$ where $s = 2p/(1+p)$. □

Lemma 2.2. *Under assumptions of Lemma* 2.1*, if* $\mu_n(z) \to \mu(z)$ *and* $\nu_n(z) \to \nu(z)$ *a.e. in* D*, then the limit function* f *is a* $W^{1,s}_{\rm loc}$ *solution to the equation* (1.1) *with* $s = 2p/(1+p)$.

Proof. We set $\zeta(z) = \overline{\partial} f(z) - \mu(z)\,\partial f(z) - \nu(z)\,\overline{\partial f(z)}$ and, assuming that $\mu_n(z) \to \mu(z)$ and $\nu_n(z) \to \nu(z)$ a.e. in D, we will show that $\zeta(z) = 0$ a.e. in D. Indeed, for every disk B with $\overline{B} \subset D$, by the triangle inequality

$$\left| \int\limits_B \zeta(z)\,dx\,dy \right| \le I_1(n) + I_2(n) + I_3(n)$$

where

$$I_1(n) = \left| \int\limits_B \left(\overline{\partial} f(z) - \overline{\partial} f_n(z)\right)\,dx\,dy \right|,$$

$$I_2(n) = \left| \int\limits_B \left(\mu(z)\,\partial f(z) - \mu_n(z)\,\partial f_n(z)\right)\,dx\,dy \right|,$$

$$I_3(n) = \left| \int\limits_B \left(\overline{\nu(z)}\,\partial f(z) - \overline{\nu_n(z)}\,\partial f_n(z)\right)\,dx\,dy \right|.$$

By Lemma 2.1, $\overline{\partial} f_n$ and ∂f_n converge weakly in $L^s_{\rm loc}(D)$ to the corresponding generalized derivatives of f. Hence, by the result on the representation of linear continuous functionals in L^p, $p \in [1,\infty)$, in terms of functions in L^q, $1/p + 1/q = 1$, see [12, IV.8.1 and IV.8.5], we see that $I_1(n) \to 0$ as $n \to \infty$. Note that $I_2(n) \le I_2'(n) + I_2''(n)$, where

$$I_2'(n) = \left| \int\limits_B \mu(z)(\partial f(z) - \partial f_n(z))\,dx\,dy \right|$$

and

$$I_2''(n) = \left| \int\limits_B (\mu(z) - \mu_n(z))\partial f_n(z)\,dx\,dy \right|,$$

and we see that $I_2'(n) \to 0$ as $n \to \infty$ because $\mu \in L^\infty$. In order to estimate the second term, we make use of the fact that the sequence $|\partial f_n|$ is weakly compact in $L^s_{\rm loc}$, see e.g. [12, IV.8.10], and hence $|\partial f_n|$ is absolutely equicontinuous in $L^1_{\rm loc}$, see e.g. [12, IV.8.11]. Thus, for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\int_E |\partial f_n(z)|\,dx\,dy < \varepsilon, \quad n = 1, 2, \ldots,$$

whenever E is measurable set in B with $|E| < \delta$. On the other hand, by the Egoroff theorem, see e.g. [12, III.6.12], $\mu_n(z) \to \mu(z)$ uniformly on some set $S \subset \overline{B}$ such that $|E| < \delta$ where $E = B \setminus S$. Now $|\mu_n(z) - \mu(z)| < \varepsilon$ on S for large n and consequently

$$\begin{aligned} I_2''(n) \le \int_S & |\mu(z) - \mu_n(z)| \cdot |\partial f_n(z)|\,dx\,dy \\ &+ \int_E |\mu(z) - \mu_n(z)| \cdot |\partial f_n(z)|\,dx\,dy \\ &\le \varepsilon \int_B |\partial f_n(z)|\,dx\,dy + 2\int_E |\partial f_n(z)|\,dx\,dy \\ &\le \varepsilon \left(\|Q\|^{1/2} \cdot |f(\lambda B)|^{1/2} + 2 \right) \end{aligned}$$

for large enough n, i.e. $I_2''(n) \to 0$ as $n \to \infty$ because $\varepsilon > 0$ is arbitrary. The fact that $I_3(n) \to 0$ as $n \to \infty$ is handled similarly. Thus, $\int_B \zeta(z)\,dx\,dy = 0$ for all disks B with $\overline{B} \subset D$. By the Lebesgue theorem on differentiability of the indefinite integral, see e.g. [34, IV(6.3)], $\zeta(z) = 0$ a.e. in D. □

Remark 2.1. Lemma 2.1 and Lemma 2.2 extend the well known convergence theorem where $Q(z) \in L^\infty$, see Lemma 4.2 in [6], and [4].

Recall that a doubly-connected domain in the complex plane is called a ring domain and the modulus of a ring domain E is the number $\mod E$ such that E is conformally equivalent to the annulus $\{1 < |z| < e^{\mod E}\}$. We write $A = A(r, R; z_0)$, $0 < r < R < \infty$, for the annulus $r < |z - z_0| < R$.

Let Γ be a family of Jordan arcs or curves in the plane. A nonnegative and Borel measurable function ρ defined in $\mathbb{C}$ is called admissible for the family Γ if the relation

$$\int_\gamma \rho\,ds \ge 1 \tag{2.2}$$

holds for every locally rectifiable $\gamma \in \Gamma$. The quantity

$$M(\Gamma) = \inf_\rho \int_{\mathbb{C}} \rho^2\,dx, \tag{2.3}$$

where the infimum is taken over all ρ admissible for the family Γ is called the modulus of the family Γ, see, e.g., [1, p. 16], [24]. It is well known that this quantity is a conformal invariant. Moreover, in these terms the conformal modulus of a ring domain E has the representation, see, e.g., [24],

$$\operatorname{mod} E = \frac{2\pi}{M(\Gamma)} \tag{2.4}$$

where Γ is the family of curves joining the boundary components of E in E. Note also, that this modulus $M(\Gamma)$ coincides with the conformal capacity of E. Recall that the reciprocal to $M(\Gamma)$ is usually called the extremal length of Γ, however, in what follows, we will don't make use of this concept.

