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A TWO-RADII THEOREM FOR WEIGHTED BALL MEANS ON SYMMETRIC SPACES

Generalizations of functions with zero ball means on Riemannian symmetric spaces $X = G/K$ are studied. An analog of the local two-radii theorem for symmetric spaces of noncompact type with a complex group G is obtained.

Key words: *symmetric spaces, zero spherical means, two-radii theorem.*

1. Introduction. The first example of nonzero function with vanishing integrals over balls of fixed radius in $\mathbb{R}^n$ was considered by Chakalov [1] (see also [2] and [3] for the case of spherical averages). A very utilizing description of all such functions was obtained in [4].

In connection with non-triviality of the corresponding class the following problem arises. Let f be a locally integrable function in $\mathbb{R}^n$ and let E be a given set of positive numbers. Assume that for all $r \in E$ and $x \in \mathbb{R}^n$,

$$\int_{|u|\leqslant r} f(x+u)du = 0.$$

For what sets E does this imply that f vanishes identically? The well known two-radii theorem states that if E consists of two positive numbers r_1 and r_2 such that r_1/r_2 is not the ratio of two zeros of the Bessel function $J_{n/2}$ then $f = 0$ and this condition on the ratio r_1/r_2 is necessary (see [5], [6]). The first local version of the two-radii theorem is given by Smith [7], and Berenstein and Gay [8]. A definitive version of the local two-radii theorem are due to V.V. Volchkov [4].

The results obtained suggest the general problem of investigating functions with zero ball means on homogeneous spaces. Functions with vanishing integrals over all geodesic balls in symmetric spaces with radii in a fixed set have been studied by many authors. The case of Riemannian symmetric rank-one spaces has been treated more detail. In particular, analogues of the two-radii theorem were established for spaces of constant curvature and, after that, for rank-one symmetric spaces (see [9–11] and the references therein). In the present paper we study an analog of the two-radii problem for symmetric spaces $X = G/K$ of noncompact type with a complex group G.

2. Auxiliary constructions. As regards basic notations and facts from the theory of symmetric spaces, see, for instance, [12–14].

Let $X = G/K$ be a symmetric space of noncompact type, G being a connected semisimple Lie group with finite center and K a maximal compact subgroup. The Lie algebras of G and K are respectively denoted by $\mathfrak{g}$ and $\mathfrak{k}$. The adjoint representations of $\mathfrak{g}$ and G are respectively denoted by ad and Ad. Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition, $\mathfrak{p}$ being the orthogonal complement to $\mathfrak{k}$ with respect to the Killing form

$\langle\ ,\ \rangle$ of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, $\mathfrak{a}^*$ its dual and $\mathfrak{a}^*_{\mathbb{C}}$ the space of $\mathbb{R}$-linear maps of $\mathfrak{a}$ into $\mathbb{C}$. Let Σ denote the system of retricted roots on $\mathfrak{a}$, Σ^+ the set of $\alpha \in \Sigma$ which are positive on some fixed Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and put

$$\rho = \frac{1}{2} \sum_{\alpha\in\Sigma^+} m_\alpha \alpha,$$

m_α being the multiplicity of α. For $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha$ denote the corresponding root space and let

$$\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha.$$

Put $N = \exp\mathfrak{n}$, $A = \exp\mathfrak{a}$, $A^+ = \exp\mathfrak{a}^+$ with exp denoting the exponential mapping of $\mathfrak{g}$ into G. As usual we set

$$\mathrm{Exp}P = (\exp P)K \in X$$

for each $P \in \mathfrak{p}$. Let M and M', respectively, denote the centralizer and normalizer of A in K. The Weyl group W is M'/M and we put $\mathbb{B} = K/M$. The order of W will be denoted by $|W|$.

The dimension of $\mathfrak{a}$ is called the rank of X and we shall write $\operatorname{rank} X = \dim\mathfrak{a}$. Let $|\cdot|$ be the norm in $\mathfrak{a}$ induced by the Killing form on $\mathfrak{g}$, i.e. $|H| = \sqrt{\langle H, H\rangle}$ for all $H \in \mathfrak{a}$. For $\lambda \in \mathfrak{a}^*$, let $A_\lambda \in \mathfrak{a}$ be determined by

$$\lambda(H) = \langle H, A_\lambda\rangle$$

for $H \in \mathfrak{a}$. We put $|\lambda| = |A_\lambda|$ for each $\lambda \in \mathfrak{a}^*$.

