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ON LOCAL GRADIENT ESTIMATES FOR ANISOTROPIC ELLIPTIC AND PARABOLIC EQUATIONS

It is considered wide class of anisotropic elliptic and parabolic equations. Sharp local pointwise estimates for gradient of solutions of such equations are established.

Keywords: *anisotropic elliptic and parabolic equations, poinwise estimates for gradient of solutions.*

1. Introduction. We shall consider the gradient bounds of weak solutions for quasilinear anisotropic equations of the form

$$\sum_{i=1}^{n}\frac{d}{dx_i}a_i\Big(x,u,\frac{\partial u}{\partial x}\Big)=a_0\Big(x,u,\frac{\partial u}{\partial x}\Big),\qquad x\in\Omega \tag{1}$$

or

$$\frac{\partial u}{\partial t}-\sum_{i=1}^{n}\frac{d}{dx_i}a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big)=a_0\Big(x,t,u,\frac{\partial u}{\partial x}\Big),\qquad (x,t)\in\Omega_T. \tag{2}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, $\Omega_T=\Omega\times(0,T)$, $0<T<\infty$, $a_i(\cdot,u,\xi)$, $i=0,1,\ldots,n$ satisfy the Caratheodory conditions and there exist constants K_1,K_2 such that the inequalities

$$\begin{gathered}\sum_{i=1}^{n}a_i(\cdot,u,\xi)\xi_i\ge K_1\Big(\sum_{i=1}^{n}|\xi_i|^{p_i}-1\Big),\quad |a_i(\cdot,u,\xi)|\le K_2\Big(\sum_{j=1}^{n}|\xi_j|^{p_j}\Big)^{1-\frac{1}{p_i}}+K_2,\\ |a_0(\cdot,u,\xi)|\le K_2\Big(\sum_{i=1}^{n}|\xi_i|^{p_i(1-\frac{1}{p})}+1\Big),\quad i=\overline{1,n},\end{gathered} \tag{3}$$

and

$$\left.\begin{aligned}&\sum_{i,j=1}^{n}\frac{\partial a_i(\cdot,u,\xi)}{\partial\xi_j}\eta_i\eta_j\ge K_1\sum_{i=1}^{n}|\xi_i|^{p_i-2}\eta_i^2,\\ &\sum_{i,j=1}^{n}\frac{\partial a_i(\cdot,u,\xi)}{\partial\xi_j}\lambda_i\eta_j\le K_2\Big(\sum_{i=1}^{n}|\xi_i|^{p_i-2}\lambda_i^2\Big)^{\frac12}\Big(\sum_{i=1}^{n}|\xi_i|^{p_i-2}\eta_i^2\Big)^{\frac12},\\ &|\xi_s|\Big|\frac{\partial a_i(\cdot,u,\xi)}{\partial u}\Big|+\Big|\frac{\partial a_i(\cdot,u,\xi)}{\partial x_s}\Big|\le K_2|\xi_s|\,|\xi_i|^{p_i-2},\quad i,s=\overline{1,n},\\ &\Big|\frac{\partial a_0(\cdot,u,\xi)}{\partial\xi_j}\Big|\le K_2|\xi_j|^{p_j-2},\quad j=\overline{1,n},\\ &|\xi_s|\Big|\frac{\partial a_0(\cdot,u,\xi)}{\partial u}\Big|+\Big|\frac{\partial a_0(\cdot,u,\xi)}{\partial x_s}\Big|\le K_2|\xi_s|\sum_{i=1}^{n}|\xi_i|^{p_i-2},\quad s=\overline{1,n}\end{aligned}\right\} \tag{4}$$

are valid for any $\xi, \lambda, \eta \in \mathbb{R}^n$.

$$2 < p_1 \le p_2 \le \dots \le p_n < \infty, \qquad \frac{1}{p} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}, \qquad p < n. \tag{5}$$

As a simplest model example of equations (1), (2) we can keep in the mind

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) = 0, \qquad x \in \Omega, \tag{6}$$

or

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) = 0, \qquad (x,t) \in \Omega_T. \tag{7}$$

It is well known that any weak solution of equations (1), (2) in isotropic case (i.e. $p_1 = \dots = p_n = p$) belongs to the space $C^{1,\alpha}$ locally. Review of these results can be found in the monograph of E. Di Benedetto ([7]). Recently, many authors (see, for example [1, 3, 2, 4, 5, 9, 10, 11, 13, 14, 19, 21, 18, 20, 22, 23, 24, 12, 28]) studied regularity of weak solutions of equations with nonstandard growth conditions. The local boundedness to weak solutions of equation (1) under conditions (3) and additional restrictions

$$1 < p_1 \le p_2 \le \dots \le p_n \le \frac{np}{n-p} \tag{8}$$

was obtained in [10].

This assumption is significant; there are several examples (see [22, 12]) of such equation with unbounded solutions if $p_n > \frac{np}{n-p}$.

The boundedness of weak solutions of equation (2) was derived in [28] for the case

$$1 < p_1 \le p_2 \le \dots \le p_n \le p\Big(1 + \frac{2}{n}\Big), \qquad p > \frac{2n}{n+1}. \tag{9}$$

For equation (1) and the corresponding minimization problem there are known many results on L_∞-estimates for gradient of solution in the ball $B(r) = \{x : |x - x^{(0)}| \le r\}$ only (see [1, 2, 3, 18, 20, 21]). Review of this results can be found in [21].

The main distinctive feature of the present paper is an integral estimate of the directional derivatives of the gradient of the solution. We provide these estimates for directional derivatives in any directions in comparison with earlier works where the integral estimates of the total second order derivatives were usually proved.

The paper is organized as follows.

Main definitions and results are formulated in Section 2, proofs of this results are given in Sections 3 and 4.

2. Formulation of assumptions and main results. We recall the embedding theorem (see, for example, [13]).

Lemma 1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain, $u(x) \in \overset{\circ}{W}{}^1_{\bar{p}}(\Omega) = \overset{\circ}{W}{}^1_{p_1,\dots,p_n}(\Omega)$. If $1 < p < n$, then $u(x) \in L_q(\Omega)$, $q = \frac{np}{n-p}$ and the following estimate holds*

$$\|u\|_{L_q(\Omega)} \leq c \prod_{i=1}^{n} \bigg(\int\limits_{\Omega} \Big| \frac{\partial u}{\partial x_i} \Big|^{p_i} dx \bigg)^{\frac{1}{np_i}}, \tag{10}$$

where the constant c depends only on $n, p_1, \dots, p_n$.

Definition 1. We shall say that function $u(x)$ is generalized solution of equation (1) in Ω if $u(x) \in W^1_{\bar{p},loc}(\Omega) = W^1_{p_1,\dots,p_n,loc}(\Omega)$ and the following integral identity holds

$$\sum_{i=1}^{n} \int\limits_{\Omega} a_i\Big(x, u, \frac{\partial u}{\partial x}\Big) \frac{\partial \varphi}{\partial x_i}\, dx + \int\limits_{\Omega} a_0\Big(x, u, \frac{\partial u}{\partial x}\Big)\, \varphi\, dx = 0 \tag{11}$$

for an arbitrary function $\varphi(x) \in \overset{\circ}{W}{}^1_{\bar{p},loc}(\Omega)$.

