

# Asymptotic formulas for integer partitions within the approach of microcanonical ensemble

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The problem of integer partitions is addressed using the microcanonical approach which is based on the analogy between this problem in the number theory and the calculation of microstates of a many-boson system. For ordinary (one-dimensional) partitions, the correction to the leading asymptotic is obtained. The estimate for the number of two-dimensional (plane) partitions coincides with known asymptotic results.

**Key words:** *integer partitions, plane partition, bosonic systems*

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## 1. Introduction

In this paper, we address the problem of integer partitions using the physical approach based on microcanonical treatment. Partitioning of integers is a problem in the number theory that originated in the works by Leibniz [1] and Euler [2]. A partition of a positive integer  $n$  is a way of writing  $n$  as a sum of positive integers, where the order of the summands is insignificant. The number of partitions  $p(n)$  is called a *partition function* [3, Ch. 1] Further in this work we refer to  $p(n)$  simply as *the number of partitions* to avoid confusion with the respective physical term.

To clarify the notion of partitions, let us consider the number 5. It can be represented as the following sums:

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.$$

Therefore, one has for the number of partitions  $p(5) = 7$ .

Generalization for higher-dimensional partitions is made in the following fashion. In the  $D$ -dimensional case, an integer  $n$  is represented as a sum of positive integers  $n_{i_1 \dots i_D}$ :

$$n = \sum_{i_1, \dots, i_D \geq 0} n_{i_1 \dots i_D}, \quad (1.1)$$

where  $n_{i_1 \dots i_D} \geq n_{j_1 \dots j_D}$  whenever  $i_1 \leq j_1, i_2 \leq j_2, \dots, i_D \leq j_D$  [3, p. 179].

For instance, in the case of two-dimensional (plane) partitions of 3, one has the following possibilities:

$$\begin{array}{rcccccc} 3 & 2 + 1 & 2 & 1 + 1 + 1 & 1 + 1 & 1 \\ & & + & & + & + \\ & & 1 & & 1 & 1 \\ & & & & & + \\ & & & & & 1 \end{array} \quad (1.2)$$

yielding the number of plane partitions  $p_2(3) = 6$  [3, 4]. This can be easily shown by considering the numbers  $n_{i_1 i_2}$  to be elements of square  $1 \times 1, 2 \times 2, 3 \times 3$ , and so on, matrices. To obtain the sum of matrix

elements equal to 3 (provided that they are sorted in a non-ascending order in both rows and columns, according to the above definition), we have:

$$(3), \quad \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that zero elements are skipped when writing multidimensional partitions.

## 2. Physical analogy

There is a straight analogy between the number of partitions and the number of microstates of a many-boson system of harmonic oscillators. The approach based on such an analogy was suggested as early as in 1950 by Nanda [5], whose paper is now frequently overlooked, and was later utilized by a number of authors to solve related problems [6–8].

Let the system of one-dimensional harmonic oscillators with spectrum  $\varepsilon_j = \hbar\omega j$  have an energy  $E$ :

$$E = \hbar\omega \sum_i j_i, \quad (2.1)$$

where  $j_i$  denotes the quantum number of the  $i$ th particle. The summation runs only over the excited states and thus the particles in the ground state (with zero energy) can be arbitrary in number. The set  $\{j_1, j_2, \dots\}$  corresponds to a certain microstate of the system. In the quantum case, the particles are indistinguishable, so the permutation of  $\{j_1, j_2, \dots\}$  does not lead to a new microstate. At this point, we can match the set  $\{j_1, j_2, \dots\}$  and a partition of the number  $n = E/\hbar\omega$ . Thus, the number of microstates  $\Gamma(E)$  is equal to the number of partitions of  $n$ . In the set  $\{j_1, j_2, \dots\}$ , any of the numbers  $j_i$  can be equal, so one must consider bosonic oscillators.

To make an asymptotic estimation of the number of microstates  $\Gamma(E)$ , one can use the well-known Hardy–Ramanujan formula [9] for integer partitions:

$$p^{\text{HR}}(n) = \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2/3}\sqrt{n}}, \quad (2.2)$$

therefore,

$$\Gamma(E) = \frac{1}{4\sqrt{3}E/\hbar\omega} e^{\pi\sqrt{2/3}\sqrt{E/\hbar\omega}}. \quad (2.3)$$

In the next section, we will derive this expression from physical considerations and obtain the first correction to this asymptotic result. For simplicity, we further put the unit of energy  $\hbar\omega = 1$ .

The above considerations might be extended – with minor reservations – to higher-dimensional partitions, the two-dimensional (plane) ones being best studied. The respective analysis is presented in section 4.

