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ON THE EQUIVALENCE OF CERTAIN SEMINORMS ON SOME WEIGHTED HÖLDER SPACES

The present paper is devoted to studying of some weighted Hölder spaces. These spaces are designed in the way to serve as a framework for studying different statements for the thin film equations in weighted classes of smooth functions in the multidimensional setting. These spaces can serve also for considering of other equations with the degeneration on the boundary of the domain of definition. We prove the equivalence of certain metrics on these spaces.

Keywords: *weighted Hölder spaces, degenerate parabolic equations, equivalent metrics.*

The present paper is devoted to studying of some weighted Hölder spaces $C_{n,\omega\gamma}^{m+\gamma,\frac{m+\gamma}{m}}$. These spaces were introduced in [1] and they are designed in the way to serve as a framework for consideration of different statements for the thin film equations in weighted classes of smooth functions in the multidimensional setting. These spaces can serve also for considering of other equations with the degeneration on the boundary of the domain of definition, for example, in the spirit of [2].

The literature on the subject of the thin film equations is very numerous but almost all results with sufficient regularity are devoted to the case of one spatial variable. As a possible target for an application of the spaces $C_{n,\omega\gamma}^{m+\gamma,\frac{m+\gamma}{m}}$ we only mention the papers [3]–[17].

The spaces $C_{n,\omega\gamma}^{m+\gamma,\frac{m+\gamma}{m}}$ arise at the considering linearised version of the thin film equations. Let us explain this on the example for the thin film equation in the case of partial wetting (see, for example, [3] for the accurate statement). Consider the thin film equation of fourth order for an unknown function $h(x,t)$ (compare [18])

$$\frac{\partial h}{\partial t} + \nabla \left(h^n \nabla \Delta h - \beta \nabla h\right) = f(x,t) \ \text{ in } \ \Omega, \tag{1}$$

where $n > 0$ is fixed, Ω is a half space $\Omega = \{(x,t) : x = (x', x_N) \in R^N, x_N > 0, t > 0\}$. Consider also partial wetting conditions at $x_N = 0$

$$h(x',0,t) = 0, \ \ \frac{\partial h}{\partial x_N}(x',0,t) = 1 \tag{2}$$

and an initial condition

$$h(x,0) = w(x). \tag{3}$$

From (2) it follows that we must have for $w(x)$

$$w(x',0) = 0, \ \ \frac{\partial w}{\partial x_N}(x',0) = 1. \tag{4}$$

Consequently, we have

$$w(x) \sim x_N, \quad x_N \to 0. \tag{5}$$

The linearization of equation (1) at the initial datum $w(x)$ means that we denote in (1) $h = w + u$ and extract linear with respect to u part (we also drop lower order terms). Formally, one can just replace h^n by w^n in (1) and replace h by u in other places of this equation. Taking into account (5) and replacing w by just x_N, we arrive at

$$\frac{\partial u}{\partial t} + \nabla(x_N^n \nabla\Delta u - \beta\nabla u) = f(x,t) \text{ in } \Omega. \tag{6}$$

For second order equations this procedure is described in details in, for example, [19], [20], [2], and for fourth order see [17], [3], [4] formula (13), [5] formula (7).

If we are going to consider equations (6) (and correspondingly (1)) in classes of Hölder functions we have to consider $f(x,t)$ in (6) from some (may be weighted) Hölder class. This leads to the consideration of $\nabla(x_N^n \nabla\Delta u)$ from the same weighted Hölder class. In our definition below this will be the class $C_{n,(n/4)\gamma}^{m+\gamma,\frac{m+\gamma}{m}}$. In the case of second order equations such classes were used in fact in [21]–[23], [2], where the papers [21]–[23] are based on the Carnot–Carathéodory metric and the paper [2] is based on classes $C_{n,\omega\gamma}^{m+\gamma,\frac{m+\gamma}{m}}$. Note that we consider the framework of classes $C_{n,\omega\gamma}^{m+\gamma,\frac{m+\gamma}{m}}$ as an alternative for considering the Carnot–Carathéodory metric for studying degenerate equations in classes of smooth functions – [21]–[23], [17]. Therefor, in this paper we are going to prove the equivalence of the Carnot–Carathéodory metric in spaces $C_{n,\omega\gamma}^{m+\gamma,\frac{m+\gamma}{m}}$ to some another weighted metric in these spaces.

