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ON THE Γ-COMPACTNESS OF INTEGRAL FUNCTIONALS WITH DEGENERATE LOCALLY LIPSCHITZ INTEGRANDS AND VARYING DOMAINS OF DEFINITION

In this Note, we announce a new Γ-compactness result for a sequence of integral functionals defined on varying weighted Sobolev spaces. The result concerns the case where the degenerate integrands of the functionals satisfy a local Lipschitz condition but in general may not be convex with respect to the variable corresponding to the gradient of functions from domains of definition of the functionals.

Keywords: *varying weighted Sobolev spaces, integral functional, degenerate integrand, varying domains, Γ-convergence, Γ-compactness.*

1. Introduction. In this Note, we announce a Γ-compactness theorem for a sequence of integral functionals defined on varying weighted Sobolev spaces. The theorem concerns the case where the degenerate integrands of the functionals satisfy a local Lipschitz condition. The given result is a close analogue of the Γ-compactness theorem obtained in [4, 9] in the case where the integrands of integral functionals, defined on the same spaces, are convex with respect to the variable corresponding to the gradient of functions from domains of definition of the functionals.

In this connection we note that besides the mentioned works, the questions related to the Γ-convergence of integral functionals, defined on varying weighted Sobolev spaces, and the convergence of minimizers of the corresponding variational problems were studied in [5-8].

2. Preliminaries. Let $n \in \mathbb{N}$, $n \geqslant 2$, and let Ω be a bounded domain of $\mathbb{R}^n$. Let $p \in (1, n)$. Let ν be a nonnegative function on Ω with the properties: $\nu > 0$ almost everywhere in Ω and

$$\nu \in L^1_{\mathrm{loc}}(\Omega), \qquad \left(\frac{1}{\nu}\right)^{1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega). \tag{1}$$

We denote by $L^p(\nu, \Omega)$ the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that the function $\nu|u|^p$ is summable in Ω. $L^p(\nu, \Omega)$ is a Banach space with the norm

$$\|u\|_{L^p(\nu,\Omega)} = \left(\int_\Omega \nu |u|^p \, dx\right)^{1/p}.$$

We observe that by virtue of Young's inequlity and the second inclusion of (1) we have $L^p(\nu, \Omega) \subset L^1_{\mathrm{loc}}(\Omega)$.

We denote by $W^{1,p}(\nu, \Omega)$ the set of all functions $u \in L^p(\nu, \Omega)$ such that for every $i \in \{1, \ldots, n\}$ there exists the weak derivative $D_i u$, $D_i u \in L^p(\nu, \Omega)$. $W^{1,p}(\nu, \Omega)$ is a reflexive Banach space with the norm

$$\|u\|_{1,p,\nu} = \left(\int_\Omega \nu|u|^p\,dx + \sum_{i=1}^n \int_\Omega \nu|D_iu|^p\,dx\right)^{1/p}.$$

Due to the first inclusion of (1) we have $C_0^\infty(\Omega) \subset W^{1,p}(\nu,\Omega)$. We denote by $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in $W^{1,p}(\nu,\Omega)$.

Next, let $\{\Omega_s\}$ be a sequence of domains of $\mathbb{R}^n$ which are contained in Ω.

By analogy with the spaces introduced above we define the functional spaces corresponding to the domains Ω_s.

Let $s \in \mathbb{N}$. We denote by $L^p(\nu,\Omega_s)$ the set of all measurable functions $u : \Omega_s \to \mathbb{R}$ such that the function $\nu|u|^p$ is summable in Ω_s. $L^p(\nu,\Omega_s)$ is a Banach space with the norm

$$\|u\|_{L^p(\nu,\Omega_s)} = \left(\int_{\Omega_s} \nu|u|^p\,dx\right)^{1/p}.$$

By virtue of the second inclusion of (1) we have $L^p(\nu,\Omega_s) \subset L^1_{\text{loc}}(\Omega_s)$. We denote by $W^{1,p}(\nu,\Omega_s)$ the set of all functions $u \in L^p(\nu,\Omega_s)$ such that for every $i \in \{1,\dots,n\}$ there exists the weak derivative D_iu, $D_iu \in L^p(\nu,\Omega_s)$. $W^{1,p}(\nu,\Omega_s)$ is a Banach space with the norm

$$\|u\|_{1,p,\nu,s} = \left(\int_{\Omega_s} \nu|u|^p\,dx + \sum_{i=1}^n \int_{\Omega_s} \nu|D_iu|^p\,dx\right)^{1/p}.$$