The next lemma deals with modulus estimates for quasiconformal mappings in the plane, cf. [17, 29].

Lemma 2.3. *Let $f : A \to \mathbb{C}$ be a quasiconformal mapping. Then for each nonnegative measurable functions $\rho(t)$, $t \in (r, R)$ and $p(\theta)$, $\theta \in (0, 2\pi)$, such that*

$$\int\limits_r^R \rho(t)\, dt = 1, \qquad \frac{1}{2\pi}\int\limits_0^{2\pi} p(\theta)\, d\theta = 1,$$

the following inequalities hold

$$2\pi\left[\frac{1}{2\pi}\iint\limits_A p^2(\theta) D_{-\mu,z_0}(z)\frac{dx\,dy}{|z-z_0|^2}\right]^{-1} \le M(f(\Gamma))$$

$$\le \iint\limits_A \rho^2(|z-z_0|) D_{\mu,z_0}(z)\, dx\,dy, \tag{2.5}$$

where Γ stands for the family of curves joining the boundary components of $A(r, R; z_0)$ in $A(r, R; z_0)$,

$$D_{\mu,z_0}(z) = \frac{\left|1-\mu(z)e^{-2i\theta}\right|^2}{1-|\mu(z)|^2} \tag{2.6}$$

and $\theta = \arg(z - z_0)$.

Remark 2.2. By (2.4), the inequalities (2.5) can be written in the following equivalent form

$$\left[\frac{1}{2\pi}\iint\limits_A \rho^2(|z-z_0|) D_{\mu,z_0}(z)\, dx\,dy\right]^{-1} \le \quad \operatorname{mod} f(A)$$

$$\leq \frac{1}{2\pi} \iint\limits_{A} p^2(\theta) D_{-\mu,z_0}(z) \frac{dx\,dy}{|z-z_0|^2}. \quad (2.7)$$

Proof. Let Γ be a family of curves joining the boundary components of $A = A(r, R; z_0)$ in A, and let ρ satisfies the assumption of the lemma. Denote by Γ^* the family of all rectifiable paths $\gamma^* \in f(\Gamma)$ for which f^{-1} is absolutely continuous on every closed subpath of γ^*. Then $M(f(\Gamma)) = M(\Gamma^*)$ by the Fuglede theorem, see e.g. [24, pp. 135 and 170].

Fix $\gamma^* \in \Gamma^*$ and let $\gamma = f^{-1} \circ \gamma^*$. Denote by s and s^* the natural (length) parameters of γ and γ^*, respectively. Note that the correspondence $s^*(s)$ between the natural parameters is strictly monotone function and we may assume that $s^*(s)$ is increasing. For $\gamma^* \in \Gamma^*$, the inverse function $s(s^*)$ has the (N) - property and $s^*(s)$ is differentiable a.e. as a monotone function. Thus, $ds^*/ds \neq 0$ a.e. on γ by [28]. Let s be such that $z = \gamma(s)$ is a regular point for f and suppose that $\gamma(s)$ is differentiable at s with $ds^*/ds \neq 0$. Set $r = |z - z_0|$ and let ω be the unit tangential vector to the curve γ at the point $z = \gamma(s)$. Then

$$\left|\frac{dr}{ds^*}\right| = \frac{dr}{ds} \Big/ \frac{ds^*}{ds} = \frac{|\langle \omega, \omega_0 \rangle|}{|\partial_\omega f(z)|}$$

where $\omega_0 = (z-z_0)/|z-z_0|$. Let now ρ satisfies the assumption of Lemma 2.3. Without loss of generality, by Lusin theorem, we can assume that ρ is a Borel function. We set

$$\rho^*(w) = \left\{ \rho(|z-z_0|) \left(\frac{D_{\mu,z_0}(z)}{J_f(z)} \right)^{1/2} \right\} \circ f^{-1}(w) \quad (2.8)$$

if f is differentiable and $D_{\mu,z_0}(z)/J_f(z) \neq 0$ at the point $z = f^{-1}(w)$ and $\rho^*(w) = \infty$ otherwise at $w \in f(A)$, and $\rho^*(w) = 0$ outside $f(A)$. Then ρ^* is also a Borel function and we will show that ρ^* is admissible for the family $\Gamma^* = f(\Gamma)$. Indeed, the function $z = \gamma(s(s^*))$ is absolutely continuous and hence so is $r = |z - z_0|$ as a function of the parameter s^*. Then

$$\int\limits_{\gamma^*} \rho^*\, ds^* = \int\limits_{\gamma^*} \left\{ \rho(|z-z_0|) \left(\frac{D_{\mu,z_0}(z)}{J_f(z)} \right)^{1/2} \right\} \circ f^{-1}(w)\, ds^*$$

$$\geq \int\limits_{\gamma^*} \rho(r) \left| \frac{dr}{ds^*} \right| ds^* \geq \int\limits_{r_1}^{r_2} \rho(r)\, dr = 1,$$

because of the inequality

$$\left(\frac{D_{\mu,z_0}(z)}{J_f(z)} \right)^{1/2} \circ f^{-1}(\gamma^*(s^*)) \geq \left| \frac{dr}{ds^*} \right|. \quad (2.9)$$