Note that in the case of elliptic equations more simple weighted Hölder classes with unweighted Hölder constants can be used – [24], [25]. The reason is that in the elliptic case no agreement between smoothness in x-variables and t- variable is needed.

Let us turn now to exact definitions and to the main results.

Denote $H = \{x = (x', x_N) \in R^N : x_N > 0\}$, $Q = \{(x,t) : x \in H, -\infty < t < \infty\}$. And we note at once that all the reasoning and statement below are valid in evident way also for $Q^+ = \{(x,t) : x \in H, t \geq 0\}$ instead of Q. Let m be a positive integer and let n be a positive number, $n < m$. Denote

$$\omega = n/m < 1.$$

Let $C_{\omega\gamma}^{\gamma}(\overline{H})$, $\gamma \in (0,1)$, be the weighted Hölder space of continuous functions $u(x)$ with the finite norm

$$|u|_{\omega\gamma,\overline{H}}^{(\gamma)} \equiv \|u\|_{C_{\omega\gamma}^{\gamma}(\overline{H})} \equiv |u|_{\overline{H}}^{(0)} + \langle u\rangle_{\omega\gamma,\overline{H}}^{(\gamma)}, \tag{7}$$

where

$$|u|_{\overline{H}}^{(0)} = \max_{x\in\overline{H}} |u(x)|,\ \langle u\rangle_{\omega\gamma,\overline{H}}^{(\gamma)} = \sup_{x,\overline{x}\in\overline{H}} (x_N^*)^{\omega\gamma} \frac{|u(x)-u(\overline{x})|}{|x-\overline{x}|^{\gamma}},\ x_N^* = \max\{x_N, \overline{x}_N\}. \tag{8}$$

Thus $\langle u\rangle^{(\gamma)}_{\omega\gamma,\overline{H}}$ represents a weighted Hölder constant of the function $u(x)$. We suppose that

$$n < m, \quad , \text{ if } n \text{ is a noninteger} \quad (1-\omega)\gamma = \gamma\left(1-\frac{n}{m}\right) < \min(\{n\}, 1-\{n\}), \tag{9}$$

where for a real number a ,$\{a\}$ is the fractional part of a, $[a]$ is the integer part of a. This assumption is technical and it allows us, for example, to consider the functions x_N^{n-j} as elements of $C^{\gamma}_{\omega\gamma}(\overline{H})$ for all integer $j < n$.

In the similar way we define the Hölder seminorms with respect to each variable separately

$$\langle u\rangle^{(\gamma)}_{\omega\gamma,x_i,\overline{H}} = \sup_{x,\overline{x}\in\overline{H}} (x_N^*)^{\omega\gamma}\frac{|u(x)-u(\overline{x})|}{h^{\gamma}}, \; x_N^* = \max\{x_N, \overline{x}_N\}, i = \overline{1,N}, \tag{10}$$

where $x = (x_1, ...x_i, ..., x_N)$, $\overline{x} = (x_1, ...x_i + h, ..., x_N)$, $h > 0$.

In the standard way we denote by $\langle u\rangle^{(\gamma)}_{x_i,\overline{H}}$, $\langle u\rangle^{(\gamma)}_{x',\overline{H}}$, and $\langle u\rangle^{(\gamma)}_{x,\overline{H}}$ usual unweighted Hölder seminorms with respect to each variable separately, with respect to $x' = (x_1, ..., x_{N-1})$ or with respect to all x-variables.

Note that in terms of the Carnot-Carathéodory metric seminorm (8) is equivalent to

$$\langle u\rangle^{(\gamma)}_{\omega\gamma,\overline{H}} \simeq \sup_{x,\overline{x}\in\overline{H}}\frac{|u(x)-u(\overline{x})|}{s(x,\overline{x})^{\gamma}},$$

where the Carnot–Carathéodory distance is defined as

$$s(x,\overline{x}) = \frac{|x-\overline{x}|}{|x-\overline{x}|^{\omega} + x_N^{\omega} + \overline{x}_N^{\omega}}. \tag{11}$$

In the case of $m = 2$, $n \in (0, 1)$ this was proved in [2] and in the general case we have the following theorem which is the main result of the present paper.