Definition 2. We shall say that function $u(x,t)$ is generalized solution of equation (2) in Ω_T if $u(x,t) \in V_{2,\bar{p},loc}(\Omega_T) \equiv C_{loc}\big(0,T; L_{2,loc}(\Omega)\big) \cap L_{\bar{p}}\big(0,T; W^1_{\bar{p},loc}(\Omega)\big)$ and the following integral identity holds

$$\int\limits_{\Omega} u(x,t)\varphi(x,t)\, dx \Big|_{t_1}^{t_2} + \int\limits_{t_1}^{t_2} \int\limits_{\Omega} \Big\{ - u\varphi_t + \sum_{i=1}^{n} a_i\Big(x,t,u,\frac{\partial u}{\partial x}\Big) \frac{\partial \varphi}{\partial x_i}$$
$$- a_0\Big(x,t,u,\frac{\partial u}{\partial x}\Big)\varphi \Big\} dx\, dt = 0 \tag{12}$$

for all intervals $[t_1,t_2] \subset [0,T)$ and an arbitrary function $\varphi(x,t) \in W^1_{2,loc}\big(0,T; L_2(\Omega)\big) \cap L_{\bar{p},loc}\big(0,T; \overset{\circ}{W}{}^1_{\bar{p},loc}(\Omega)\big)$.

Theorem 1. *Let $u(x)$ be an arbitrary solution of equation (1) in Ω. Assume that structural conditions (3), (4) are satisfied. Suppose also that parameters p_i satisfy conditions*

$$2 < p_1 \leq p_2 \leq \dots \leq p_n < \frac{np_1}{n - p_1}, \tag{13}$$

then $u(x) \in W^1_{\infty,loc}(\Omega)$. Moreover, for any set $B(r,\theta) = \{x_i : |x_i - x_i^{(0)}| \leq \theta^{\alpha \frac{\beta - p_i}{p_i}} r^{\frac{p}{p_i}},\ i = \overline{1,n}\} \subset \Omega$, there exists positive constant C depending only on $n, p_1, \dots, p_n, K_1, K_2$ such that for any $1 \leq s \leq n$ and any $\alpha, \beta, \theta > 0$ the inequality holds

$$\operatorname*{ess\,sup}_{B(\frac{r}{2},\theta)} \Big[\Big| \frac{\partial u}{\partial x_s} \Big|^{p_s} \theta^{\alpha\beta} r^p \Big]^{1 - n(\frac{1}{p_1} - \frac{1}{p_n})} \leq C \bigg(1 + \theta^{-n\alpha\frac{\beta - p}{p} + \alpha\beta} r^{-n+p} \sum_{i=1}^{n} \int\limits_{B(r,\theta)} \Big| \frac{\partial u}{\partial x_i} \Big|^{p_i} dx \bigg). \tag{14}$$

Theorem 2. *Let $u(x,t)$ be an arbitrary solution of equation* (2) *in Ω_T. Assume that structural conditions* (3), (4) *are satisfied. Suppose also that parameters p_i satisfy conditions*

$$2 < p_1 \le p_2 \le \dots \le p_n < \frac{n+4}{n+2} p_1, \tag{15}$$

then $\dfrac{\partial u}{\partial x} \in L_{\infty,loc}(\Omega_T)$. Moreover, for any $Q(r,\theta,\sigma) = B(r,\theta) \times (t_0-\sigma, t_0+\sigma)$, there exists positive constant C depending only on $n, p_1, \dots, p_n, K_1, K_2$ such that for any $1 \le s \le n$ and any $\alpha, \beta, \theta > 0$ the inequality holds

$$\operatorname*{ess\,sup}_{Q(\frac{r}{2},\theta,\frac{\sigma}{2})} \Big[\Big|\frac{\partial u}{\partial x_s}\Big|^{p_s} \theta^{\alpha\beta} r^p\Big]^{\frac{n+4}{p_n} - \frac{n+2}{p_1}}$$
$$\le C\big(\sigma^{-1} + \theta^{-\alpha(\beta-2)} r^{-p}\big)^{\frac{n+2}{2}} \theta^{\alpha\beta\frac{p-2}{p}\frac{n}{2}} r^{(p-2)\frac{n}{2}} \iint\limits_{Q(r,\theta,\sigma)} \Big(1 + \theta^{\alpha\beta} r^p \sum_{k=1}^n \Big|\frac{\partial u}{\partial x_k}\Big|^{p_k}\Big) dx\, dt. \tag{16}$$

Precision of the estimates (13), (15) follows, for example, from sharp estimates for fundamental solutions to anisotropic elliptic and parabolic equations (see [26, 27]).

3. Proof of Theorem 1. Further we will denote by c, c_i different positive constants depending on $n, p_1, \dots, p_n$, K_1, K_2 only.

Taking the x_s-derivative of equation (1) and integrating over Ω we obtain

$$\sum_{i=1}^n \int\limits_\Omega \frac{da_i(x,u,\frac{\partial u}{\partial x})}{dx_s} \frac{\partial \varphi}{\partial x_i} dx + \int\limits_\Omega \frac{da_0(x,u,\frac{\partial u}{\partial x})}{dx_s} \varphi\, dx = 0 \quad \text{for every } s,\ 1 \le s \le n \tag{17}$$

Here $\varphi(x)$ is an arbitrary sufficiently smooth function vanishing on $\partial\Omega$,

$$\frac{dA\big(x,u,\frac{\partial u}{\partial x}\big)}{dx_s} = \sum_{j=1}^n \frac{\partial A}{\partial u_{x_j}} \frac{\partial^2 u}{\partial x_j \partial x_s} + \frac{\partial A}{\partial u}\frac{\partial u}{\partial x_s} + \frac{\partial A}{\partial x_s}.$$

Without loss of generality it can be assumed that $x_0 = 0$ and r satisfies the condition

$$\theta^{\alpha\frac{\beta-p_i}{p_i}} r^{\frac{p}{p_i}} \le \min\big\{1, \operatorname{dist}(0,\partial\Omega)\big\}, \quad i = \overline{1,n}, \quad \theta^{\alpha\beta} r^p \le 1 \tag{18}$$

thus $B(r,\theta) \subset \Omega$.

Let us introduce the nonnegative cut-off function $\xi(x)$ for the set $B(r,\theta)$,

$$\xi(x) \equiv 1 \quad \text{in} \quad B\Big(\frac{r}{2},\theta\Big), \quad 0 \le \xi(x) \le 1,$$
$$\Big|\frac{\partial \xi(x)}{\partial x_i}\Big| \le c\theta^{-\alpha\frac{\beta-p_i}{p_i}} r^{-\frac{p}{p_i}} \qquad i = 1, \dots, n.$$

Denote also

$$b = \theta^{\alpha\beta} r^p, \qquad w(x) = 1 + \sum_{k=1}^n \Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}} b^{\frac{1}{2}}. \tag{19}$$

Lemma 2. *Let all conditions of Theorem 1 are satisfied. Then there exists a positive constant C_1 depending only on $n, p_1, \dots, p_n, K_1, K_2$, such that for any $1 \le s \le n,\ m \ge 0,\ l \ge n$ the following inequality holds*

$$\sum_{i=1}^{n} \int_{\Omega} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2} \Big(\frac{\partial^2 u}{\partial x_i \partial x_s}\Big)^2 w^m(x)\xi^l(x)\,dx$$
$$\le C_1(l+m)^2 b^{-1}\theta^{-2\alpha\frac{\beta-p_s}{p_s}} r^{-2\frac{p}{p_s}} \int_{\Omega} w^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x)\xi^{l-2}(x)\,dx. \qquad (20)$$

Proof. We can assume without loss of generality that

$$\sum_{i,s=1}^{n} \int_{\Omega} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2} \Big(\frac{\partial^2 u}{\partial x_i \partial x_s}\Big)^2 w^m(x)\xi^l(x)\,dx < \infty,$$

$$\int_{\Omega} w^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x)\xi^{l-2}(x)\,dx < \infty.$$

Later we shall prove that there exist positive numbers m_0, l_0, such that the integrals in (20) are finite for all $m \ge m_0,\ l \ge l_0$.

