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# UNIQUENESS OF APPROXIMATE SOLUTIONS OF THE BELTRAMI EQUATIONS

We introduce a notion of an approximate solution to the Beltrami equations, obtain some properties of such solutions and show that the approximate solution is unique up to pre-composition with a conformal mapping.

**1. Introduction.** Let $D$ be a domain in the complex plane $\mathbb{C}$, i.e., a connected and open subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. The **Beltrami equation** is the equation of the form

$$f_{\overline{z}} = \mu(z) \cdot f_z \tag{1}$$

where $f_{\overline{z}} = \overline{\partial} f = (f_x + i f_y)/2$, $f_z = \partial f = (f_x - i f_y)/2$, $z = x + iy$, and $f_x$ and $f_y$ are partial derivatives of $f$ in $x$ and $y$, correspondingly. The function $\mu$ is called the **complex coefficient** and

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{2}$$

the **maximal dilatation** or in short the **dilatation** of the equation (1). The Beltrami equation (1) is said to be **degenerate** if $ess\, sup\, K_\mu(z) = \infty$.

There are numerous old and recent works devoted to the existence problem for degenerate Beltrami equations, see e.g. [2], [7]–[12], [18], [21]–[23], [25], [28]–[30], [32], [38]–[42], [48]–[49]. In almost all these works one actually proves just the existence of the approximate solution for (1). However, the problem of uniqueness of solutions for (1) is insufficiently known explored. To the moment it is known the Stoilow factorization only for narrow special cases of solutions and $\mu$. In this paper we show that if $K_\mu \in L^1_{loc}$, then the approximate solution of Beltrami equation (1) is unique up to pre-composition with a conformal mapping.

Given $z_0 \in \overline{D}$, the **tangential dilatation** of (1) with respect to $z_0$ is

$$K^T_\mu(z, z_0) = \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2},$$

see [40]–[41], cf. the corresponding terms and notations in [3]–[5], [18], [25] and [34].

Recall also that a function $f : D \to \mathbb{C}$ is **absolutely continuous on lines**, abbr. $f \in$**ACL**, if, for every closed rectangle $R$ in $D$ whose sides are parallel to the coordinate axes, $f|R$ is absolutely continuous on almost all line segments in $R$ which are parallel to the sides of $R$. In particular, $f$ is ACL (possibly modified on a set of Lebesgue measure zero) if it belongs to the Sobolev class $W^{1,1}_{loc}$ of locally integrable functions with locally integrable first generalized derivatives and, conversely, if $f \in \mathrm{ACL}$ has locally integrable first partial derivatives, then $f \in W^{1,1}_{loc}$, see e.g. 1.2.4 in [31]. Note that, if $f \in \mathrm{ACL}$, then $f$ has partial derivatives $f_x$ and $f_y$ a.e. and, for a sense-preserving ACL homeomorphism $f : D \to \mathbb{C}$, the Jacobian $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2$ is nonnegative a.e. In this case, the **complex dilatation** $\mu_f$ of $f$ is the ratio $\mu(z) = f_{\overline{z}}/f_z$, if $f_z \neq 0$ and $\mu(z) = 0$ otherwise, and the **dilatation** $K_f$ of $f$ is $K_\mu(z)$, see (2). Note that $|\mu(z)| \leq 1$ a.e. and $K_\mu(z) \geq 1$ a.e.

Recall that, given a family of paths $\Gamma$ in $\overline{\mathbb{C}}$, a Borel function $\rho : \overline{\mathbb{C}} \to [0, \infty]$ is called **admissible** for $\Gamma$, abbr. $\rho \in adm\,\Gamma$, if

$$\int\limits_{\gamma} \rho(z)\,|dz| \ \geq \ 1 \tag{3}$$

for each $\gamma \in \Gamma$. The **modulus** of $\Gamma$ is defined by

$$M(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_{\mathbb{C}} \rho^2(z)\ dxdy\ . \tag{4}$$