Denote

$$[u]^{(\gamma)}_s \equiv \sup_{x,\overline{x}\in\overline{H}}\frac{|u(x)-u(\overline{x})|}{s(x,\overline{x})^{\gamma}}, \tag{12}$$

where $s(x,\overline{x})$ is defined in (11)

Theorem. *Seminorm* (10) *is equivalent to seminorm* (12). *This means that there are constants C_1 and C_2 with the property*

$$[f]^{(\gamma)}_s \le C_1\,\langle f\rangle^{(\gamma)}_{\omega\gamma,\overline{H}} \le C_2[f]^{(\gamma)}_s \tag{13}$$

for any continuous in $\overline{H}$ function $f(x)$.

Proof. Let the seminorm $[f]^{(\gamma)}_s$ is finite. We show, that then

$$\langle f\rangle^{(\gamma)}_{\omega\gamma,\overline{H}} \le C[f]^{(\gamma)}_s. \tag{14}$$

Let $\varepsilon_0 \in (0,1)$ is small and fixed. Let $x = (x', x_N)$ and let first

$$|x' - \overline{x}'| \geq \varepsilon_0 x_N. \tag{15}$$

Then the more

$$|x - \overline{x}| \geq |x' - \overline{x}'| \geq \varepsilon_0 x_N. \tag{16}$$

Under this condition

$$s(x,\overline{x}) = \frac{|x - \overline{x}|}{x_N^\omega + \overline{x}_N^\omega + |x' - \overline{x}'|^\omega} \leq$$

$$\leq C \frac{|x - \overline{x}|}{|x - \overline{x}|^\omega + \overline{x}_N^\omega + |x' - \overline{x}'|^\omega} \leq C|x - \overline{x}|^{1-\omega}.$$

Therefore, denoting $\beta = \gamma(1-\omega)$,

$$\frac{|f(x,t) - f(\overline{x},t)|}{|x - \overline{x}|^\beta} \leq C \frac{|f(x,t) - f(\overline{x},t)|}{s(x,\overline{x})^\gamma} \leq C[u]_s^{(\gamma)}. \tag{17}$$

Besides, because of (16), and then of (17),

$$x_N^{\gamma\omega} \frac{|f(x,t) - f(\overline{x},t)|}{|x - \overline{x}|^\gamma} \leq \frac{x_N^{\gamma\omega}}{(\varepsilon_0 x_N)^{\gamma\omega}} \frac{|f(x,t) - f(\overline{x},t)|}{|x - \overline{x}|^{\gamma(1-\omega)}} \leq C[u]_s^{(\gamma)}. \tag{18}$$

Let now

$$|x' - \overline{x}'| \leq \varepsilon_0 x_N. \tag{19}$$

Under this condition, as it easy to see,

$$s(x,\overline{x}) \sim C x_N^{-\omega} |x - \overline{x}|. \tag{20}$$

Consequently,

$$x_N^{\gamma\omega} \frac{|f(x,t) - f(\overline{x},t)|}{|x - \overline{x}|^\gamma} \leq C \frac{|f(x,t) - f(\overline{x},t)|}{s(x,\overline{x})^\gamma} \leq C[u]_s^{(\gamma)}. \tag{21}$$

We estimate now the unweighted Hölder constant of the function f with the exponent β.

To estimate it we consider the two cases.

If

$$|x_N - \overline{x}_N| \geq \varepsilon_0 x_N,$$

then

$$|x - \overline{x}| \geq |x_N - \overline{x}_N| \geq \varepsilon_0 x_N$$

and therefore, as it was above,

$$s(x,\overline{x}) \leq \frac{|x - \overline{x}|}{(|x - \overline{x}|/\varepsilon_0)^\omega} \leq C|x - \overline{x}|^{1-\omega},$$

so that, as above,

$$\frac{|f(x,t)-f(\overline{x},t)|}{|x-\overline{x}|^{\beta}} \leq C\frac{|f(x,t)-f(\overline{x},t)|}{s(x,\overline{x})^{\gamma}} \leq C[u]_s^{(\gamma)}. \tag{22}$$