Let us verify the later inequality. Since $|dr/ds^*| = |\langle\omega, \omega_0\rangle|/|\partial_\omega f(z)|$ and $\partial_\omega f(z) = f_z(z)(1+\mu(z)\bar{\omega}^2)$, we see that

$$\min_{|\omega|=1} \frac{|\partial_\omega f(z)|}{|\langle\omega, \omega_0\rangle|} = 2|f_z| \cdot \min_{|w|=1} \left|\frac{w+a}{w+1}\right|,$$

where $\omega_0 = (z-z_0)/|z-z_0|$, $a = \mu\bar{\omega}_0^2$, $w = \omega^2\bar{\omega}_0^2$ and ω is an arbitrary unit vector. The Möbius mapping $\varphi(w) = \frac{w+a}{w+1}$ transforms the unit circle into a straight line with the unit normal vector $\vec{n} = \frac{a-1}{|a-1|} = (\varphi(0) - \varphi(1))/|\varphi(0) - \varphi(1))|$. Then the required distance from the straight line to the origin is calculated as $|\langle\varphi(1), \vec{n}\rangle|$. Hence,

$$\begin{aligned}\min_{|\omega|=1} \frac{|\partial_\omega f(z)|}{|\langle\omega, \omega_0\rangle|} &= 2|f_z| \left\langle \frac{1+a}{2}, \frac{1-a}{|1-a|} \right\rangle \\ &= |f_z| \operatorname{Re}\left\{\frac{(1+a)(1-\bar{a})}{|1-a|}\right\} \\ &= |f_z| \frac{1-|\mu(z)|^2}{|1-\mu(z)\frac{\bar{z}-\bar{z}_0}{z-z_0}|} = \left(\frac{J_f(z)}{D_{\mu,z_0}(z)}\right)^{1/2}\end{aligned}$$

and we arrive at the inequality (2.9).

Since f and f^{-1} are locally absolutely continuous in their domains, we can perform the following change of variables

$$\int_{f(A)} \rho^{*2}(w)\, du\, dv = \int_A \rho^2(|z-z_0|) D_{\mu,z_0}(z)\, dx\, dy.$$

Thus, we arrive at the inequality

$$M(f(\Gamma)) \le \int_A \rho^2(|z-z_0|) D_{\mu,z_0}(z)\, dx\, dy,$$

completing the first part of the proof.

In order to get the left inequality in (2.5), we take into account that $p(\theta)$ satisfies the assumption of Lemma 2.3 and show that the function

$$\rho^*(w) = \frac{1}{2\pi}\left\{\frac{p(\theta)}{|z-z_0|}\left(\frac{D_{-\mu,z_0}(z)}{J_f(z)}\right)^{1/2}\right\} \circ f^{-1}(w) \tag{2.10}$$

is admissible for the family $G^* = f(G)$, where G is the family of curve that separates the boundary components of A in A. Omitting the regularity arguments similar to those of given in the first part of the proof, we see that

$$\int_{\gamma^*} \rho^*\, ds^* \ge \frac{1}{2\pi} \int_{\gamma^*} p(\theta)\left|\frac{d\theta}{ds^*}\right| ds^* \ge \frac{1}{2\pi} \int_0^{2\pi} p(\theta)\, d\theta = 1$$

because of the inequality

$$\frac{1}{r}\cdot\left(\frac{D_{-\mu,z_0}(z)}{J_f(z)}\right)^{1/2}\circ f^{-1}(\gamma^*(s^*))\geq\left|\frac{d\theta}{ds^*}\right|,\quad r=|z-z_0|. \tag{2.11}$$

Indeed, since

$$\left|\frac{d\theta}{ds^*}\right|=\left|\frac{d\theta}{ds}\right|\Big/\frac{ds^*}{ds}=\frac{|\sqrt{1-\langle\omega,\omega_0\rangle^2}|/r}{|\partial_\omega f(z)|},$$

and $\partial_\omega f(z)=f_z(z)(1+\mu(z)\bar{\omega}^2)$, we see that

$$\min_{|\omega|=1}\frac{|\partial_\omega f(z)|}{|\sqrt{1-\langle\omega,\omega_0\rangle^2}|}=2|f_z|\cdot\min_{|w|=1}\left|\frac{w+a}{w-1}\right|,$$

where $\omega_0=(z-z_0)/|z-z_0|$, $a=\mu\bar{\omega}_0^2$, $w=\omega^2\bar{\omega}_0^2$ and ω is an arbitrary unit vector. The Möbius conformal mapping $\varphi(w)=\frac{w+a}{w-1}$ transforms the unit circle into a straight line with the unit normal vector $\vec{n}=\frac{1+a}{|1+a|}=(\varphi(-1)-\varphi(0))/|\varphi(-1)-\varphi(0)|$. Then the required distance from the straight line to the origin is calculated as $|\langle\varphi(-1),\vec{n}\rangle|$. Hence,

$$\begin{aligned}\min_{|\omega|=1}&\frac{|\partial_\omega f(z)|}{|\sqrt{1-\langle\omega,\omega_0\rangle^2}|}\\&=2|f_z|\left\langle\frac{1-a}{2},\frac{1+a}{|1+a|}\right\rangle=|f_z|\operatorname{Re}\left\{\frac{(1+a)(1-\bar{a})}{|1+a|}\right\}\\&\qquad=|f_z|\frac{1-|\mu(z)|^2}{|1+\mu(z)\frac{\bar{z}-\bar{z}_0}{z-z_0}|}=\left(\frac{J_f(z)}{D_{-\mu,z_0}(z)}\right)^{1/2}\end{aligned}$$

and we get the inequality (2.11). Performing the change of variable, we have that

$$Mf((G))\leq\int\limits_{f(A)}\rho^{*2}(w)\,du\,dv=\frac{1}{4\pi^2}\int\limits_{A}p^2(\theta)D_{-\mu,z_0}(z)\frac{dx\,dy}{|z-z_0|}.$$

Noting that $Mf((G))=1/Mf((\Gamma))$, we arrive at the required left inequality (2.5) and thus complete the proof. □

Let us consider an application of Lemma 2.3 to the case when the angular dilatation coefficient $D_{\mu,z_0}(z)$ is dominated by a BMO function. To this end, we need the following auxiliary result, see [32, Lemma 2.21].