Given a domain $D$ and two sets $E$ and $F$ in $\overline{\mathbb{C}}$, $\Delta(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{C}}$ which join $E$ and $F$ in $D$, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. Motivated by the ring definition of quasiconformality in [16], we introduced the following notion in [40]. Let $D$ be a domain in $\mathbb{C}$, $z_0 \in D$, and $Q : D \to [0, \infty]$ a measurable function. A homeomorphism $f : D \to \overline{\mathbb{C}}$ is called a **ring $Q-$homeomorphism** at the point $z_0$ if

$$M(\Delta(fC_1, fC_2, fD)) \ \leq \int\limits_{A} Q(z) \cdot \eta^2(|z - z_0|)\ dxdy \tag{5}$$

for every circular ring $A \subset D$ centered at $z_0$,

$$A = A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}, \quad 0 < r_1 < r_2 < \infty\ ,$$

and every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\ dr \ = \ 1 \tag{6}$$

and where $C_1 = \{z \in \mathbb{C} : |z - z_0| = r_1\}$ and $C_2 = \{z \in \mathbb{C} : |z - z_0| = r_2\}$.

Now, given a domain $D$ in $\mathbb{C}$ and a measurable function $Q : D \to [0, \infty]$, we say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a **ring $Q-$homeomorphism at a boundary point** $z_0$ of the domain $D$ if

$$M(\Delta(fC_1, fC_2, fD)) \ \leq \int\limits_{A \cap D} Q(z) \cdot \eta^2(|z - z_0|)\ dxdy \tag{7}$$

for every ring $A = A(z_0, r_1, r_2)$ and every continua $C_1$ and $C_2$ in $D$ which belong to the different components of the complement to the ring $A$ in $\overline{\mathbb{C}}$ containing $z_0$ and $\infty$, correspondingly, and for every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ satisfying the condition (6).

An ACL homeomorphism $f : D \to \mathbb{C}$ is called a **strong ring solution** of the Beltrami equation (1) with a complex coefficient $\mu$ if $f$ satisfies (1) a.e., $f^{-1} \in W^{1,2}_{loc}(f(D))$ and $f$ is a ring $Q$–homeomorphism at every point $z_0 \in \overline{D}$ with $Q(z) = Q_{z_0}(z) := K^T_\mu(z, z_0) \leq K_\mu(z)$. In fact, if $Q \in L^1_{loc}(D)$, then similarly to [44] one can prove that the single condition (5) implies $f \in ACL$, furthermore, $f \in W^{1,1}_{loc}(D)$, $J_f(z) \neq 0$ a.e., see e.g. [45].

Following to [8], we call a homeomorphism $f \in W^{1,1}_{loc}(D)$ a **regular solution** of (1) if $f$ satisfies (1) a.e. and $J_f(z) \neq 0$ a.e.

Note that above the condition $f^{-1} \in W^{1,2}_{loc}(f(D))$ implies that $f$ has $(N^{-1})$-property and a.e. point $z$ is a **regular point** for the mapping $f$, i.e., $f$ is differentiable at $z$ with $J_f(z) \neq 0$, see e.g. [26], p.121, 128–130 and 150, and Theorem 1 in [33]. Conversely, if $f \in W^{1,1}_{loc}(D)$, $K_f \in L^1_{loc}(D)$ and $J_f(z) \neq 0$ a.e., then $f^{-1} \in W^{1,2}_{loc}(f(D))$, see e.g. [19]. Moreover, by [19] $g_w = 0 = g_{\overline{w}}$ for a.e. $w$ where $J_g(w) = 0$, $g = f^{-1}$. Note also that the condition $K_\mu \in L^1_{loc}(D)$ is necessary for a homeomorphic ACL solution $f$ of (1) to have the property $g = f^{-1} \in W^{1,2}_{loc}(f(D))$ because this property implies that

$$\int\limits_C K_\mu(z)\, dxdy \leq 4 \int\limits_C \frac{dxdy}{1 - |\mu(z)|^2} = 4 \int\limits_{f(C)} |\partial g|^2\, dudv < \infty$$

for every compact set $C \subset D$. The change of variables is correct here, say by Lemmas III.2.1 and III.3.2 and Theorems III.3.1 and III.6.1 in [26], cf. also I.C(3) in [1].