The form $\langle\ ,\ \rangle$ induces a G-invariant Riemannian structure on X with the corresponding distance function $d(\cdot,\cdot)$ and the Riemannian measure dx. As usual L denote the Laplace-Beltrami operator on X. Let $o = \{K\}$ be the origin in X. For $R \geqslant 0$ and $y \in X$, we denote

$$B_R(y) = \{x \in X\colon d(x, y) < R\}, \quad B_R = B_R(o).$$

We normalize the Haar measure dk on K such that the total measure is 1.

Let $g = k(g)\exp H(g)n(g)$ denote the factoring of an element $g \in G$ according to the Iwasawa decomposition $G = KAN$ and for $x \in X$, $b \in \mathbb{B}$ let $A(x, b) \in \mathfrak{a}$ be defined by

$$A(gK, kM) = -H(g^{-1}k).$$

Let $\mathcal{O}$ be a non-empty open K-invariant subset of $X = G/K$, $\mathcal{D}'(\mathcal{O})$ be the space of distributions on $\mathcal{O}$, and $\mathfrak{W}(\mathcal{O})$ be an arbitrary subset of $\mathcal{D}'(\mathcal{O})$. We shall write $\mathfrak{W}_\natural(\mathcal{O})$ for the set of all K-invariant distributions in $\mathfrak{W}(\mathcal{O})$.

Let $\mathbf{D}(G)$ denote the algebra of left invariant differential operators on G and $\mathbf{D}(X)$ the algebra of G-invariant differential operators on X. We recall that a function $\varphi \in C^\infty(X)$ is called a spherical function if φ is K-invariant, $\varphi(o) = 1$, and for each $D \in \mathbf{D}(X)$ there exists $\lambda_D \in \mathbb{C}$ such that $D\varphi = \lambda_D\varphi$.

The spherical functions on X are given by Harish-Chandra's formula

$$\varphi_\lambda(gK) = \int_K e^{(i\lambda-\rho)(H(gk))}dk, \quad g \in G,$$

λ running through $\mathfrak{a}^*_{\mathbb{C}}$. Also

$$\varphi_\mu \equiv \varphi_\lambda \quad \text{if and only if} \quad \mu = s\lambda \quad \text{for some} \quad s \in W.$$

The symmetry identity for φ_λ has the form

$$\varphi_\lambda(h^{-1}gK) = \int_K e^{(-i\lambda+\rho)(A(kh))}e^{(i\lambda+\rho)(A(kg))}dk, \quad h, g \in G,$$

where $A(g) = -H(g^{-1})$, see [13, Chapter 4, Lemma 4.4]. In particular,

$$\varphi_\lambda(gK) = \varphi_{-\lambda}(g^{-1}K). \tag{1}$$

Next, if the group G is complex then

$$\varphi_\lambda(\text{Exp}P) = J^{-1/2}(P)\int_K e^{i\langle A_\lambda, \text{Ad}(k)P\rangle}dk, \quad P \in \mathfrak{p}, \tag{2}$$

where J is defined by

$$J(P) = \det\left(\frac{\sinh \text{ad}P}{\text{ad}P}\right)$$

(see [13, Chapter 4, Propositions 4.8 and 4.10]). We note that

$$\int_X f(x)dx = \int_{\mathfrak{p}} f(\text{Exp}P)J(P)dP, \quad f \in (L^1 \cap \mathcal{E}')(X)$$

(see [13, Chapter 2, § 3, (69)]).

Denote by $\mathcal{E}'(\mathcal{O})$ the set of all distributions on $\mathcal{O}$ with compact supports. For any distribution $f \in \mathcal{E}'(\mathrm{X})$ we define the Fourier transform $\widetilde{f}$ by letting

$$\widetilde{f}(\lambda, b) = \left\langle f, e^{(-i\lambda+\rho)(A(x,b))}\right\rangle, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ b \in \mathbb{B}.$$

It follows that if $f \in \mathcal{E}'_\natural(X)$ then

$$\widetilde{f}(\lambda, b) = \langle f, \varphi_{-\lambda}(x)\rangle \tag{3}$$

for all $(\lambda, b) \in \mathfrak{a}^*_{\mathbb{C}} \times \mathbb{B}$. We write $\widetilde{f}(\lambda)$ for the right hand side of (3). This function is called the spherical transform of $f \in \mathcal{E}'_\natural(X)$.