Lemma 2.4. *Let Q be a non-negative BMO function in the disk $B = \{z : |z| < 1\}$, and for $0 < t < e^{-2}$, let $A(t) = \{z : t < |z| < e^{-1}\}$. Then*

$$\eta(t) := \iint\limits_{A(t)} \frac{Q(z)\,dx\,dy}{|z|^2(\log|z|)^2} \leq c \log\log 1/t \tag{2.12}$$

where c is a constant which depends only on the average Q_1 of Q over $|z| < e^{-1}$ and on the BMO norm $\|Q\|_*$ of Q in B.

For the sake of completeness, we give a short proof.

Proof of Lemma 2.4. Fix $t \in (0, e^{-2})$. For $n = 1, 2, \ldots,$ let $t_n = e^{-n}$, $A_n = \{z : t_{n+1} < |z| < t_n\}, B_n = \{z : |z| < t_n\}$ and Q_n the mean value of $Q(z)$ in B_n. Now choose an integer N, such that $t_{N+1} \leq t < t_N$. Then $A(t) \subset A(t_{N+1}) = \cup_{n=1}^{N+1} A_n$, and

$$\eta(t) \leq \iint\limits_{A(t_{N+1})} \frac{Q(z)}{|z|^2(\log|z|)^2}\,dx\,dy = S_1 + S_2 \tag{2.13}$$

where

$$S_1 = \sum_{n=1}^{N} \iint\limits_{A_n} \frac{Q(z) - Q_n}{|z|^2(\log|z|)^2}\,dx\,dy \tag{2.14}$$

and

$$S_2 = \sum_{n=1}^{N} Q_n \iint\limits_{A_n} \frac{dx\,dy}{|z|^2(\log|z|)^2} \tag{2.15}$$

Since $A_n \subset B_n$, and for $z \in A_n$, $|z|^{-2} \leq \pi e^2/|B_n|$ and $\log 1/|z| > n$, it follows that

$$|S_1| \leq \sum_{n=1}^{N} \iint\limits_{A_n} \frac{|Q(z) - Q_n|}{|z|^2(\log|z|)^2}\,dx\,dy$$
$$\leq \pi \sum_{n=1}^{N} \frac{e^2}{n^2} \left(\frac{1}{|B_n|} \iint\limits_{B_n} |Q(z) - Q_n|\,dx\,dy \right).$$

Hence,

$$|S_1| \leq 2\pi e^2 \|Q\|_*. \tag{2.16}$$

Now, note that

$$
\begin{aligned}
|Q_k - Q_{k-1}| = \frac{1}{|B_k|}&\left|\iint\limits_{B_k} (Q(z) - Q_{k-1})\, dx\, dy\right| \\
&\leq \frac{1}{|B_k|} \iint\limits_{B_k} |Q(z) - Q_{k-1}|\, dx\, dy \\
&= \frac{e^2}{|B_{k-1}|} \iint\limits_{B_k} |Q(z) - Q_{k-1}|\, dx\, dy \\
&\leq \frac{e^2}{|B_{k-1}|} \iint\limits_{B_{k-1}} |Q(z) - Q_{k-1}|\, dx\, dy \leq e^2 \|Q\|_*.
\end{aligned}
$$

Thus, by the triangle inequality,

$$
Q_n \leq Q_1 + \sum_{k=2}^{n} |Q_k - Q_{k-1}| \leq Q_1 + ne^2 \|Q\|_*, \tag{2.17}
$$

and, since

$$
\iint\limits_{A_n} \frac{dx\, dy}{|z|^2 (\log |z|)^2} \leq \frac{1}{n^2} \iint\limits_{A_n} \frac{dx\, dy}{|z|^2} = \frac{2\pi}{n^2},
$$

it follows by (2.15), that

$$
S_2 \leq 2\pi \sum_{n=1}^{N} \frac{Q_n}{n^2} \leq 2\pi Q_1 \sum_{n=1}^{N} \frac{1}{n^2} + 2\pi e^2 \|Q\|_* \sum_{1}^{N} \frac{1}{n}. \tag{2.18}
$$

Finally, $\sum_{n=1}^{N} 1/n^2$ is bounded, and $\sum_{n=1}^{N} 1/n < 1 + \log N < 1 + \log\log 1/t$, and, thus, (2.12) follows from (2.13), (2.16) and (2.18). $\square$

Lemma 2.5. *Let $f : D \to \mathbb{C}$ be a quasiconformal mapping with complex dilatation $\mu(z) = f_{\bar z}(z)/f_z(z)$, such that $K_f(z) = K_{\mu,0}(z) \leq Q(z) \in BMO$ a.e. in D. Then for every annulus $A(r, Re^{-1}; z_0)$, $r < Re^{-2}$, contained in D,*

$$
M(f(\Gamma)) \leq \frac{c}{\log\log(R/r)} \tag{2.19}
$$

where Γ stands for the family of curves joining the boundary components of $A(r, Re^{-1}; z_0)$ in $A(r, Re^{-1}; z_0)$ and c is the constant in Lemma 2.4 associated with the function $Q(Rz + z_0)$.