For $n \in \mathbb{N}$, define $\mu_n : D \to \mathbb{C}$ by letting $\mu_n(z) = \mu(z)$ if $|\mu(z)| \leq 1 - 1/n$ and 0 otherwise. Let $f_n : D \to \mathbb{C}$ be a homeomorphic ACL solution of (1) with $\mu_n$ instead of $\mu$. We call a homeomorphism $f$ an **approximate solution** of (1) if there exists such a sequence $\{f_n\}$ converged to $f$ uniformly on each compact set in $D$. We call such a sequence $\{f_n\}$ an **approximating sequence** for $f$.

In the classical case when $\|\mu\|_\infty < 1$, equivalently, when $K_\mu \in L^\infty(D)$, every ACL homeomorphic solution $f$ of the Beltrami equation (1) is in the class $W^{1,2}_{loc}(D)$ together with its inverse mapping $f^{-1}$, and hence $f$ is a strong ring solution of (1) by Theorem 1 below. In the case $\|\mu\|_\infty = 1$ with $K_\mu \leq Q \in$ BMO, again $f^{-1} \in W^{1,2}_{loc}(f(D))$ and $f$ belongs to $W^{1,s}_{loc}(D)$ for all $1 \leq s < 2$ but not necessarily to $W^{1,2}_{loc}(D)$, see e.g. [38]. However, there is a varity of degenerate Beltrami equations for which strong ring solutions exist as shown in the paper [42]. The inequalities (5) and (7), which holds for the strong ring solutions, is an important tool in deriving various local and boundary properties of such solutions, see e.g. [27], [37] and [46], cf. also [36].

**2. Preliminaries.** We consider the extended complex plane $\overline{\mathbb{C}}$ as a metric space with the spherical (chordal) metric:

$$s(z, \zeta) = \frac{|z - \zeta|}{\sqrt{1 + |z|^2}\sqrt{1 + |\zeta|^2}}\,, \quad z \neq \infty \neq \zeta\,; \quad s(z, \infty) = \frac{1}{\sqrt{1 + |z|^2}}\,.$$

The **kernel of a sequence of open sets** $\Omega_n \subseteq \overline{\mathbb{C}}$, $n = 1, 2, \ldots$ is the open set

$$\Omega_0 \;=\; \text{Kern } \Omega_n \;:\;=\; \bigcup_{m=1}^{\infty} \text{ Int} \left( \bigcap_{n=m}^{\infty} \Omega_n \right)$$

where Int $A$ denotes the set consisting of all inner points of $A$, in other words, Int $A$ is the union of all open disks in $A$ with respect to the spherical distance.

Proposition 2.1. Let $h_n : D \to D'_n$, $D'_n = h_n(D)$, be a sequence of homeomorphisms given in a domain $D \subseteq \overline{\mathbb{C}}$. If $h_n$ converge as $n \to \infty$ locally uniformly with respect to the spherical (chordal) metric to a homeomorphism $h : D \to D' \subseteq \overline{\mathbb{C}}$, then $D' = h(D) \subseteq$ Kern $D'_n$.

This is Proposition 3.6 in [8]. Later on, we apply also the following useful.

Remark 2.1. It's well known that every metric space is $\mathcal{L}^*$-space, i.e. a space with a convergence, see e.g. Theorem 2.1.1 in [24], and in the compact spaces the Uhryson axiom says: $x_n \to x_0$ as $n \to \infty$ if and only if, for every convergent subsequence $x_{n_k} \to x_*$, the equality $x_* = x_0$ holds, see the definition 20.1.3 in [24].

To prove that an approximate solution is a strong ring solution we need the following two auxiliary statements. The next proposition can be found as Theorem 2.16 in [42], cf. the corresponding result for inner points in [39].