Let $\mathcal{E}'_{\natural\natural}(X)$ be the set of all nonzero distributions $T \in \mathcal{E}'_\natural(X)$ with the following property: there exists an even entire function $\overset{\circ}{T} : \mathbb{C} \to \mathbb{C}$ such that

$$\widetilde{T}(\lambda) = \overset{\circ}{T}(\sqrt{\langle\lambda, \lambda\rangle}) \quad \text{for all} \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}}.$$

From the Paley-Wiener theorem for the spherical transform it follows that the class $\mathcal{E}'_{\natural\natural}(X)$ is broad enough. We point out that

$$\mathcal{E}'_{\natural\natural}(X) = \mathcal{E}'_{\natural}(X) \quad \text{provided that} \quad \operatorname{rank} X = 1.$$

Let $\times$ denote the convolution on X. We recall from [13, Chapter 2, § 5, (12)] that if $f \in \mathcal{D}'(X)$ and $T \in \mathcal{E}'(X)$ then

$$\langle f \times T, u\rangle = \left\langle T(g_2K), \left\langle f(g_1K), \int_K u(g_1kg_2K)dk \right\rangle\right\rangle, \quad u \in \mathcal{D}(X). \tag{4}$$

3. Statement and proof of the main result. Suppose that $r_1, r_2 > 0$ and $R > \max\{r_1, r_2\}$. Let $\mathfrak{N}_{r_1,r_2}(B_R)$ denotes the set of all functions $f \in L^{1,\text{loc}}(B_R)$ satisfying the conditions

$$\int_{B_{r_1}} f(gx)d\mu(x) \leqslant 0, \quad g \in G, \; d(o, go) < R - r_1, \tag{5}$$

$$\int_{B_{r_2}} f(gx)d\mu(x) \geqslant 0, \quad g \in G, \; d(o, go) < R - r_2, \tag{6}$$

where

$$d\mu(x) = (J(\operatorname{Exp}^{-1}x))^{-1/2}dx.$$

We consider the following problem. Let $f \in \mathfrak{N}_{r_1,r_2}(B_R)$. Under what conditions it follows that $f = 0$? We shall give an answer to this questions for symmetric spaces $X = G/K$ of noncompact type with a complex group G.

Let $l = \frac{1}{2}\dim X$. Denote by E_X the set of all numbers of the form α/β where $\alpha, \beta > 0$ and $J_l(\alpha) = J_l(\beta) = 0$ (here J_l is the Bessel function of order l). For the first time, this set was introduced by Zalcman [5] in relation to the Euclidean version of two-radii theorem. As is known, E_X is countable and everywhere dense in $(0, +\infty)$.

We say that a number $\tau > 0$ is well approximated by elements of E_X if for each $m > 0$ there exist positive numbers α, β such that $J_l(\alpha) = J_l(\beta) = 0$ and

$$|\tau - \alpha/\beta| < (2 + \beta)^{-N}.$$

Let WA_X be the set of all points well approximated by elements of E_X. We point out the following properties of the set WA_X (see [11, Proposition 2.1.10]):

(a) $\tau \in \mathrm{WA}_X$ if and only if $\tau^{-1} \in \mathrm{WA}_X$.

(b) WA_X is of zero Lebesque measure in $(0, +\infty)$.

(c) The intersection of WA_X with any interval $(a, b) \subset (0, +\infty)$ is uncountable.

(d) $\tau \in \mathrm{WA}_X$ if and only if for each $m > 0$ there exists $\gamma > 0$ such that $J_l(\gamma) = 0$ and $J_l(\tau\gamma) < (2 + \gamma)^{-m}$.

The main result of this paper is as follows.

Theorem 1. *Let $X = G/K$ be a symmetric space of noncompact type with a complex group G and assume that $r_1, r_2 > 0$, $R > \max\{r_1, r_2\}$. Then the following assertions hold.*