Proof. Since

$$
\int\limits_{\gamma} \frac{ds}{|z - z_0| \log(R/|z - z_0|)} \geq \int\limits_{r}^{R/e} \frac{dt}{t \log R\, /t} = \log\log R\, /r = a
$$

we see that the function $\rho(|z - z_0|) = 1/a(|z - z_0| \log R/|z - z_0|)$ is admissible for the family Γ. By Lemma 2.3, and the inequality

$$D_{\mu,z_0}(z) \leq K_{\mu,0}(z) \leq Q(z),$$

we get that

$$M(f(\Gamma)) \leq \frac{1}{a^2} \iint_{r<|z-z_0|<Re^{-1}} \frac{Q(z)\, dx\, dy}{|z - z_0|^2 \log^2(R/|z - z_0|)}.$$

Performing the change of variable $z \mapsto Rz + z_0$, and making use of Lemma 2.4, we have

$$M(f(\Gamma)) \leq \iint_{r/R<|z|<e^{-1}} \frac{Q(Rz + z_0)\, dx\, dy}{|z|^2 \log^2(1/|z|)} \leq \frac{c}{\log\log(R/r)}.$$

$\square$

Remark 2.3. Note that it is not possible, in general, to replace the BMO bound in the previous results by a simpler requirement that the maximal dilatation itself belongs to BMO. For example, consider the functions $Q(x, y) = 1+|\log |y||$, $(x, y) \in \mathbb{R}^2$ and $u(x, y) = Q(x, y)$ if $y > 0$ and $u(x, y) = 1$ if $y \leq 0$. Then $u \leq Q$ and $Q \in BMO$ but u does not belong to BMO.

Lemma 2.6. *There exists a universal constant $C_0 > 0$ with the property that for a ring domain B in $\mathbb{C}$ with $\bmod B > C_0$ which separates a point z_0 from ∞ we can choose an annulus A in B of the form $A = A(r_1, r_2; z_0)$, $r_1 < r_2$, so that $\bmod A \geq \bmod B - C_0$.*

For the proof of the above statement, see [18], where the authors assert that one can take $C_0 = \pi^{-1} \log 2(1 + \sqrt{2}) = 0.50118\ldots$. In fact, it essentially follows from the famous Teichmüller lemma on his extremal ring domain.

3. Proof of main theorem

In view of Lemma 2.1 and Lemma 2.2, the problem of the existence of an $W^{1,1}_{\mathrm{loc}}$ homeomorphic solution to the Beltrami equation (1.1) can be reduced, by a suitable approximation procedure, to the problem of normality of certain families of quasiconformal mappings. By the well-known Arzela–Ascoli theorem, the latter is related to appropriate oscillation estimates.

Proof of Theorem 1.2. We split the proof of Theorem 1.2 into three parts. Given μ, ν, we first generate a sequence of quasiconformal mappings, corresponding to a suitable truncation of the above Beltrami coefficients, and show, making use of Lemma 2.5, that the chosen sequence is normal with respect to the locally uniform convergence. Then we prove that the limit mappings are univalent, belong to the Sobolev space $W^{1,s}_{\mathrm{loc}}(D)$, $s \in [1,2)$, and satisfy the differential equation (1.1) a.e. in D. Finally we deduce the regularity properties of the required solution to the equation (1.1).

$\mathbf{n^0 1}$. Let μ, ν, be Beltrami coefficients defined in D with $|\mu| + |\nu| < 1$ a.e. in D. For $n = 1, 2, \ldots$, we set in $D_n = D \bigcap B(n)$

$$\mu_n(z) = \mu(z), \quad \text{if } |\mu(z)| \leq 1 - 1/n, \tag{3.1}$$

$$\nu_n(z) = \nu(z), \quad \text{if } |\nu(z)| \leq 1 - 1/n, \tag{3.2}$$

and $\mu_n(z) = \nu_n(z) = 0$ otherwise, including the points $z \in B(n) \setminus D_n$. Here $B(n)$ stands for the disk $|z| < n$. The coefficients μ_n, ν_n now are defined in the disk $B(n)$ and satisfy the strong ellipticity condition $|\mu_n(z)| + |\nu_n(z)| \leq q_n < 1$. Therefore, by Theorem 1.1, there exists a quasiconformal mapping $f_n(z) = \omega_n(z/n)/|\omega_n(1/n)|$ of $B(n)$ onto $B(R_n)$ for some $R_n = 1/|\omega_n(1/n)| > 1$ satisfying a.e in $B(n)$ the equation

$$f_{n\bar{z}} - \mu_n(z) f_{nz} - \nu_n(z) \overline{f_{nz}} = 0 \tag{3.3}$$

and normalized by $f_n(0) = 0$, $|f_n(1)| = 1$. We extend f_n over $\partial B(n)$ to the complex plane $\overline{\mathbb{C}}$ by the symmetry principle. It implies, in particular, that $f_n(\infty) = \infty$. We will call such f_n the canonical approximating sequence. It follows from (3.3) and the symmetry principle that f_n satisfies a.e. in $\mathbb{C}$ the Beltrami equation

$$f_{n\bar{z}} = \mu_n^*(z) f_{nz}$$

where

$$\mu_n^*(z) = \begin{cases} \tilde{\mu}_n(z), & \text{if } z \in B(n), \\ \overline{\tilde{\mu}_n(n^2/\bar{z})} z^2/\bar{z}^2, & \text{if } z \in \mathbb{C} \setminus B(n), \end{cases}$$

and

$$\tilde{\mu}_n(z) = \mu_n(z) + \nu_n(z) \cdot \frac{\overline{f_{nz}}}{f_{nz}}.$$

Note that $K_{\mu,\nu}(z) \le Q(z)$ a.e. in $B(n)$.

Our immediate task now is to show that the canonical approximating sequence of quasiconformal mappings $f_n : \mathbb{C} \to \mathbb{C}$ forms a normal family of mappings with respect to the locally uniform convergence in $\mathbb{C}$. To this end, we first prove that the family is equicontinuous locally uniformly in $\mathbb{C}$. More precisely, we show that for every compact set $E \subset \mathbb{C}$

$$|f_n(z') - f_n(z'')| \le C \left(\log \frac{1}{|z' - z''|} \right)^{-\alpha}, \tag{3.4}$$

for every $n \ge N$ and $z', z'' \in E$ such that $|z' - z''|$ is small enough. Here C is an absolute positive constant and $\alpha > 0$ depends only on E and Q.