We denote by $\widetilde{C}_0^\infty(\Omega_s)$ the set of all restrictions on Ω_s of functions from $C_0^\infty(\Omega)$. Due to the first inclusion of (1) we have $\widetilde{C}_0^\infty(\Omega_s) \subset W^{1,p}(\nu,\Omega_s)$. We denote by $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ the closure of the set $\widetilde{C}_0^\infty(\Omega_s)$ in $W^{1,p}(\nu,\Omega_s)$.

Observe that if $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and $s \in \mathbb{N}$, then $u|_{\Omega_s} \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$.

Definition 1. If $s \in \mathbb{N}$, then $q_s : \overset{\circ}{W}{}^{1,p}(\nu,\Omega) \to \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ is the mapping such that for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, $q_su = u|_{\Omega_s}$.

Definition 2. Let for every $s \in \mathbb{N}$, I_s be a functional on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$, and let I be a functional on $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. We say that the sequence $\{I_s\}$ Γ-converges to the functional I if the following conditions are satisfied:

1) for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ there exists a sequence $w_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that $\lim\limits_{s\to\infty} \|w_s - q_su\|_{L^p(\nu,\Omega_s)} = 0$ and $\lim\limits_{s\to\infty} I_s(w_s) = I(u)$;

2) for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and for every sequence $u_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that $\lim\limits_{s\to\infty} \|u_s - q_su\|_{L^p(\nu,\Omega_s)} = 0$ we have $\liminf\limits_{s\to\infty} I_s(u_s) \geqslant I(u)$.

3. Statement of the main result. Let $b \in L^1(\Omega)$, $b \geqslant 0$ in Ω, and let $\{\psi_s\}$ be a sequence of functions satisfying the following conditions: for every $s \in \mathbb{N}$, $\psi_s \in L^1(\Omega_s)$ and $\psi_s \geqslant 0$ in Ω_s; for every open cube $Q \subset \mathbb{R}^n$ we have

$$\limsup_{s\to\infty} \int_{Q\cap\Omega_s} \psi_s\,dx \leqslant \int_{Q\cap\Omega} b\,dx.$$

Next, let c_i, $i = 1, \dots, 4$, be positive constants, and let $f_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$, $s \in \mathbb{N}$, be a sequence of functions such that: for every $s \in \mathbb{N}$ and for every $\xi \in \mathbb{R}^n$ the function $f_s(\cdot, \xi)$ is measurable in Ω_s; for every $s \in \mathbb{N}$, for almost every $x \in \Omega_s$ and for every $\xi \in \mathbb{R}^n$,

$$c_1\nu(x)|\xi|^p - \psi_s(x) \leqslant f_s(x, \xi) \leqslant c_2\nu(x)|\xi|^p + \psi_s(x); \tag{2}$$

for every $s \in \mathbb{N}$, for almost every $x \in \Omega_s$ and for every $\xi, \xi' \in \mathbb{R}^n$,

$$|f_s(x,\xi) - f_s(x,\xi')| \leqslant c_3\nu(x)(1+|\xi|+|\xi'|)^{p-1}|\xi-\xi'| + c_4[\nu(x)]^{1/p}[\psi_s(x)]^{(p-1)/p}|\xi-\xi'|. \tag{3}$$

Obviously, for every $s \in \mathbb{N}$, f_s is a Carathéodory function, and if $s \in \mathbb{N}$ and $u \in W^{1,p}(\nu, \Omega_s)$, we have $f_s(x, \nabla u) \in L^1(\Omega_s)$.