Proposition 2.2. Let $f : D \to \mathbb{C}$ be a sense-preserving homeomorphism of the class $W^{1,2}_{loc}(D)$ such that $f^{-1} \in W^{1,2}_{loc}(f(D))$. Then at every point $z_0 \in \overline{D}$ the mapping $f$ is a ring Q-homeomorphism with $Q(z) \;=\; K^T_\mu(z, z_0)$ where $\mu(z) = \mu_f(z)$.

The following proposition was proved in [43] as Theorem 4.1.

Proposition 2.3. Let $f_n : D \to \overline{\mathbb{C}}$, $n = 1, 2, \ldots$ be a sequence of ring $Q$-homeomorphisms at a point $z_0 \in \overline{D}$. If $f_n$ converges locally uniformly to a homeomorphism $f : D \to \overline{\mathbb{C}}$, then $f$ is also a ring $Q$-homeomorphism at $z_0$.

We also need the following convergence theorem for the Beltrami equations, see Theorem 3.1 in [43].

Proposition 2.4. Let $D$ be a domain in $\mathbb{C}$ and let $f_n : D \to \mathbb{C}$ be a sequence of sense-preserving ACL homeomorphisms with complex dilatations $\mu_n$ such that

$$\frac{1 + |\mu_n(z)|}{1 - |\mu_n(z)|} \;\leq\; Q(z) \in L^1_{\text{loc}}(D) \qquad \forall\, n = 1, 2, \ldots \tag{8}$$

If $f_n \to f$ uniformly on each compact set in $D$, where $f : D \to \mathbb{C}$ is a homeomorphism, then $f \in$ ACL and $\partial f_n$ and $\overline{\partial} f_n$ converge weakly in $L^1_{\text{loc}}(D)$ to $\partial f$ and $\overline{\partial} f$, respectively. Moreover, if in addition $\mu_n \to \mu$ a.e., then $\overline{\partial} f = \mu \partial f$ a.e.

Remark 2.2. In fact, it is easy to show that under the condition (8) $f_n$ as well as $f$ belong to $W^{1,1}_{\text{loc}}(D)$. Moreover, if in addition $Q \in L^p_{\text{loc}}(D)$, then $f_n$ and $f$ belong to $W^{1,s}_{\text{loc}}(D)$, $\partial f_n \to \partial f$ and $\overline{\partial} f_n \to \overline{\partial} f$ weakly in $L^s_{\text{loc}}(D)$, where $s = 2p/(1 + p)$, see e.g. Lemma 2.2 in [7].

## 3. On convergence of inverse homeomorphisms.

**Lemma 3.1.** *Let $D$ be a domain in $\overline{\mathbb{C}}$ and let $f_n : D \to \overline{\mathbb{C}}$ be a sequence of homeomorphisms from $D$ into $\overline{\mathbb{C}}$ such that $f_n \to f$ as $n \to \infty$ locally uniformly with respect to the spherical metric to a homeomorphism $f$ from $D$ into $\overline{\mathbb{C}}$. Then $f_n^{-1} \to f^{-1}$ locally uniformly in $f(D)$, too.*

*Proof.* Set $g_n = f_n^{-1}$ and $g = f^{-1}$. The locally uniform convergence $g_n \to g$ is equivalent to the so-called continuous convergence, meaning that $g_n(w_n) \to g(w_0)$ for every convergent sequence $w_n \to w_0$ in $f(D)$, see e.g. [13], p.268. So, let $w_n \in f(D)$, $n = 0, 1, 2, \ldots$ and $w_n \to w_0$ as $n \to \infty$. Let us show that $z_n := g(w_n) \to z_0 := g(w_0)$ as $n \to \infty$. By Remark 2.1 it suffices to prove that for every convergent subsequence $z_{n_k} \to z_*$ as $k \to \infty$, the equality $z_* = z_0$ holds. Let $D_0$ be a subdomain of $D$ such that $z_0 \in D_0$ and $\overline{D}_0$ is a compact subset of $D$. Then by Proposition 2.1 $f(D_0) \subseteq \operatorname{Kern} f_{n_k}(D_0)$ and hence $w_0$ together with its neighborhood belongs to $f_{n_k}(D_0)$ for all $k \geq K$. Thus, with no loss of generality we may assume that $w_{n_k} \in f_{n_k}(D_0)$, i.e. $z_{n_k} \in D_0$ for all $k = 1, 2, \ldots$ and, consequently, $z_* \in D$. Then, by the continuous convergence $f_n \to f$, we have that $f_{n_k}(z_{n_k}) \to f(z_*)$, i.e. $f_{n_k}(g_{n_k}(w_{n_k})) = w_{n_k} \to f(z_*)$. The latter implies that $w_0 = f(z_*)$, i.e. $z_* = z_0$. The proof is complete. □

