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GENERALIZED ACTION PRINCIPLES IN MECHANICS

In this paper, we begin with the Lagrangian $L = T - V$, the difference between the kinetic and potential energies, by introducing some constraints, and applying the Lagrange multiplier, we obtain various forms of generalized action principles including the Hamilton's principle and Schwinger's principle, and some unknown action principles.

Introduction. We begin with the definition of the action functional as time integral over the Lagrangian L of a dynamical system:

$$J(x_i) = \int_{t_1}^{t_2} L dt. \tag{1}$$

Here the Lagrangian is defined as follows

$$L = \frac{1}{2} m x_i'^2 - V(x_i), \qquad x_i' = dx_i/dt. \tag{2}$$

Newton's motion equation can be obtained from the stationary condition of the functional (1), which reads $m x_i'' + \dfrac{\partial V}{\partial x_i} = 0$.

We can introduce some constraints to the action functional (1), leading to various principles required. For example, if the total energy is a conserved quantity, i.e. $T + V = \text{const}$, which is considered as a constraint of the functional (1), then we obtain the Euler-Maupertuis principle (principle of least action) [1]:

$$\int_{t_1}^{t_2} (2T - \text{const}) dt \to \min, \quad \text{or} \quad \int_{t_1}^{t_2} T dt \to \min.$$

In this paper we will obtain Hamiltonian and other actions from the Lagrangian (2) by introducing some constraints.

1. Hamiltonian. Now we introduce a generalized velocity [1, 2]:

$$p_i = \frac{\partial L}{\partial x_i'} = m x_i'. \tag{3}$$

We consider equation (3) as a constraint of the action functional (1), accordingly the Lagrangian (2) can be written as follows $L(x_i, p_i) = \dfrac{1}{2m} p_i^2 - V(x_i)$. By Lagrange multiplier, we have the following generalized Lagrangian

$$L_1(x_i, p_i, \lambda_i) = \frac{1}{2m} p_i^2 - V(x_i) + \lambda_i (p_i - m x_i'). \tag{4}$$

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The multiplier can be readily identified, which reads

$$\lambda_i = -\frac{1}{m} p_i. \tag{5}$$

Substituting the identified multiplier into (4) results in

$$L_1(x_i, p_i) = \frac{1}{2m} p_i^2 - V(x_i) - \frac{1}{m} p_i (p_i - m x_i') = p_i x_i' - H,$$

where $H(x_i, p_i)$ is a Hamiltonian [1,2]: $H(x_i, p_i) = \frac{1}{2m} p_i^2 + V(x_i)$. Now we introduce a new variable u_i defined as

$$u_i = x_i'. \tag{6}$$

We consider the equation (6) as a constraint of the action functional (1), in such case, the Lagrangian (2) can be rewritten as $L(x_i, u_i) = \frac{1}{2} m u_i^2 - V(x_i)$. By Lagrange multiplier, we have the following generalized Lagrangian

$$L_2(x_i, u_i, \lambda_i) = \frac{1}{2} m u_i^2 - V(x_i) + \lambda_i (u_i - x_i'). \tag{7}$$

The multiplier can be readily identified, which reads

$$\lambda_i = -m u_i. \tag{8}$$

Substituting the identified multiplier into (7) results in

$$L_2(x_i, u_i) = \frac{1}{2} m u_i^2 - V(x_i) - m u_i (u_i - x_i') = m u_i x_i' - \tilde{H}(x_i, u_i),$$

where $\tilde{H}(x_i, u_i)$ is given by $\tilde{H}(x_i, p_i) = \frac{1}{2} m u_i^2 + V(x_i)$.

2. Schwinger's action. From the equation (8), we know that the multiplier is actually the generalized velocity:

$$\lambda_i = -m u_i = -p_i. \tag{9}$$

Substituting (9) into (7), and keeping p_i an independent variable, we have

$$L_3(x_i, u_i, p_i) = \frac{1}{2} m u_i^2 - V(x_i) - p_i (u_i - x_i') = p_i x_i' - \overline{H}(x_i, u_i, p_i), \tag{10}$$

where $\overline{H}(x_i, u_i, p_i) = -\frac{1}{2} m u_i^2 + V(x_i) + p_i u_i$.

The equation (10) is called Schwinger action [1].

By same manipulation, from the equation (5), the multiplier can be also determined as $\lambda_i = -\frac{1}{m} p_i = -u_i$. So we obtain another action like Schwinger's, which reads

$$L_4(x_i, p_i, u_i) = \frac{1}{2m} p_i^2 - V(x_i) - u_i (p_i - m x_i') = m u_i x_i' - \hat{H}(x_i, u_i, p_i),$$

where $\hat{H}(x_i, u_i, p_i) = -\frac{1}{2m} p_i^2 + V(x_i) + u_i p_i$.

3. More generalized action. A more generalized action can be obtained by linear combination of $L_1(x_i, u_i)$ and $L_2(x_i, p_i)$ [3, 4]:

$$L_5(x_i, u_i, p_i) = L_1(x_i, u_i) + L_2(x_i, p_i) = -\frac{1}{2m}p_i^2 - \frac{1}{2}mu_i^2 + (p_i + mu_i)x_i' - 2V(x_i) = \\ = (p_i + mu_i)x_i' - H_e(x_i, u_i, p_i), \tag{11}$$

where $H_e(x_i, u_i, p_i) = \frac{1}{2m}p_i^2 + \frac{1}{2}mu_i^2 + 2V(x_i)$.

The Euler equations can be readily obtained, which read

$$\begin{aligned} \delta x_i: &\quad \frac{d}{dt}(p_i + mu_i) + \frac{\partial H_e}{\partial x_i} = \frac{d}{dt}(p_i + mu_i) + 2\frac{\partial V}{\partial x_i} = 0, \\ \delta u_i: &\quad mx_i' - \frac{\partial H_e}{\partial u_i} = mx_i' - mu_i = 0, \\ \delta p_i: &\quad x_i' - \frac{\partial H_e}{\partial p_i} = x_i' - \frac{1}{m}p_i = 0. \end{aligned}$$

In a more general form, the equation (11) can be written as

$$L_5(x_i, u_i, p_i) = \alpha L_1(x_i, u_i) + \beta L_2(x_i, p_i),$$

where α and β are constants.

Linearly combining L_i $(i = 1, 2, 3, 4, 5)$, we have

$$L_6(x_i, u_i, p_i) = \sum_{i=1}^{5} \alpha_i L_i,$$

where α_i are constants, and we often set $\sum_{i=1}^{5} \alpha_i = 1$.

Conclusion. In this paper, we obtained some generalized action functionals including known and unknown ones. As it is well known that the action principles are the foundation of the Lagrangian mechanics and Hamiltonian mechanics, therefore, the new obtained generalized action functionals might also lead to some new kinds of mechanics. The applications of the new obtained action functionals will be discussed in detail in author's forthcoming publications.

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