Definition 3. If $s \in \mathbb{N}$, then J_s is the functional on $\widetilde{W}_0^{1,p}(\nu, \Omega_s)$ such that for every function $u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$,

$$J_s(u) = \int_{\Omega_s} f_s(x, \nabla u)\, dx.$$

We denote by $\mathcal{F}_{\mathrm{Lip}}$ the set of all functions $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions: for every $\xi \in \mathbb{R}^n$ the function $f(\cdot, \xi)$ is measurable in Ω; for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$,

$$-b(x) \leqslant f(x, \xi) \leqslant c_2\nu(x)|\xi|^p + b(x);$$

there exist positive constants c' and c'' such that for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^n$,

$$|f(x,\xi) - f(x,\xi')| \leqslant c'\nu(x)(1 + |\xi| + |\xi'|)^{p-1}|\xi - \xi'| + c''[\nu(x)]^{1/p}[b(x)]^{(p-1)/p}|\xi - \xi'|.$$

It is easy to see that for every $f \in \mathcal{F}_{\mathrm{Lip}}$ and for every $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ the function $f(x, \nabla u)$ is summable in Ω.

Definition 4. If $f \in \mathcal{F}_{\mathrm{Lip}}$, then J^f is the functional on $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ such that for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$,

$$J^f(u) = \int_{\Omega} f(x, \nabla u)\, dx.$$

Theorem 1. *Assume that* $\nu \in L^1(\Omega)$, *and let* $g_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$, $s \in \mathbb{N}$, *be a sequence of functions satisfying the following conditions*: *for every* $s \in \mathbb{N}$ *and for every* $\xi \in \mathbb{R}^n$ *the function* $g_s(\cdot, \xi)$ *is measurable in* Ω_s; *for every* $s \in \mathbb{N}$ *and for almost every* $x \in \Omega_s$ *the function* $g_s(x, \cdot)$ *is convex in* $\mathbb{R}^n$; *if* $\varepsilon > 0$, *then there exists* $\sigma_\varepsilon > 0$ *such that for every* $s \in \mathbb{N}$, *for almost every* $x \in \Omega_s$ *and for every* $\xi \in \mathbb{R}^n$,

$$|f_s(x, \xi) - g_s(x, \xi)| \leqslant \varepsilon\nu(x)|\xi|^p + \sigma_\varepsilon\psi_s(x). \tag{4}$$

Next, suppose that there exists a sequence of nonempty open sets $\Omega^{(k)}$ *of* $\mathbb{R}^n$ *such that*:

a) *for every* $k \in \mathbb{N}$, $\overline{\Omega^{(k)}} \subset \Omega^{(k+1)} \subset \Omega$;

b) $\lim\limits_{k\to\infty} \operatorname{meas}(\Omega \setminus \Omega^{(k)}) = 0$;

c) *for every $k \in \mathbb{N}$ the functions ν and b are bounded in $\Omega^{(k)}$.*

Then there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $f \in \mathcal{F}_{\mathrm{Lip}}$ such that the sequence $\{J_{s_j}\}$ Γ-converges to the functional J^f.

The proof of the theorem will be published in a forthcoming authors' article. We only note that the proof is analogous to that of the Γ-compactness result given in [4, 9] for the case where the integrands of integral functionals defined on the spaces $\widetilde{W}_0^{1,p}(\nu, \Omega_s)$ are convex with respect to the variable corresponding to the gradient of functions in these spaces. At the same time, some additional difficulties are connected exactly with the fact that in the present article for the integrands f_s conditions (3) and (4) are used instead of the convexity of the functions $f_s(x, \cdot)$ for every $s \in \mathbb{N}$ and for almost every $x \in \Omega_s$ which was assumed in [4, 9].

We observe that some Γ-compactness results for integral functionals with degenerate integrands and an unvarying domain of definition were established in [1–3] for the case where the integrands are convex with respect to the variable corresponding to the gradient of functions from the domain of definition of the functionals.