Let us substitute in integral identity (17) the test function $\varphi(x) = \dfrac{\partial u}{\partial x_s} w_\varepsilon^m(x)\xi^l(x)$, $m \ge 0,\ l \ge n$, $\varepsilon > 0$, $w_\varepsilon(x) = 1 + \sum_{k=1}^{n}(|\frac{\partial u}{\partial x_k}|^2 + \varepsilon^2)^{\frac{p_k}{4}} b^{\frac{1}{2}}$. Then for every s, $1 \le s \le n$, we get

$$\sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial a_i}{\partial u_{x_j}} \frac{\partial^2 u}{\partial x_i \partial x_s} \frac{\partial^2 u}{\partial x_j \partial x_s} w_\varepsilon^m(x)\xi^l(x)\,dx$$
$$= -\sum_{i,j=1}^{n} \int_{\Omega} \frac{\partial a_i}{\partial u_{x_j}} \frac{\partial^2 u}{\partial x_j \partial x_s} \Big\{ l \frac{\partial u}{\partial x_s} w_\varepsilon^m(x) \frac{\partial \xi(x)}{\partial x_i} \xi^{l-1}(x)$$
$$+ m b^{\frac{1}{2}} \frac{\partial u}{\partial x_s} w_\varepsilon^{m-1}(x) \sum_{k=1}^{n} \frac{p_k}{2} \frac{\partial u}{\partial x_k} \Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2 + \varepsilon^2\Big)^{\frac{p_k}{4}-1} \frac{\partial^2 u}{\partial x_k \partial x_i} \xi^l(x) \Big\} dx$$
$$- \sum_{i=1}^{n} \int_{\Omega} \Big(\frac{\partial a_i}{\partial u} \frac{\partial u}{\partial x_s} + \frac{\partial a_i}{\partial x_s} \Big) \Big\{ \frac{\partial^2 u}{\partial x_i \partial x_s} w_\varepsilon^m(x)\xi^l(x)$$
$$+ m b^{\frac{1}{2}} \frac{\partial u}{\partial x_s} w_\varepsilon^{m-1}(x) \sum_{k=1}^{n} \frac{p_k}{2} \frac{\partial u}{\partial x_k} \Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2 + \varepsilon^2\Big)^{\frac{p_k}{4}-1} \frac{\partial^2 u}{\partial x_k \partial x_i} \xi^l(x)$$
$$+ l \frac{\partial u}{\partial x_s} w_\varepsilon^m(x) \frac{\partial \xi(x)}{\partial x_i} \xi^{l-1}(x) \Big\} dx - \int_{\Omega} \frac{da_0}{dx_s} \frac{\partial u}{\partial x_s} w_\varepsilon^m(x)\xi^l(x)\,dx. \qquad (21)$$

Estimating integrals in the right-hand side of (21) due to conditions (4) and applying Young's inequality, we get

$$\sum_{i=1}^{n}\int\limits_{\Omega}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big(\frac{\partial^2 u}{\partial x_i\partial x_s}\Big)^2 w_\varepsilon^m(x)\xi^l(x)\,dx$$

$$\le c_1 b(l+m)^2\sum_{i,k=1}^{n}\int\limits_{\Omega}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big|\frac{\partial u}{\partial x_s}\Big|^2\Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2+\varepsilon^2\Big)^{\frac{p_k}{2}-1}\Big(\frac{\partial^2 u}{\partial x_i\partial x_k}\Big)^2 w_\varepsilon^{m-2}(x)\xi^l(x)\,dx$$

$$+c_1(l+m)^2\sum_{i=1}^{n}\int\limits_{\Omega}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big|\frac{\partial u}{\partial x_s}\Big|^2\Big|\frac{\partial \xi(x)}{\partial x_i}\Big|^2 w_\varepsilon^m(x)\xi^{l-2}(x)\,dx$$

$$+c_1(l+m)^2\sum_{i=1}^{n}\int\limits_{\Omega}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big|\frac{\partial u}{\partial x_s}\Big|^2 w_\varepsilon^m(x)\xi^l(x)\,dx=\sum_{t=1}^{3}I_t. \tag{22}$$

After simple computations we have

$$I_1\le c_2 b^{-\frac{2}{p_s}+\frac12}(l+m)^2\sum_{i,k=1}^{n}\int\limits_{\Omega}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2+\varepsilon^2\Big)^{\frac{p_k}{4}-1}\Big(\frac{\partial^2 u}{\partial x_i\partial x_k}\Big)^2 w_\varepsilon^{m-1+\frac{4}{p_1}}\xi^2(x)\,dx. \tag{23}$$

Taking into account the definition of number b , we obtain

$$I_2\le c_3 b^{-1-\frac{2}{p_s}}(l+m)^2\sum_{i=1}^{n}b^{\frac{2}{p_i}}\int\limits_{\Omega}w_\varepsilon^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}\Big|\frac{\partial\xi(x)}{\partial x_i}\Big|^2\xi^{l-2}(x)\,dx$$

$$\le c_4 b^{-1}\theta^{-2\alpha\frac{\beta-p_s}{p_s}}r^{-2\frac{p}{p_s}}(l+m)^2\int\limits_{\Omega}w_\varepsilon^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x)\xi^{l-2}(x)\,dx. \tag{24}$$

In the same way, using also condition (18), we deduce

$$I_3\le c_5 b^{-1}\theta^{-2\alpha\frac{\beta-p_s}{p_s}}r^{-2\frac{p}{p_s}}(l+m)^2\int\limits_{\Omega}w_\varepsilon^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x)\xi^{l-2}(x)\,dx. \tag{25}$$