Indeed, let E be a compact set of $\mathbb{C}$ and $z', z'' \in E$ be a pair of points satisfying $|z' - z''| < e^{-4}$. If we choose N such that $\operatorname{dist}(E, \partial B(N)) > 1$, then we see that the annulus

$$A = \{z \in \mathbb{C} : |z' - z''| < |z - z'| < |z' - z''|^{1/2} \cdot e^{-1}\}$$

is contained in $B(N)$. Moreover, at least one of the points 0 or 1 lies outside of the annulus A and belongs to the unbounded component of its complement.

Let Γ be the family of curves joining the circles $|z - z'| = |z' - z''| = r$ and $|z - z'| = |z' - z''|^{1/2} e^{-1} = Re^{-1}$ in A. The complement of the ring domain $f_n(A)$ to the complex plane has the bounded and unbounded components Δ_n and Ω_n, respectively. Then, by the well-known Gehring's lemma, see [14],

$$M(f_n(\Gamma)) \ge \frac{2\pi}{\log(\lambda/\delta_n \delta_n^*)}$$

where δ_n and δ_n^* stand for the spherical diameters of Δ_n and Ω_n. Since for small enough $|z' - z''|$ $\delta_n^* \ge 1/\sqrt{2}$, we get that

$$\delta_n \le \sqrt{2} \lambda e^{-2\pi/M(f_n(\Gamma))}$$

where λ is an absolute constant. On the other hand, by Lemma 2.5, we have

$$M(f_n(\Gamma)) \leq \frac{c}{\log\log(1/|z'-z''|^{1/2})} \tag{3.5}$$

where the positive constant c depends only on E and Q. If $|z'-z''|$ is small enough, then $2\delta_n \geq |f_n(z')-f_n(z'')|$ and hence

$$|f_n(z')-f_n(z'')| \leq 2\sqrt{2}\lambda e^{-2\pi/M(f_n(\Gamma))}. \tag{3.6}$$

Combining the estimate (3.6) with the inequality (3.5), we arrive at (3.4).

The required normality of the family $\{f_n\}$ with respect to the spherical metric in $\overline{\mathbb{C}}$ now follows by the Ascoli–Arzela theorem, see e.g. [37, 20.4]. Thus, we complete the first part of the proof.

$\mathbf{n^0 2}$. Now we show that the limit mapping f is injective. To this end, without loss of generality, we may assume that the sequence f_n converges locally uniformly in $\mathbb{C}$ to a limit mapping f which is not a constant because of the chosen normalization. Since the mapping degree is preserved under uniform convergence, f has degree 1, see e.g., [15]. We now consider the open set $V = \{z \in \mathbb{C} : f \text{ is locally constant at } z\}$. First we show that if $z_0 \in \mathbb{C} \setminus V$, then $f(z) \neq f(z_0)$ for $z \in \mathbb{C} \setminus \{z_0\}$. Picking a point $z^* \neq z_0$, we choose a small positive number R so that $|z^*-z_0| > R/e$. Then, by Lemma 2.5

$$\operatorname{mod} f_n(A(r,R/e;z_0) = \frac{2\pi}{M(f_n(\Gamma))} \geq \frac{2\pi}{c}\log\log(R/r) > C_0$$

for sufficiently small $0 < r < R/e^2$, where C_0 is the constant in Lemma 2.6. By virtue of Lemma 2.6, we can find an annulus $A_n = \{w : r_n < |w-f_n(z_0)| < r'_n\}$ in the ring domain $f_n(A(r,R/e;z_0))$ for n large enough. Since f is not locally constant at z_0, there exists a point z' in the disk $|z-z_0| < r$ with $f(z_0) \neq f(z')$. The annulus A_n separates $f_n(z_0), f_n(z')$ from $f_n(z^*)$, so we obtain $|f_n(z')-f_n(z_0)| \leq r_n$ and $r'_n \leq |f_n(z^*)-f_n(z_0)|$. In particular, $|f_n(z')-f_n(z_0)| \leq |f_n(z^*)-f_n(z_0)|$ for n large enough. Letting $n \to \infty$, we obtain $0 < |f(z')-f(z_0)| \leq |f(z^*)-f(z_0)|$, and hence $f(z^*) \neq f(z_0)$.

We next show that the set V is empty. Indeed, suppose that V has a non-empty component V_0. Then f takes a constant value, say b, in V_0. If $z^* \in \partial V_0$, then by continuity, we have $f(z^*) = b$. On the other hand, $z^* \notin V$ and therefore $f(z) \neq f(z^*) = b$ for any point z other than z_0, which contradicts the fact that $f = b$ in V_0. We conclude that V is empty, namely, f is not locally constant at any point and hence $f(z) \neq f(\zeta)$ if $z \neq \zeta$. Thus, the injectivity of f follows.

Now we have that the sequence f_n of quasiconformal mappings converges locally uniformly in $\mathbb{C}$ to a limit function f. On the other hand, $K_{\mu_n,\nu_n}(z) \le Q(z) \in L^p(B(n))$ for every $p > 1$ and $n = 1, 2, \ldots$, and $\mu_n(z) \to \mu(z)$ and $\nu_n(z) \to \nu(z)$ a.e. in D and to 0 in $\mathbb{C} \setminus \overline{D}$ as $n \to \infty$. Then, by Lemma 2.2, we arrive at the conclusion that the limit mapping f is homeomorphic solution for the equation $\overline{\partial} f = \mu(z)\partial f + \nu\overline{\partial f}$ in D of the class $W^{1,s}_{\rm loc}(D)$, $s = 2p/(1+p)$, and moreover this solution f admits a conformal extension to $\mathbb{C} \setminus \overline{D}$. Furthermore, the infinity is the removable singularity for the limit mapping by Theorem 6.3 in [32]. Thus, the mapping f admits extension to a self homeomorphism of $\overline{\mathbb{C}}$, $f(\infty) = \infty$, which is conformal in $\overline{\mathbb{C}} \setminus \overline{D}$.