If now, under the condition (19), we have

$$|x_N-\overline{x}_N| \leq \varepsilon_0 x_N, \tag{23}$$

then in this case

$$|x-\overline{x}| \leq |x'-\overline{x}'|+|x_N-\overline{x}_N| \leq 2\varepsilon_0 x_N. \tag{24}$$

Therefore, in the force of (20),

$$s(x,\overline{x}) \leq Cx_N^{-\omega}|x-\overline{x}| \leq$$
$$\leq Cx_N^{-\omega}(2\varepsilon_0 x_N)^{\omega}|x-\overline{x}|^{1-\omega} = C|x-\overline{x}|^{1-\omega}. \tag{25}$$

Consequently, in this case

$$\frac{|f(x,t)-f(\overline{x},t)|}{|x-\overline{x}|^{\beta}} \leq C\frac{|f(x,t)-f(\overline{x},t)|}{s(x,\overline{x})^{\gamma}} \leq C[u]_s^{(\gamma)}. \tag{26}$$

The estimate (14) follows now from (17), (18), (21), (22) and (26).

Further, let now the seminorm $\langle f\rangle_{\omega\gamma,\overline{H}}^{(\gamma)}$ is finite. Let us prove the following estimate

$$[f]_s^{(\gamma)} \leq C\,\langle f\rangle_{\omega\gamma,\overline{H}}^{(\gamma)}. \tag{27}$$

Let first

$$|x'-\overline{x}'| \leq \varepsilon_0 x_N, \quad x_N>0. \tag{28}$$

Then

$$s(x,\overline{x}) \geq \nu\frac{|x-\overline{x}|}{x_N^{\omega}},$$

and consequently

$$\frac{|f(x,t)-f(\overline{x},t)|}{s(x,\overline{x})^{\gamma}} \leq Cx_N^{\gamma\omega}\frac{|f(x,t)-f(\overline{x},t)|}{|x-\overline{x}|^{\gamma}} \leq C\,\langle f\rangle_{\omega\gamma,\overline{H}}^{(\gamma)}. \tag{29}$$

In the particular case $x_N=0$ we have $\overline{x}_N=0$ and therefore

$$s(x,\overline{x}) = |x'-\overline{x}'|^{1-\omega} = |x-\overline{x}|^{1-\omega},$$

and so again

$$\frac{|f(x,t)-f(\overline{x},t)|}{s(x,\overline{x})^{\gamma}} = \frac{|f(x,t)-f(\overline{x},t)|}{|x-\overline{x}|^{\beta}} \leq C\,\langle f\rangle_{\omega\gamma,\overline{H}}^{(\gamma)}. \tag{30}$$

Let now we have

$$|x'-\overline{x}'| \geq \varepsilon_0 x_N. \tag{31}$$

Then

$$s(x,\overline{x}) \geq \nu \frac{|x-\overline{x}|}{|x'-\overline{x}'|^{\omega}} \geq \nu |x-\overline{x}|^{1-\omega}, \tag{32}$$

and consequently,

$$\frac{|f(x,t)-f(\overline{x},t)|}{s(x,\overline{x})^{\gamma}} \leq C \frac{|f(x,t)-f(\overline{x},t)|}{|x-\overline{x}|^{\beta}} \leq C \left\langle f \right\rangle^{(\gamma)}_{\omega\gamma,\overline{H}}. \tag{33}$$

Thus, (27) follows from (29), (30), (32). And so the theorem is proved. □

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С. П. Дегтярев

Об эквивалентности некоторых полунорм в весовых пространствах Гельдера.

Данная статья посвящена изучению некоторых весовых пространств Гельдера. Эти пространства являются естественными классами гладких функций для изучения уравнений типа уравнений тонких пленок в многомерном случае. Эти классы могут быть применены также для изучения других уравнений с вырождением на границе области определения. Мы доказываем эквивалентность некоторых различных метрик в этих пространствах.

Ключевые слова: *весовые пространства Гельдера, вырождающиеся параболические уравнения, эквивалентные метрики.*

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