To estimate the right-hand side of (23) let us substitute into integral identity (17) the test function

$$\varphi(x)=\frac{p_k}{2}\frac{\partial u}{\partial x_k}\Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2+\varepsilon^2\Big)^{\frac{p_k}{4}-1}w^{m-1+\frac{4}{p_1}}\xi^l(x),$$

summing over all $k=1,\dots,n$, we have

$$I_4+I_5=\sum_{i,j,k=1}^{n}\frac{p_k}{2}\int\limits_{\Omega}\frac{\partial a_i}{\partial u_{x_j}}\frac{\partial^2 u}{\partial x_i\partial x_k}\frac{\partial^2 u}{\partial x_j\partial x_k}\Big\{\Big(\frac{p_k}{2}-1\Big)\Big|\frac{\partial u}{\partial x_k}\Big|^2+\varepsilon^2\Big\}$$

$$\times\Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2+\varepsilon^2\Big)^{\frac{p_k}{4}-2}w_\varepsilon^{m-1+\frac{4}{p_1}}(x)\xi^l(x)\,dx$$

$$+ b^{\frac{1}{2}}\Big(m-1+\frac{4}{p_1}\Big)\sum_{i,j,k,t=1}^{n}\frac{p_k}{2}\frac{p_t}{2}\int\limits_{\Omega}\frac{\partial a_i}{\partial u_{x_j}}\frac{\partial^2 u}{\partial x_i\partial x_t}\frac{\partial^2 u}{\partial x_j\partial x_k}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_t}$$
$$\times\Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2+\varepsilon^2\Big)^{\frac{p_k}{4}-1}\Big(\Big|\frac{\partial u}{\partial x_t}\Big|^2+\varepsilon^2\Big)^{\frac{p_t}{4}-1}w_\varepsilon^{m-2+\frac{4}{p_1}}(x)\xi^l(x)\,dx$$
$$= -l\sum_{i,j,k=1}^{n}\frac{p_k}{2}\int\limits_{\Omega}\frac{\partial a_i}{\partial u_{x_j}}\frac{\partial^2 u}{\partial x_j\partial x_k}\frac{\partial u}{\partial x_k}\Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2+\varepsilon^2\Big)^{\frac{p_k}{4}-1}w_\varepsilon^{m-1+\frac{4}{p_1}}(x)\frac{\partial\xi(x)}{\partial x_i}\xi^l(x)\,dx$$
$$-\sum_{i,k=1}^{n}\frac{p_k}{2}\int\limits_{\Omega}\Big(\frac{\partial a_i}{\partial u}\frac{\partial u}{\partial x_k}+\frac{\partial a_i}{\partial x_k}\Big)\frac{\partial}{\partial x_i}\Big\{\frac{\partial u}{\partial x_k}\Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2+\varepsilon^2\Big)^{\frac{p_k}{4}-1}w_\varepsilon^{m-1+\frac{4}{p_1}}(x)\xi^l(x)\Big\}dx$$
$$-\sum_{k=1}^{n}\frac{p_k}{2}\int\limits_{\Omega}\frac{da_0}{dx_k}\frac{\partial u}{\partial x_k}\Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2+\varepsilon^2\Big)^{\frac{p_k}{4}-1}w_\varepsilon^{m-1+\frac{4}{p_1}}\xi^l(x)\,dx=\sum_{h=6}^{8}I_h. \qquad (26)$$

Conditions (4) imply

$$I_9 = c_6\min\Big(\frac{p_1}{2}-1,1\Big)\sum_{i,k=1}^{n}\int\limits_{\Omega}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2+\varepsilon^2\Big)^{\frac{p_k}{4}-1}$$
$$\times\Big(\frac{\partial^2 u}{\partial x_i\partial x_k}\Big)^2 w_\varepsilon^{m-1+\frac{4}{p_1}}(x)\xi^l(x)\,dx\le I_4+I_5. \qquad (27)$$

Using conditions (4) and Young's inequality, we get

$$|I_6|\le\frac{1}{8}I_9$$
$$+c_7(l+m)^2\sum_{i,k=1}^{n}\int\limits_{\Omega}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big(\Big|\frac{\partial u}{\partial x_k}\Big|^2+\varepsilon^2\Big)^{\frac{p_k}{4}}w_\varepsilon^{m-1+\frac{4}{p_1}}(x)\Big|\frac{\partial\xi(x)}{\partial x_i}\Big|^2\xi^{l-2}(x)\,dx$$
$$\le\frac{1}{8}I_9+c_8(l+m)^2 b^{-\frac{3}{2}}\sum_{i=1}^{n}b^{\frac{2}{p_i}}\int\limits_{\Omega}w_\varepsilon^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x)\Big|\frac{\partial\xi(x)}{\partial x_i}\Big|^2\xi^{l-2}(x)\,dx. \qquad (28)$$

By the same argument

$$|I_7|+|I_8|\le\frac{1}{8}I_9+c_9(l+m)^2 b^{-\frac{3}{2}}\sum_{i=1}^{n}b^{\frac{2}{p_i}}\theta^{-2\alpha\frac{\beta-p_i}{p_i}}r^{-2\frac{p}{p_i}}\int\limits_{\Omega}w^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x)\xi^{l-2}(x)\,dx. \quad (29)$$

Now from (22) due to estimates (23)–(29), conditions (18), the choice of b and letting $\varepsilon\to 0$ we get inequality (20). This completes the proof of Lemma 2. □

The application of embedding Lemma 2.1 with $p_1 = \dots = p_n = 2$ yields

$$\Big(\int\limits_\Omega w^m(x)\xi^l(x)\,dx\Big)^{\frac{n-2}{n}}$$
$$\le c_{10}\prod_{s=1}^n\Big(\int\limits_\Omega w^{m\frac{n-2}{n}-2}(x)\Big|\frac{\partial w}{\partial x_s}\Big|^2\xi^{l\frac{n-2}{n}}(x)\,dx+\int\limits_\Omega w^{m\frac{n-2}{n}}(x)\Big|\frac{\partial\xi(x)}{\partial x_s}\Big|^2\xi^{l\frac{n-2}{n}-2}(x)\,dx\Big)^{\frac1n}$$
$$\le c_{11}(m+l)^2\prod_{s=1}^n\Big(b\sum_{i=1}^n\int\limits_\Omega\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big(\frac{\partial^2 u}{\partial x_i\partial x_s}\Big)^2 w^{m\frac{n-2}{n}-2}(x)\xi^{l\frac{n-2}{n}}(x)\,dx$$
$$+\int\limits_\Omega w^{m\frac{n-2}{n}}(x)\Big|\frac{\partial\xi(x)}{\partial x_s}\Big|^2\xi^{l\frac{n-2}{n}-2}(x)\,dx\Big)^{\frac1n}. \qquad (30)$$

Combining (20) and (30), we obtain

$$\Big(\int\limits_\Omega w^m(x)\xi^l(x)\,dx\Big)^{\frac{n-2}{n}}$$
$$\le c_{12}(m+l)^c\Big(\prod_{s=1}^n\theta^{-2\alpha\frac{\beta-p_s}{p_s}}r^{-2\frac{p}{p_s}}\Big)^{\frac1n}\int\limits_\Omega w^{m\frac{n-2}{n}+\frac{4}{p_1}-\frac{4}{p_n}}(x)\xi^{l\frac{n-2}{n}-2}(x)\,dx.$$

This inequality implies

$$\int\limits_\Omega w^m(x)\xi^l(x)\,dx\le c_{13}(m+l)^c\Big(\theta^{-2\alpha\frac{\beta-p}{p}}r^{-2}\int\limits_\Omega w^{m\frac{n-2}{n}+\frac{4}{p_1}-\frac{4}{p_n}}(x)\xi^{l\frac{n-2}{n}-2}(x)\,dx\Big)^{\frac{n}{n-2}}. \qquad (31)$$

Taking into account (31), we derive form (31) necessary estimate (20). This conclude the proof of Lemma 2.

Now Moser's iterative arguments give us the boundedness of $|\frac{\partial u}{\partial x}|$ and estimate (14). This proves Theorem 1.