$\mathbf{n^0 3}$. The mappings f_n, $n = 1, 2, \ldots$, are homeomorphic and therefore $g_n := f_n^{-1} \to g := f^{-1}$ as $n \to \infty$ locally uniformly in $\mathbb{C}$, see [11, p. 268]. By the change of variables, that is correct because f_n and $g_n \in W^{1,2}_{\rm loc}$, we obtain under large n

$$\int\limits_{D'_*} |\partial g_n|^2 \, du \, dv = \int\limits_{g_n(D'_*)} \frac{dx\, dy}{1 - |\mu_n(z)|^2} \le \int\limits_{D_*} Q(z)\, dx\, dy < \infty$$

for bounded domains $D_* \subset \mathbb{C}$ and relatively compact sets $D'_* \subset \mathbb{C}$ with $g(\overline{D'_*}) \subset D_*$. The latter estimate means that the sequence g_n is bounded in $W^{1,2}(D'_*)$ for large n and hence $g \in W^{1,2}_{\rm loc}(\mathbb{C})$. Moreover, $\partial g_n \to \partial g$ and $\overline{\partial} g_n \to \overline{\partial} g$ weakly in $L^2_{\rm loc}$, see e.g. [31, III.3.5]. The homeomorphism g has $(N)-$property because $g \in W^{1,2}_{\rm loc}$, see e.g. [24, Theorem 6.1 of Chapter III], and hence $J_f(z) \neq 0$ a.e., see [28].

Finally, the right inequality in (1.6) follows from (3.4). In order to get the left inequality we make use of the length-area argument, see, e.g. [35], p. 75. Let E be a compact set in $\mathbb{C}$ and $E' = f(E)$. Next, let w', w'' be a pair of points in E' with $|w' - w''| < 1$. Consider the family of circles $\{S(w', r)\}$ centered at w' of radius r,

$$r_1 = |w' - w''| < r < r_2 = |w' - w''|^{1/2}.$$

Since $g = f^{-1}(w) \in W^{1,2}_{\rm loc}(\mathbb{C})$, we can apply the standard oscillation estimate

$$\int\limits_{r_1}^{r_2} {\rm osc}^2(g, S(w', r)) \cdot \frac{dr}{r} \le c \iint\limits_{|w-w'|<r_2} |\nabla g|^2 \cdot dx\, dy$$

where $S(w', r)$ stands for the circle $|w - w'| = r$. It yields the estimate

$$\inf_{r\in(r_1,r_2)} \operatorname{osc}(g, S(w',r)) \le c_1 \log^{-1/2}\frac{1}{|w'-w''|}.$$

The mapping g is a homeomorphism, so $\operatorname{osc}(g, B(w',r)) \le \operatorname{osc}(g_n, S(w',r))$ for every $r \in (r_1, r_2)$ where $B(w',r) = \{w : |w - w'| < r\}$. Thus, we get the inequality

$$|g(w') - g(w'')| < C_1 \log^{-1/2}\frac{1}{|w'-w''|}. \tag{3.7}$$

Setting $w' = f(z')$ and $w'' = f(z'')$, we arrive at the required estimate

$$|f(z') - f(z'')| > Ce^{-a/|z'-z''|^2}. \tag{3.8}$$

The last result can be deduced from Gehring's oscillation inequality, see, e.g., [14]. □

Remark 3.1. The first two parts of the proof for Theorem 1.2 are based on Lemma 2.1, Lemma 2.2, the modulus estimate

$$\operatorname{mod} f_n(A(r, R/e; z_0)) \ge C \log\log(R/r), \tag{3.9}$$

as well as on the fact that the right hand side of (3.9) approaches ∞ as $r \to 0$. Recall that the proof of inequality (3.9) is based on Lemma 2.3 and the estimate (1.5). More refined results, based on Lemma 2.3, can be obtained for the degenerate Beltrami equation (1.1) if we replace the basic assumption (1.5) by another one, say, by the inequality

$$\frac{\left(\left|1-\mu(z)\frac{\bar z-\bar z_0}{z-z_0}\right| + |\nu(z)|\right)^2}{1-(|\mu(z)|+|\nu(z)|)^2} \le Q_{z_0}(z) \tag{3.10}$$

where $Q_{z_0}(z) \in \mathrm{BMO}$ for every $z_0 \in D$. We also can replace (1.5) by the inequality

$$\iint\limits_D \frac{e^{H(K_{\mu,\nu}(z))}\,dx\,dy}{(1+|z|^2)^2} < M \tag{3.11}$$

where H stands for a dominating factor of divergence type, see for details [16]. Notice, that typical choices for $H(x)$ are ηx and $\eta x/(1+\log^+ x)$ for a positive constant η. However, we will not pursue these directions here and have an intention to publish the corresponding results elsewhere.

Acknowledgments. The research of the third author was partially supported by the Ukrainian State Foundation of Fundamental Investigations (FFI), Grant number F25.1/055.

References

[1] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, van Nostrand, 1966.

[2] K. Astala, T. Iwaniec, P. Koskela, and G. J. Martin, *Mappings of BMO-bounded distortion* // Math. Ann. **317** (2000), 703–726.

[3] K. Astala, V. Nesi, *Composites and quasiconformal mappings: new optimal bounds in two dimensions* // Calc. Var. Partial Differential Equations, **18** (2003), 335–355.

[4] L. Bers, *Uniformization by Beltrami equation* // Commun. in Pure and Appl. Math. **14** (1961), 215–228.

[5] B. V. Bojarski, *On solutions of elliptic systems in the plane* // Doklady Akad. Nauk SSSR (N.S.) **102** (1955), 871–874 (in Russian).

[6] B. V. Bojarski, *Generalized solutions of a system of differential equations of the first order of the elliptic type with discontinuous coefficients* // Mat. Sb. **43 (85)** (1957), 451–503 (in Russian).