**4. Properties of approximate solutions.** In this section we show that an approximate solution to (1) is its regular solution and also a strong ring solution for any complex coefficient $\mu$ with $K_\mu \in L^1_{loc}$.

**Theorem 4.1.** *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}(D)$. Then any approximate solution to the Beltrami equation (1) is a regular solution.*

*Proof.* Let $f$ be an approximate solution of the Beltrami equation (1) and let $\{f_n\}$ be its approximating sequence. Then $f \in W^{1,1}_{loc}$ by Proposition 2.4.

Now, set $g_n = f_n^{-1}$ and $g = f^{-1}$. By Lemma 3.1 we have that $g_n \to g$ locally uniformly in $f(D)$. Moreover, by a change of variables which is permitted because $f_n$ and $g_n$ are in $W^{1,2}_{loc}$, see e.g. Lemmas III.2.1 and III.3.2 and Theorems III.3.1 and III.6.1 in [26], cf. also I.C(3) in [1], we obtain that for large $n$

$$\int\limits_B |\partial g_n|^2 \, dudv \; = \int\limits_{g_n(B)} \frac{dxdy}{1 - |\mu_n(z)|^2} \; \leq \int\limits_{B^*} K_\mu \, dxdy \; < \; \infty \tag{9}$$

where $B^*$ and $B$ are relatively compact domains in $D$ and $f(D)$, respectively, such that $g(\bar{B}) \subset B^*$. The relation (9) implies that the sequence $g_n$ is bounded in $\mathrm{W}^{1,2}(B)$, and hence $f^{-1} \in \mathrm{W}^{1,2}_{loc}(f(D))$, see e.g. Lemma III.3.5 in [35] or Theorem 4.6.1 in [15]. The latter condition brings in turn that $f$ has $(N^{-1})-$property, see e.g. Theorem III.6.1 in [26], and hence $J_f(z) \neq 0$ a.e., see Theorem 1 in [33]. Thus, $f$ is a regular solution of (1). □

**Theorem 4.2.** *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}(D)$. Then any approximate solution to the Beltrami equation (1) is a strong ring solution.*

*Proof.* Let $\{f_n\}$ be an approximating sequence for $f$. By Proposition 2.2 the mapping $f_n$ is a ring $Q$-homeomorphism with $Q(z) = K^T_\mu(z, z_0)$ where $\mu(z) = \mu_f(z)$. Then by the Proposition 2.3 we obtain that $f$ is a ring $Q$–homeomorphism with $Q(z) = K^T_\mu(z, z_0)$ at every point $z_0 \in \overline{D}$. We have already shown under the proof of Theorem 4.1 that $f \in W^{1,1}_{loc}$ and $f^{-1} \in \mathrm{W}^{1,2}_{loc}(f(D))$. Thus, $f$ is a strong ring solution of (1). $\Box$

**5. Factorization theorem.**

**Theorem 5.1.** *Let $f : D \to \mathbb{C}$ be an approximate solution to the Beltrami equation (1) with measurable $\mu : D \to \mathbb{C}$ such that $|\mu(z)| < 1$ a.e. and*

$$\frac{1 + |\mu(z)|}{1 - |\mu(z)|} \le Q(z) \in L^1_{\mathrm{loc}} \qquad \forall\ n = 1, 2, \ldots$$