4. Proof of Theorem 2. Without loss of generality it can be assumed that $\dfrac{\partial u}{\partial t}\in L_{2,loc}(\Omega_T)$, otherwise we will consider the Steklov average (see, for example, [7]).

Taking the x_s-derivative of equation (2) and integrating over Ω_T we deduce

$$\iint\limits_{\Omega_T}\frac{\partial^2 u}{\partial t\partial x_s}\varphi\,dx\,dt+\sum_{i=1}^n\iint\limits_{\Omega_T}\frac{da_i(x,t,u,\frac{\partial u}{\partial x})}{dx_s}\frac{\partial\varphi}{\partial x_i}\,dx\,dt=\iint\limits_{\Omega_T}\frac{da_0(x,t,u,\frac{\partial u}{\partial x})}{dx_s}\varphi\,dx\,dt, \quad (32)$$

for every s, $1\le s\le n$. Here $\varphi(x,t)$ is an arbitrary sufficiently smooth function vanishing on $\partial\Omega\times(0,T)$.

Assume that θ, r satisfy condition (18) and

$$0 < \sigma < \min(t_0, T - t_0). \tag{33}$$

Therefore, $Q(r,\theta) \subset \Omega$.

Let us introduce the nonnegative cut-off function $\psi(t)$ for interval $(t_0 - \sigma, t_0 + \sigma)$, $0 \le \psi(t) \le 1$, $\psi(t) \equiv 1$ in $(t_0 - \frac{\sigma}{2}, t_0 + \frac{\sigma}{2})$ and $\left|\frac{d\psi(t)}{dt}\right| \le c\sigma^{-1}$.

Let us denote also

$$b = \theta^{\alpha\beta} r^p, \qquad w(x,t) = 1 + \sum_{k=1}^{n} \left|\frac{\partial u}{\partial x_k}\right|^{\frac{p_k}{2}} b^{\frac{1}{2}}. \tag{34}$$

Lemma 3. *Let all the conditions of Theorem 2 are satisfied. Then there exists a positive constant C_2, depending only on $n, p_1, \dots, p_n, K_1, K_2$, such that the following inequality*

$$\begin{aligned} &\operatorname*{ess\,sup}_t \int_\Omega w^{m+\frac{4}{p_n}}(x,t)\xi^l(x)\psi^l(t)\,dx \\ &+ b^{\frac{2}{p_s}} \sum_{i=1}^{n} \iint_{\Omega_T} \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \Big(\frac{\partial^2 u}{\partial x_i \partial x_s}\Big)^2 w^m(x,t)\xi^l(x)\psi^l(t)\,dx\,dt \\ &\le C_2(l+m)^2[\sigma^{-1} + \theta^{-\alpha(\beta-2)}r^{-p}] \iint_{\Omega_T} w^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x,t)\xi^{l-2}(x)\psi^{l-2}(t)\,dx\,dt. \end{aligned} \tag{35}$$

is valid for any $1 \le s \le n, m \ge 0,\ l \ge n$.

Proof. We can assume without loss of generality that

$$\begin{aligned} &\operatorname*{ess\,sup}_t \int_\Omega w^{m+\frac{4}{p_n}}(x,t)\xi^l(x)\psi^l(t)\,dx \\ &+ \sum_{i,s=1}^{n} \iint_{\Omega_T} \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \Big(\frac{\partial^2 u}{\partial x_i \partial x_s}\Big)^2 w^m(x,t)\xi^l(x)\psi^l(t)\,dx\,dt < \infty, \\ &\iint_{\Omega_T} w^{m+2+\frac{4}{p_1}-\frac{4}{p_n}}(x,t)\xi^{l-2}(x)\psi^{l-2}(t)\,dx\,dt < \infty. \end{aligned}$$

Later we shall prove that there exist positive numbers $m_0,\ l_0$, such that the integrals in (35) are finite for all $m \ge m_0,\ l \ge l_0$.

It can be assumed also that $p_1 \ge 4$, in the general case we consider $w_\varepsilon(x,t) = 1 + (\sum_{k-1}^{n} |\frac{\partial u}{\partial x_k}|^2 + \varepsilon^2)^{\frac{p_k}{4}} b^{\frac{1}{2}}$, $\varepsilon > 0$ instead of $w(x,t)$. The proof is analogous and we omit the details. Let us substitute in integral identity (32) the test function

$$\varphi(x,t) = \frac{\partial u}{\partial x_s} w^m(x,t)\xi^l(x)\psi^l(t), \qquad m \ge 0,\ l \ge n,$$

were function $\xi(x)$ was defined in Section 3.

Evaluating the first integral of (32) in the right-hand side and due to definition of the test function we have for any $\tau \in (t_0-\sigma, t_0+\sigma)$

$$\iint\limits_{\Omega_T} \frac{\partial^2 u}{\partial t \partial x_s}\varphi(x,t)\,dx\,dt = \frac{1}{2}b^{-\frac{2}{p_s}} \int\limits_{\Omega}\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^m(x,\tau)\xi^l(x)\psi^l(\tau)\,dx$$
$$-\frac{l}{2}b^{-\frac{2}{p_s}}\iint\limits_{\Omega_T}\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^m(x,t)\xi^l(x)\frac{d\psi(t)}{dt}\psi^{l-1}(t)\,dx\,dt \tag{36}$$
$$-\frac{m}{4}b^{-\frac{2}{p_s}+\frac{1}{2}}\sum_{k=1}^{n}\iint\limits_{\Omega_T} p_k \frac{\partial^2 u}{\partial t\partial x_k}\frac{\partial u}{\partial x_k}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2}\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^{m-1}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt.$$

Let us substitute into integral identity (32) the test function

$$\varphi = \frac{m}{4}b^{-\frac{2}{p_s}+\frac{1}{2}}p_k\frac{\partial u}{\partial x_k}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2}\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^{m-1}(x,t)\xi^l(x)\psi^l(t),$$

and sum over $k=1,2,\ldots,n$ we obtain

$$-\frac{m}{4}b^{-\frac{2}{p_s}+\frac{1}{2}}\sum_{k=1}^{n}\iint\limits_{\Omega_T} p_k \frac{\partial^2 u}{\partial t\partial x_k}\frac{\partial u}{\partial x_k}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2}\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^{m-1}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt$$
$$= \frac{m}{4}b^{-\frac{2}{p_s}+\frac{1}{2}}\sum_{i,k=1}^{n} p_k\Big(\frac{p_k}{2}-1\Big)\iint\limits_{\Omega_T}\frac{da_i}{dx_k}\frac{\partial^2 u}{\partial x_i\partial x_k}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2}$$
$$\times\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^{m-1}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt$$
$$+\frac{m}{2}b^{\frac{1}{2}}\sum_{i,k=1}^{n} p_k\iint\limits_{\Omega_T}\frac{da_i}{dx_k}\frac{\partial^2 u}{\partial x_i\partial x_s}\frac{\partial u}{\partial x_s}\frac{\partial u}{\partial x_k}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2} w^{m-1}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt$$
$$+\frac{m(m-1)}{8}b^{-\frac{2}{p_s}+1}\sum_{i,k,q=1}^{n} p_k p_q\iint\limits_{\Omega_T}\frac{da_i}{dx_k}\frac{\partial^2 u}{\partial x_i\partial x_q}\frac{\partial u}{\partial x_k}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2}\frac{\partial u}{\partial x_q}\Big|\frac{\partial u}{\partial x_q}\Big|^{\frac{p_q}{2}-2}$$
$$\times\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^{m-2}(x,t)\xi^l(x)\psi^l(t)\,ds\,dt$$
$$+\frac{ml}{4}b^{-\frac{2}{p_s}+\frac{1}{2}}\sum_{i,k=1}^{n} p_k\iint\limits_{\Omega_T}\frac{da_i}{dx_k}\frac{\partial u}{\partial x_k}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2}$$
$$\times\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^{m-1}(x,t)\frac{\partial\xi(x)}{\partial x_i}\xi^{l-1}(x)\psi^l(x)\,dx\,dt$$