[7] M. A. Brakalova and J. A. Jenkins, *On solutions of the Beltrami equation* // J. Anal. Math. **76** (1998), 67–92.

[8] Z.-G. Chen, *Estimates on $\mu(z)$-homeomorphisms of the unit disk* // Israel J. Math. **122** (2001), 347–358.

[9] Z.-G. Chen, *$\mu(z)$-homeomorphisms of the plane* // Michigan Math. J. **51** (2003), N 3, 547–556.

[10] G. David, *Solutions de l'équation de Beltrami avec $\|\mu\|_\infty = 1$* // Ann. Acad. Sci. Fenn. Ser. A I Math. **13** (1988), 25–70.

[11] J. Dugunji, *Topology*, Boston: Allyn & Bacon, Inc., 1966.

[12] N. Dunford and J. Schwartz, *Linear Operators*, Part I: General Theory, New York, London: Interscience Publishers, 1958.

[13] C. Fefferman and E. M. Stein, *H_p spaces of several variables* // Acta Math. **129** (1972), 137–193.

[14] F. W. Gehring, *A remark on the moduli of rings* // Comment. Math. Helv. **36** (1962), 42–46.

[15] V. Ya. Gutlyanskii, O. Martio, V. I. Ryazanov and M. Vuorinen, *On local injectivity and asymptotic linearity of quasiregular mappings* // Studia Mathematica, **128** (1998), N 3, 243–271.

[16] V. Gutlyanskii, O. Martio, T. Sugawa and M. Vuorinen, *On the degenerate Beltrami equation* // Trans. Amer. Math. Soc., **357** (2005), N 3, 875–900.

[17] V. Gutlyanskii and T. Sugawa, *On Lipschitz continuity of quasiconformall mappings* // Report. Univ. Juväskylä, **83** (2001), 91–108.

[18] D. A. Herron, X. Liu, and D. Minda, *Ring domains with separating circles or separating annuli* // J. Analyse Math. **53** (1989), 233–252.

[19] T. Iwaniec and G. Martin, *Geometrical Function Theory and Non–linear Analysis*, Clarendon Press, Oxford, 2001.

[20] T. Iwaniec and G. Martin, *The Beltrami equation* // Memoires of the AMS, **893** (2008), 1–92.

[21] F. John and L. Nirenberg, *On functions of bounded mean oscillation* // Comm. Pure Appl. Math. **14** (1961), 415—426.

[22] M. A. Lavrent'ev, *Variational method for boundary value problems for systems of elliptic type equations*, (Russian), Izdat. Acad. Nauk SSSR, Moskva, 1962.

[23] O. Lehto, *Remarks on generalized Beltrami equations and conformal mappings*, Proceedings of the Romanian-Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings (Braşov, 1969), Publ. House of the Acad. of the Socialist Republic of Romania, Bucharest, 1971, pp. 203–214.

[24] O. Lehto and K. Virtanen, *Quasiconformal Mappings in the Plane*, Springer, New York etc., 1973.

[25] O. Martio and V. Miklyukov, *On existence and uniqueness of degenerate Beltrami equation*, Complex Var. Theory Appl. 49 (2004), no. 7-9, 647–656.

[26] S. G. Mihlin, *Linear partial differential equations*, (Russian), Izdat. "Vyss. Skola", Moscow, 1977, 431 pp.

[27] I. N. Pesin, *Mappings which are quasiconformal in the mean* // (Russian), Dokl. Akad. Nauk SSSR **187** (1969), 740–742, English translation in Soviet Math. Dokl. **10** (1969), 939–941.

[28] S. P. Ponomaryev, *The N^{-1}-property of mappings and Lusin's (N) condition* // Mat. Zametki 58 (1995), 411–418; transl. in Math. Notes **58** (1995), 960–965.

[29] E. Reich and H. Walczak, *On the behavior of quasiconformal mappings at a point*, Trans. Amer. Math. Soc. **117** (1965), 338–351.

[30] H. M. Reimann and F. Rushener, *Funktionen Beschränkter Mittlerer Oscillation,* Lecture Notes in Math., **487**, Springer Verlag, 1975.

[31] Yu. G. Reshetnyak, *Space Mappings with Bounded Distortion*, Providence, Rhode Island: American Mathematical Society, 1989.

[32] V. Ryazanov, U. Srebro and E. Yakubov, BMO-*quasiconformal mappings* // J. d'Anal. Math. **83** (2001), 1–20.

[33] V. Ryazanov, U. Srebro and E. Yakubov, *The Beltrami equation and ring homeomorphisms* // Ukrainian Math. Bull., **4**, No. 1 (2007), 79–115.

[34] S. Saks, *Theory of the Integral*, New York, Dover Publ. Inc., 1964.

[35] G. D. Suvorov, *Families of plane topological mappings*, (Russian), Izdat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1965.

[36] P. Tukia, *Compactness properties of μ-homeomorphisms* // Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), 47–69.

[37] J. Väisälä, *Lectures on n-Dimensional Quasiconformal Mappings* // Lecture Notes in Math. **229** (1971), Springer-Verlag, Berlin-New York, 1–144.

[38] I. N. Vekua, *Generalized analytic functions*, (Russian), Fizmatgiz, Moscow, 1959.

Contact information

Bogdan V. Bojarski Institute of Mathematics
Polish Academy of Sciences,
ul. Sniadeckich 8, P.O. Box 21,
00-956 Warszawa,
Poland
E-Mail: `B.Bojarski@impan.gov.pl`

Vladimir Ya. Gutlyanskiĭ,
Vladimir I. Ryazanov

Institute of Applied Mathematics
and Mechanics, NAS of Ukraine,
ul. Roze Luxemburg 74,
83114, Donetsk,
Ukraine
E-Mail: gut@iamm.ac.donetsk.ua,
vlryazanov1@rambler.ru