4. An example. It is not difficult to verify that condition (2) along with the additional requirement that

$$\text{for every } s \in \mathbb{N} \text{ and for almost every } x \in \Omega_s \text{ the function } f_s(x, \cdot) \text{ is convex in } \mathbb{R}^n \qquad (5)$$

implies that conditions (3) and (4) are satisfied, and in this case the constants c_3 and c_4 depend only on p and c_2, and for every $s \in \mathbb{N}$, $g_s = f_s$.

The following example shows that sequences of integrands satisfying conditions (2)–(4) may not satisfy condition (5).

In fact, let $p \geqslant 2$, and let for every $s \in \mathbb{N}$ and for every $(x, \xi) \in \Omega_s \times \mathbb{R}^n$,

$$f_s(x, \xi) = \nu(x)|\xi|^p - [\nu(x)]^{(p-1)/p}[\psi_s(x)]^{1/p}|\xi|^{p-1}.$$

Then for every $s \in \mathbb{N}$, for every $x \in \Omega_s$ and for every $\xi \in \mathbb{R}^n$ inequality (2) holds with $c_1 = 1/p$ and $c_2 = 1$. Moreover, for every $s \in \mathbb{N}$, for every $x \in \Omega_s$ and for every $\xi, \xi' \in \mathbb{R}^n$ inequality (3) holds with $c_3 = 2(p-1)$ and $c_4 = 1$. Finally, defining for every $s \in \mathbb{N}$ the function $g_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ by $g_s(x, \xi) = \nu(x)|\xi|^p$, $(x, \xi) \in \Omega_s \times \mathbb{R}^n$, we establish that the following properties hold: for every $s \in \mathbb{N}$ and for every $\xi \in \mathbb{R}^n$ the function $g_s(\cdot, \xi)$ is measurable in Ω_s; for every $s \in \mathbb{N}$ and for every $x \in \Omega_s$ the function $g_s(x, \cdot)$ is convex in $\mathbb{R}^n$; if $\varepsilon > 0$, then for every $s \in \mathbb{N}$, for every $x \in \Omega_s$ and for every $\xi \in \mathbb{R}^n$ inequality (4) holds with $\sigma_\varepsilon = \varepsilon^{1-p}$. On the other hand, supposing that for every $s \in \mathbb{N}$, $\psi_s > 0$ a. e. in Ω_s, we find that for every $s \in \mathbb{N}$ and for almost every $x \in \Omega_s$ the function $f_s(x, \cdot)$ is not convex in $\mathbb{R}^n$, and therefore, property (5) does not hold.

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О Γ-компактности интегральных функционалов с вырождающимися локально липшицевыми интегрантами и переменными областями определения.

В заметке анонсирован новый результат о Γ-компактности последовательности интегральных функционалов, определенных на переменных весовых пространствах Соболева. Этот результат относится к случаю, когда вырождающиеся интегранты функционалов удовлетворяют локальному условию Липшица, но, вообще говоря, могут не быть выпуклыми относительно переменной, соответствующей градиенту функций из областей определения функционалов.

Ключевые слова: *переменные весовые пространства Соболева, интегральный функционал, вырождающийся интегрант, переменные области,* Γ*-сходимость,* Γ*-компактность.*

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Про Γ-компактність інтегральних функціоналів з виродними локально ліпшіцівими інтегрантами і змінними областями визначення.

В замітці анонсований новий результат про Γ-компактність послідовності інтегральних функціоналів, визначених на змінних вагових просторах Соболєва. Цей результат належить випадку, коли виродні інтегранти функціоналів задовольняють локальну умову Ліпшіца, але, взагалі кажучи, можуть не бути опуклими відносно змінної, що відповідає градієнту функцій з областей визначення функціоналів.

Ключові слова: *змінні вагові простори Соболєва, інтегральний функціонал, виродний інтегрант, змінні області,* Γ*-збіжність,* Γ*-компактність.*

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