$$- \frac{m}{4} b^{-\frac{2}{p_s}+\frac{1}{2}} \sum_{k=1}^{n} p_k \iint\limits_{\Omega_T} \frac{da_0}{dx_k} \frac{\partial u}{\partial x_k} \Big| \frac{\partial u}{\partial x_k} \Big|^{\frac{p_k}{2}-2} \Big(1 + \Big| \frac{\partial u}{\partial x_s} \Big|^2 b^{\frac{2}{p_s}} \Big) w^{m-1}(x,t) \xi^l(x) \psi^l(t)\, dx\, dt$$

$$= \sum_{N=1}^{5} J_N. \tag{37}$$

Combining (36) and (37) from (32) we get

$$\frac{1}{2} b^{-\frac{2}{p_s}} \int\limits_{\Omega} \Big(1 + \Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^m(x,\tau) \xi^l(x) \psi^l(\tau)\, dx$$
$$+ \sum_{i,j=1}^{n} \iint_{\Omega_T} \frac{\partial a_i}{\partial u_{x_j}} \frac{\partial^2 u}{\partial x_i \partial x_s} \frac{\partial^2 u}{\partial x_j \partial x_s} w^m(x,t) \xi^l(x) \psi^l(t)\, dx\, dt$$
$$= - \sum_{i,j=1}^{n} \iint\limits_{\Omega_T} \frac{\partial a_i}{\partial u_{x_j}} \frac{\partial^2 u}{\partial x_j \partial x_s} \Big\{ l \frac{\partial u}{\partial x_s} w^m(x,t) \frac{\partial \xi(x)}{\partial x_i} \xi^{l-1}(x) \psi^l(t)$$
$$+ m b^{\frac{1}{2}} \frac{\partial u}{\partial x_s} w^{m-1}(x,t) \sum_{k=1}^{n} \frac{p_k}{2} \frac{\partial u}{\partial x_k} \Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2} \frac{\partial^2 u}{\partial x_k \partial x_i} \xi^l(x) \psi^l(t) \Big\} dx\, dt$$
$$- \sum_{i=1}^{n} \iint\limits_{\Omega_T} \Big(\frac{\partial a_i}{\partial u} \frac{\partial u}{\partial x_s} + \frac{\partial a_i}{\partial x_s} \Big) \Big\{ \frac{\partial^2 u}{\partial x_i \partial x_s} w^m(x,t) \xi^l(x) \psi^l(t)$$
$$+ l \frac{\partial u}{\partial x_s} w^m(x,t) \frac{\partial \xi(x)}{\partial x_i} \xi^{l-1}(x) \psi^l(t)$$
$$+ m b^{\frac{1}{2}} \frac{\partial u}{\partial x_s} w^{m-1}(x,t) \sum_{k=1}^{n} \frac{p_k}{2} \frac{\partial u}{\partial x_k} \Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2} \frac{\partial^2 u}{\partial x_k \partial x_i} \xi^l(x) \psi^l(t) \Big\} dx\, dt$$
$$+ \iint\limits_{\Omega_T} \frac{da_0}{dx_s} \frac{\partial u}{\partial x_s} w^m(x,t) \xi^l(x) \psi^l(t)\, dx\, dt$$
$$+ \frac{l}{2} b^{-\frac{2}{p_s}} \iint\limits_{\Omega_T} \Big(1 + \Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^m(x,t) \xi^l(x) \frac{d\psi(t)}{dt} \psi^{l-1}(t)\, dx\, dt - \sum_{N=1}^{5} J_N. \tag{38}$$

First we observe that

$$J_0 = c_{15} b^{-\frac{2}{p_s}} \int\limits_{\Omega} \Big(1 + \Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^m(x,\tau) \xi^l(x) \psi^l(\tau)\, dx$$
$$+ c_{15} \sum_{i=1}^{n} \iint\limits_{\Omega_T} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2} \Big(\frac{\partial^2 u}{\partial x_i \partial x_s}\Big)^2 w^m(x,t) \xi^l(x) \psi^l(t)\, dx\, dt$$
$$\le \frac{1}{2} b^{-\frac{2}{p_s}} \int\limits_{\Omega} \Big(1 + \Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big) w^m(x,\tau) \xi^l(x) \psi^l(\tau)\, dx$$

$$+\sum_{i,j=1}^{n}\iint\limits_{\Omega_T}\frac{\partial a_i}{\partial u_{x_j}}\frac{\partial^2 u}{\partial x_i\partial x_s}\frac{\partial^2 u}{\partial x_j\partial x_s}w^m(x,t)\xi^l(x)\psi^l(t)\,dx\,dt. \tag{39}$$

Conditions (4) and Young's inequality imply

$$\begin{aligned}
J_1 \geq{}& c_{16}b^{-\frac{2}{p_s}+\frac{1}{2}}m\sum_{i,k=1}^{n}\iint\limits_{\Omega_T}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big(\frac{\partial^2 u}{\partial x_i\partial x_k}\Big)^2\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2}\\
&\times\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big)w^{m-1}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt\\
&-c_{16}b^{-\frac{2}{p_s}+\frac{1}{2}}m\sum_{i,k=1}^{n}\iint\limits_{\Omega_T}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}}\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big)w^{m-1}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt\\
={}& c_{16}J_6-c_{16}b^{-\frac{2}{p_s}+\frac{1}{2}}m\sum_{i,k=1}^{n}\iint\limits_{\Omega_T}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}}\\
&\times\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big)w^{m-1}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt.
\end{aligned}\tag{40}$$

By the same argument we get

$$\begin{aligned}
|J_2| \leq{}& \frac{1}{8}J_0\\
&+c_{17}m^2 b\sum_{i,k=1}^{n}\iint\limits_{\Omega_T}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big|\frac{\partial u}{\partial x_s}\Big|^{2}\Big|\frac{\partial u}{\partial x_k}\Big|^{p_k-2}\Big(\frac{\partial^2 u}{\partial x_i\partial x_k}\Big)^2 w^{m-2}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt\\
&+c_{17}m^2 b\sum_{i,k=1}^{n}\iint\limits_{\Omega_T}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big|\frac{\partial u}{\partial x_s}\Big|^{2}\Big|\frac{\partial u}{\partial x_k}\Big|^{p_k}w^{m-2}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt.
\end{aligned}\tag{41}$$