*Suppose $g$ is another approximate solution to (1) in $D$. Then there is a conformal mapping $h : f(D) \to \mathbb{C}$ such that*

$$g = h \circ f.$$

*Proof.* Let $\{f_n\}$ and $\{g_n\}$ be approximating sequences for $f$ and $g$, correspondingly. Set $h_n = g_n \circ f_n^{-1}$. By uniqueness theorem for the uniformly elliptic Beltrami equations, see [26, p.183], $h_n$ is a conformal mapping for any $n \in \mathbb{N}$. Next, by Lemma 3.1 we have that $h_n = g_n \circ f_n^{-1} \to g \circ f^{-1} = h$ as $n \to \infty$ locally uniformly in $f(D)$. Thus, it remains to apply the Weierstrass theorem on the uniform convergence of sequences of analytic functions, see e.g., [17, p.17], from which we conclude that $h$ is the conformal mapping. $\Box$

**6. The main corollaries and conjectures.** Following to [20], we say that a function $\varphi : D \to \mathbb{R}$ of the class $L^1_{loc}$ has **finite mean oscillation** at a point $z_0 \in D$, write $\varphi \in FMO(z_0)$, if

$$\overline{\lim_{\varepsilon \to 0}} \ \fint_{B(z_0,\varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon(z_0)|\ dxdy \ < \ \infty$$

where

$$\overline{\varphi}_\varepsilon(z_0) = \fint_{B(z_0,\varepsilon)} \varphi(z)\ dxdy$$

is the average of $\varphi$ over the disk $B(z_0, \varepsilon) = \{z \in \mathbb{C} : \ |z - z_0| < \varepsilon\}$ with small $\varepsilon > 0$. We also write $\varphi \in FMO(D)$, or simply $\varphi \in FMO$, if $\varphi \in FMO(z_0)$ for all $z_0 \in D$.

Applying Theorem 11.6 from [29], we obtain the following corollary of Theorem 5.1.

**Corollary 6.1.** *Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}$. Suppose that every point $z_0 \in D$ has neighborhood $U_{z_0}$ such that*

$$K^T_\mu(z, z_0) \le Q_{z_0}(z) \quad a.e.$$

*for some function $Q_{z_0}(z)$ of finite mean oscillation at the point $z_0$ in the variable $z$. Then the Beltrami equation (1) has unique approximate solution up to pre-composition with a conformal mapping.*

REMARK 6.1. In particular, we obtain that the conclusion of Corollary 6.1 holds if either

$$\overline{\lim_{\varepsilon\to 0}} \quad \fint_{B(z_0,\varepsilon)} \frac{\left|1-\frac{\overline{z-z_0}}{z-z_0}\ \mu(z)\right|^2}{1-|\mu(z)|^2} \quad dxdy \ < \ \infty \qquad \forall\ z_0 \in D \tag{10}$$

or

$$K_\mu(z) \ = \ \frac{1+|\mu(z)|}{1-|\mu(z)|} \quad \le \quad Q(z) \in FMO\,(D) \ . \tag{11}$$

Similarly, by Theorem 11.10 in [29] this is valid if $K_\mu \in L^1_{loc}(D)$ and

$$\int\limits_0^{\delta(z_0)} \frac{dr}{rk^T_{z_0}(r)} = \infty \qquad\qquad \forall\ z_0 \in D \tag{12}$$

where $k^T_{z_0}(r)$ is the average of the tangential dilatation $K^T_\mu(z,z_0)$ over the circle $C(z_0,r) = \{z\in\mathbb{C} : |z-z_0| = r\}$, $\delta(z_0) < \text{dist}\ (z_0,\partial D)$, and, in particular, if

$$k^T_{z_0}(r) = O\left(\log\frac{1}{r}\right) \quad \text{as} \quad r\to\infty \quad \forall\ z_0\in D\ . \tag{13}$$

We complete our paper with the following two equivalent conjectures.

*Conjecture 1.* Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}(D)$ (for which the Beltrami equation (1) has at least one approximate solution). Then any regular solution to (1) is an approximate solution to (1).

*Conjecture 2.* Let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L^1_{loc}(D)$ (for which the Beltrami equation (1) has at least one approximate solution). Then a regular solution to (1) is unique up to pre-composition with a conformal mapping.


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