(i) *If* $f \in \mathfrak{N}_{r_1,r_2}(B_R)$*,* $r_1 + r_2 < R$ *and* $r_1/r_2 \notin E_X$ *then* $f = 0$*.*
(ii) *If* $f \in (\mathfrak{N}_{r_1,r_2} \cap L)(B_R)$*,* $r_1 + r_2 = R$ *and* $r_1/r_2 \notin E_X$ *then* $f = 0$*.*
(iii) *If* $r_1 + r_2 = R$ *and* $r_1/r_2 \notin \mathrm{WA}_X$ *then for each integer* $m \geqslant 0$ *there exists a non-trivial function* $f \in (\mathfrak{N}_{r_1,r_2} \cap C^m)(B_R)$*.*
(iv) *If* $r_1 + r_2 > R$ *then there exists a non-trivial function* $f \in (\mathfrak{N}_{r_1,r_2} \cap C^\infty)(B_R)$*.*
(v) *If* $r_1/r_2 \in E_X$ *then there exists a non-trivial real analytic function* $f \in \mathfrak{N}_{r_1,r_2}(X)$*.*

We note that the situations described in assertions (i)–(v) actually occur for suitable r_1, r_2 (see properties (a), (b), (c) of the set WA_X).

Proof. Let $g \in G$ and let χ_i be the characteristic function of the ball B_{r_i}, $i = 1, 2$. Then

$$\chi_i(g^{-1}o) = \chi_i(go). \tag{7}$$

Using (1) and (2) one infers that

$$J^{-1/2}(\mathrm{Exp}^{-1}(go))\psi_\lambda(\mathrm{Exp}^{-1}(go)) = J^{-1/2}(\mathrm{Exp}^{-1}(g^{-1}o))\psi_{-\lambda}(\mathrm{Exp}^{-1}(g^{-1}o)),$$

where

$$\psi_\lambda(P) = \int_K e^{i\langle A_\lambda, \mathrm{Ad}(k)P\rangle} dk, \quad P \in \mathfrak{p}.$$

Hence

$$J^{-1/2}(\mathrm{Exp}^{-1}(go)) = J^{-1/2}(\mathrm{Exp}^{-1}(g^{-1}o)). \tag{8}$$

In view of (7), (8) and (4) relations (5) and (6) can be written as

$$f \times T_1 \leqslant 0 \quad \text{in} \quad B_{R-r_1},$$

$$f \times T_2 \geqslant 0 \quad \text{in} \quad B_{R-r_2},$$

where

$$T_i(x) = J^{-1/2}(\mathrm{Exp}^{-1}x)\chi_i(x), \quad x \in X.$$

Lemma 2.2.9 in [11] shows that $T_i \in \mathcal{E}'_{\natural\natural}(X)$ and

$$\widetilde{T_i}(\lambda) = c_i \mathbf{I}_{l/2}(r_i^{-1}\sqrt{\langle\lambda,\lambda\rangle}), \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},$$

where the constant c_i is independent of λ and $\mathbf{I}_\nu(z) = I_\nu(z)/z^\nu$. Then

$$f \times T_1 \times T_2 \leqslant 0 \quad \text{and} \quad f \times T_1 \times T_2 \geqslant 0$$

in $B_{R-r_1-r_2}$ so that

$$f \times T_1 \times T_2 = 0 \quad \text{in} \quad B_{R-r_1-r_2}.$$

Hence $f \times T_1 = 0$ in B_{R-r_1}, and $f \times T_2 = 0$ in B_{R-r_2}. Using [11, Theorem 2.2.9] we arrive at the desired assertion. □

In connection with Theorem 1 the following problem seems to be interesting.

Problem 1. *Is the assertion of Theorem 1 true for every* $f \in L^{1,\mathrm{loc}}(B_R)$*, where* $R = r_1 + r_2$*?*

Note, in conclusion, that the method of the proof of Theorem 1 makes it possible to obtain a similar result for rank-one symmetric spaces $X = G/K$ (see [11, Theorem 2.2.8]).

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Теорема о двух радиусах для взвешенных шаровых средних на симметрических пространствах.

Изучаются обобщения функций с нулевыми шаровыми средними на римановых симметрических пространствах $X = G/K$. Получен аналог локальной теоремы о двух радиусах для симметрических пространств некомпактного типа с комплексной группой G.

Ключевые слова: *симметрические пространства, нулевые сферические средние, теорема о двух радиусах.*

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Теорема про два радіуси для зважених кульових середніх на симетричних просторах.

Вивчаються узагальнення функцій з нульовими кульовими середніми на ріманових симетричних просторах $X = G/K$. Одержано аналог локальної теореми про два радіуси для симетричних просторів некомпактного типу з комплексною групою G.

Ключові слова: *симетричні простори, нульові сферичні середні, теорема про два радіуси*

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