Using again conditions (4) and Young's inequality yields

$$\begin{aligned}
J_3 \geq{}& -\frac{1}{8}c_{16}J_6-c_{18}b^{-\frac{2}{p_s}+\frac{3}{2}}m^2\sum_{i,k,q=1}^{n}\iint\limits_{\Omega_T}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big|\frac{\partial u}{\partial x_k}\Big|^{p_k}\Big|\frac{\partial u}{\partial x_q}\Big|^{\frac{p_q}{2}}\\
&\times\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big)w^{m-3}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt.
\end{aligned}\tag{42}$$

As before

$$\begin{aligned}
|J_4|+|J_5| \leq{}& \frac{1}{8}c_{16}J_6+c_{19}b^{-\frac{2}{p_s}+\frac{1}{2}}(m+l)^2\sum_{i,k=1}^{n}\iint\limits_{\Omega_T}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}}\\
&\times\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big)w^{m-1}(x,t)\Big(1+\Big|\frac{\partial \xi(x)}{\partial x_i}\Big|^2\Big)\xi^{l-2}(x)\psi^l(t)\,dx\,dt.
\end{aligned}\tag{43}$$

Now, from (38) due to conditions (4), inequalities (23)–(26) from Section 3, (39)–(43), taking into account the choice of number b, we obtain

$$J_0 \le c_{20}(l+m)^2 b^{-\frac{2}{p_s}+\frac{1}{2}} \sum_{i,k=1}^{n} \iint\limits_{\Omega_T} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2} \Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2}$$
$$\times \Big(\frac{\partial^2 u}{\partial x_i \partial x_k}\Big)^2 w^{m-1+\frac{4}{p_1}}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt$$
$$+ c_{20}(l+m)^2 b^{-1}\theta^{-2\alpha\frac{\beta-p_s}{p_s}} r^{-2\frac{p}{p_s}} \iint\limits_{\Omega_T} w^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}\xi^{l-2}(x)\psi^{l-2}(t)\,dx\,dt$$
$$+ c_{20}(l+m)^2 b^{-\frac{2}{p_s}}\sigma^{-1} \iint\limits_{\Omega_T} w^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}\xi^{l-2}(x)\psi^{l-2}(t)\,dx\,dt. \tag{44}$$

To estimate first integral in the right-hand side of (44) we substitute into integral identity (32) the test function

$$\varphi(x,t) = b^{-\frac{2}{p_s}+\frac{1}{2}}\,\frac{p_k}{2}\frac{\partial u}{\partial x_k}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2} w^{m-1+\frac{4}{p_1}}(x,t)\xi^l(x)\psi^l(t).$$

We sum over all $k = 1, \dots, n$. Then for every $\tau \in (t_0-\sigma, t_0+\sigma)$

$$\frac{b^{-\frac{2}{p_s}}}{m+\frac{4}{p_1}} \int\limits_{\Omega} w^{m+\frac{4}{p_1}}(x,\tau)\xi^l(x)\psi^l(\tau)\,dx$$
$$+ \sum_{i,k=1}^{n} \frac{p_k}{2} b^{-\frac{2}{p_s}+\frac{1}{2}} \iint\limits_{\Omega_T} \frac{da_i}{dx_k}\frac{\partial}{\partial x_i}\Big(\frac{\partial u}{\partial x_k}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}} w^{m-1+\frac{4}{p_1}}(x,t)\xi^l(x)\psi^l(t)\Big)dx\,dt$$
$$- \sum_{k=1}^{n} \frac{p_k}{2} b^{-\frac{2}{p_s}+\frac{1}{2}} \iint\limits_{\Omega_T} \frac{da_0}{dx_k}\frac{\partial u}{\partial x_k}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2} w^{m-1+\frac{4}{p_1}}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt$$
$$= \frac{l}{m+\frac{4}{p_1}} b^{-\frac{2}{p_s}} \iint\limits_{\Omega_T} w^{m+\frac{4}{p_1}}(x,t)\xi^l(x)\frac{d\psi(t)}{dt}\psi^{l-1}(t)\,dx\,dt. \tag{45}$$

We estimate second and third integrals in the left-hand side of (45) as in (26)–(29) from Section 3. This remark gives

$$b^{-\frac{2}{p_s}+\frac{1}{2}} \sum_{i,k=1}^{n} \iint\limits_{\Omega_T} \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big|\frac{\partial u}{\partial x_k}\Big|^{\frac{p_k}{2}-2}\Big(\frac{\partial^2 u}{\partial x_i\partial x_k}\Big)^2 w^{m-1+\frac{4}{p_1}}(x,t)\xi^l(x)\psi^l(t)\,dx\,dt$$
$$\le c_{21}(l+m)^2 b^{-\frac{2}{p_s}}\sigma^{-1} \iint\limits_{\Omega_T} w^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x,t)\xi^{l-2}(x)\psi^{l-2}(t)\,dx\,dt$$
$$+ c_{21}(l+m)^2 b^{-1}\theta^{-2\alpha\frac{\beta-p_s}{p_s}} r^{-2\frac{p}{p_s}} \iint\limits_{\Omega_T} w^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x,t)\xi^{l-2}(x)\psi^{l-2}(t)\,dx\,dt. \tag{46}$$

Combining (46) with (44), we get

$$b^{-\frac{2}{p_s}}\int\limits_{\Omega}\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big)w^m(x,\tau)\xi^l(x)\psi^l(\tau)\,dx$$

$$+\sum_{i=1}^{n}\iint\limits_{\Omega_T}\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\Big(\frac{\partial^2 u}{\partial x_i\partial x_s}\Big)^2 w^m(x,t)\xi^l(x)\psi^l(t)\,dx\,dt$$

$$\le c_{22}(l+m)^2\big(b^{-\frac{2}{p_s}}\sigma^{-1}+b^{-1}\theta^{-2\alpha\frac{\beta-p_s}{p_s}}r^{-2\frac{p}{p_s}}\big)$$

$$\times\iint\limits_{\Omega_T}w^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x,t)\xi^{l-2}(x)\psi^{l-2}(t)\,dx\,dt. \tag{47}$$

It follows easily that

$$b^{\frac{2}{p_s}-1}\theta^{-2\alpha\frac{\beta-p_s}{p_s}}r^{-2(\beta-2)}=\theta^{-\alpha(\beta-2)}r^{-p}, \tag{48}$$

thus (47) implies

$$\int\limits_{\Omega}\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big)w^m(x,\tau)\xi^l(x)\psi^l(\tau)\,dx$$

$$\le c_{22}(l+m)^2\big(\sigma^{-1}+\theta^{-\alpha(\beta-2)}r^{-p}\big)\iint\limits_{\Omega_T}w^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x,t)\xi^{l-2}(x)\psi^{l-2}(t)\,dx\,dt. \tag{49}$$

We sum (49) over all $s=1,\ldots,n$ to obtain

$$\int\limits_{\Omega}w^{m+\frac{4}{p_n}}(x,\tau)\xi^l(x)\psi^l(\tau)\,dx$$

$$\le c_{23}\sum_{s=1}^{n}\int\limits_{\Omega}\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^{\frac{p_s}{2}}b^{\frac{1}{2}}\Big)^{\frac{4}{p_n}}w^m(x,\tau)\xi^l(x)\psi^l(\tau)\,dx$$

$$\le c_{24}\sum_{s=1}^{n}\int\limits_{\Omega}\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^{p_s}b\Big)^{\frac{2}{p_n}}w^m(x,\tau)\xi^l(x)\psi^l(\tau)\,dx$$

$$\le c_{25}\sum_{s=1}^{n}\int\limits_{\Omega}\Big(1+\Big|\frac{\partial u}{\partial x_s}\Big|^2 b^{\frac{2}{p_s}}\Big)w^m(x,\tau)\xi^l(x)\psi^l(\tau)\,dx$$

$$\le c_{26}(l+m)^2\big(\sigma^{-1}+\theta^{-\alpha(\beta-2)}r^{-p}\big)\iint\limits_{\Omega_T}w^{m+2-\frac{4}{p_n}+\frac{4}{p_1}}(x,t)\xi^{l-2}(x)\psi^{l-2}(t)\,dx\,dt. \tag{50}$$

Taking into account (50), we derive from (47) necessary estimate (35). This concludes the proof of Lemma 3. □

Now Moser's iterative arguments give as the boundedness of $|\frac{\partial u}{\partial x}|$ and estimate (16). This proves Theorem 2.

1. *Bhattacharya T. and Leonetti F.* Some remarks on the regularity of minimizers of integrals with anisotropic growth // Comm. Math. Univ. Carol. – 1993. – N 34. – P. 597-611.
2. *Canale A., D'Ottavia A., Leonetti F. and Longobardi M.* Differentiability for bounded minimizers of some anisotropic integrals // J. Math. Anal. Appl. – 2001. – N 253. – P. 640-650.
3. *Cavaliere P., D'Ottavia A., Leonetti F. and Longobardi M.* Differentiability for bounded minimizers of anisotropic integrals // Comm. Math. Univ. Carol. – 1998. – N 39. – P. 685-696.
4. *Chianchi A.* Local boundedness of minimizers of anisotropic integrals // Ann. Inst. H. Poincare Anal. Non Lineaire. – 2000. – N 17. – P. 147-168.
5. *Degtyarev S. P., Tedeev A. F.* $L_1 - \mathrm{L}_\infty$ bounds for a solution of the Cauchy problem for anisotropic degenerate equation with growing initial data // Math. Sbornic, (in Russian) to appear.
6. *Di Benedetto E.* $C^{1+\alpha}$ local regularity of weak solution of degenerate elliptic equations // Nonlinear Anal. – 1993. – N 7. – P. 827-850.
7. *Di Benedetto E.*, Degenerate parabolic equations. – Springer-Verlag New York. – 1993.
8. *Evans L.C.* A new proff of local $C^{1+\alpha}$-regularity for solutions of certain degenerate elliptic P. D.E. // J. Differential Equations. – 1983. – N 45. – P. 356-373.
9. *Esposito L., Leonetti F., Mingione G.* Sharp regularity for functionals with (p, q) growth // Jornal of Diff. Equations. – 2004. – N 204. – P. 5-55.
10. *Fusco N., Sbordone C.* Local boundedness of minimizers in a limit case // Manuscripta Math. – 1990. – N 69. – P. 19-25.
11. *Fusco N., Sbordone C.* Some remarks on theregularity of minima of anisotropic integrals // Commun. in P. D.E. – 1993. – vol. 18, N 1&2. – P. 153-167.
12. *Giaquinta M.* Growth conditions and regularity, a counterexample // Mamus. Math. – 1987. – N 50. – P. 245-248.
13. *Kolodij I.M.* On boundedness of weak solutions of elliptic differential equations // Moscow University Vestnik. – 1970. – N 5. – P. 44-52.
14. *Kolodij I.M.* On boundedness of weak solutions of parabolic differential equations // Moscow University Vestnik. – 1971. – N 5. – P. 25-31.
15. *Ladyzenskaya O.A. and Ural'tseva N.N.* Linear and Quasilinear Elliptic Equations. – Academic Press, N. Y. – 1968.
16. *Ladyzenskaya O.A., Solonnikov V.A., Ural'tseva N.N.* Linear and quasilinear equations of parabolic type. – Transl. Math. Mono. – 1968, vol. 23 AMS, Providence.
17. *Lewis J.L.* Regularity of the derivatives of solutions to certain degenerate elliptic equations // Indiana Univ. Math. J. – 1983. – N 32. – P. 849-858.
18. *Li J.J.* Local behavior of solutions of anisotropic elliptic equtions // Nonlinear Anal. – 1999. – N 3. – P. 617-627.
19. *Lieberman G.M.* The first initial-boundary value problem for quasilinear second order parabolic equations // Ann. Scuola Norm. Sup. Pisa. – 1986. – N 13. – P. 374-387.
20. *G. M. Lieberman* Gradient estimates for a new class of degenerate elliptic and parabolic equations // Ann. Scuola Norm. Sup. Pisa. – 1994. – N 21. – P. 497-522.
21. *G. M. Lieberman* Gradient estimates for anisotropic elliptic equations // Advances in Diff. Equations. – 2005. – vol. 10, N 7. – P. 767-812.
22. *Marcellini P.* Regularity of minimizers of integrals in the calculus of variations with nonstandard growth conditions // Arch. Rational Mech. Anal. – 1989. – N 105. – P. 267–284.
23. *Marcellini P.* Regularity and existence of solutions of elliptic equations with p, q growth conditions // J. Differential Equations. – 1991. – N 90. – P. 1-30.
24. *Mingoi Yu., Xiting L.* Boundedness of solutions of parabolic equations with anisotropic growth conditions // Can. J. Math. – 1997. – vol. 49, N 4. – P. 789-809.
25. *Moser J.* A new proof of De Giorgi's theorem concernin the regularity problem for elliptic differential equations // Comm. Pure Appl. Math. – 1960. – vol. 13, N 3. – P. 457-468.
26. *Namlyeyeva Yu.V., Shishkov A.E., Skrypnik I.I.* Isolated singulalrities of solutions of quasilinear anisotropic elliptic equations, Advanced Nonl. Studies. – 2006. – vol. 6, N 4. – P. 617-641.
27. *Namlyeyeva Yu.V., Shishkov A.E., Skrypnik I.I.* Removable isolated singulalrities for solutions of doubly nonlinear anisotropic parabolic equations // Applicable Analysis. – 2010. – vol. 89, N 10. –

P. 1559-1574.
28. *Porzio M.M.* L_∞-regularity for degenerate and singular anisotropic parabolic equations // Bollettino U.M.I. – 1997. – vol. 7, N 11A. – P. 697-707.
29. *Skrypnik I.V.* Methods for Analysis of Nonlinear Elliptic Boundary Value Problems. – Translations of Math. Monofraphs. – 1994. – 349 p.

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О локальных оценках градиентов для анизотропных эллиптических и параболических уравнений.

Рассмотрены широкие классы анизотропных эллиптических и параболических уравнений. Получены точные оценки для градиентов решений таких уравнений.

Ключевые слова: *анизотропные эллиптические и параболические уравнения, поточечные оценки градиентов решений*

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Про локальні оцінки градієнтів для анізотропних еліптичних та параболічних рівнянь.

Розглянуто широкі класи анізотропних еліптичних та параболічних рівнянь. Отримано точні оцінки градієнтів розв'язків таких рівнянь.

Ключові слова: *анізотропні еліптичні та параболічні рівняння, точкові оцінки градієнтів розв'язків*

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