## 3. Corrections to the leading asymptotics of $p(n)$

Partition function  $Z(\beta)$  can be expressed as an integral by introducing the number of microstates  $\Gamma(E)$ :

$$Z(\beta) = \sum_j e^{-\beta E_j} = \int_0^\infty \Gamma(E) e^{-\beta E} dE.$$

The above expression is nothing but the Laplace transform, so, by inverting it, we obtain:

$$\Gamma(E) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} Z(\beta) e^{\beta E} d\beta. \quad (3.1)$$

The entropy  $S(\beta)$  is equal to

$$S(\beta) = \beta E + \ln Z(\beta). \quad (3.2)$$

Thus, we have

$$Z(\beta) = e^{S(\beta)} e^{-\beta E} \quad (3.3)$$

and

$$\Gamma(E) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{S(\beta)} d\beta. \quad (3.4)$$

Using the Taylor series for entropy in the vicinity of  $\beta_0$ :

$$S(\beta) \approx S(\beta_0) + \frac{1}{2!} S''(\beta_0) (\beta - \beta_0)^2 + \frac{1}{3!} S'''(\beta_0) (\beta - \beta_0)^3, \quad (3.5)$$

where  $\beta_0$  is the stationary point,

$$S'(\beta_0) = 0, \quad (3.6)$$

for the number of microstates, one obtains:

$$\Gamma(E) \approx \frac{e^{S(\beta_0)}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \exp \left[ \frac{1}{2!} S''(\beta_0) (\beta - \beta_0)^2 + \frac{1}{3!} S'''(\beta_0) (\beta - \beta_0)^3 \right] d\beta. \quad (3.7)$$

Using the replacement  $\beta = ix + \beta_0$ , we get:

$$\begin{aligned} \Gamma(E) &\approx \frac{e^{S(\beta_0)}}{2\pi} \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{2!} S''(\beta_0) x^2 - \frac{1}{3!} S'''(\beta_0) ix^3 \right] dx \\ &= \frac{e^{S(\beta_0)}}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2!} S''(\beta_0) x^2} \left\{ \cos \left[ \frac{S'''(\beta_0)}{3!} x^3 \right] - i \sin \left[ \frac{S'''(\beta_0)}{3!} x^3 \right] \right\} dx \\ &= \frac{e^{S(\beta_0)}}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2!} S''(\beta_0) x^2} \cos \left[ \frac{S'''(\beta_0)}{3!} x^3 \right] dx. \end{aligned} \quad (3.8)$$

This integral can be expressed via the modified Bessel function of the second kind, namely:

$$\Gamma(E) \approx \frac{e^{S(\beta_0)}}{2\pi} \frac{2S''(\beta_0)}{\sqrt{3} |S'''(\beta_0)|} \exp \left\{ \frac{[S''(\beta_0)]^3}{3[S'''(\beta_0)]^2} \right\} K_{1/3} \left( \frac{[S''(\beta_0)]^3}{3[S'''(\beta_0)]^2} \right). \quad (3.9)$$

To calculate the entropy  $S(\beta)$ , we write the partition function of the oscillator system as follows:

$$Z(\beta) = \prod_{j=1}^{\infty} (1 - e^{-\beta j})^{-1}, \quad \ln Z(\beta) = - \sum_{j=1}^{\infty} \ln (1 - e^{-\beta j}). \quad (3.10)$$

Using the Euler–Maclaurin formula to calculate the sum, one obtains [7]:

$$S(\beta) = \beta E + \ln Z(\beta) = \beta E + \frac{\pi^2}{6\beta} + \frac{1}{2} \ln \beta - \frac{1}{2} \ln(2\pi) + \dots \quad (3.11)$$

Limiting ourselves to the first two terms, the stationary point yields:

$$\beta_0 = \frac{\pi}{\sqrt{6E}}. \quad (3.12)$$

With the same accuracy, we have:

$$S''(\beta_0) = \frac{2\sqrt{6}}{\pi} E^{3/2}, \quad S'''(\beta_0) = -\frac{36}{\pi^2} E^2. \quad (3.13)$$

Collecting the whole expression (3.9) together, for the number of microstates we obtain

$$\Gamma(E) = \frac{1}{18\sqrt[4]{6}E^{3/4}} \exp\left(\frac{28}{27}\pi\sqrt{\frac{2E}{3}}\right) K_{1/3}\left(\frac{1}{27}\pi\sqrt{\frac{2E}{3}}\right). \quad (3.14)$$

From this formula, there follows a correction to the main asymptotics of the number of partitions, as one substitutes  $E$  with  $n$ :

$$p(n) = \frac{1}{18\sqrt[4]{6}n^{3/4}} \exp\left(\frac{28}{27}\pi\sqrt{\frac{2n}{3}}\right) K_{1/3}\left(\frac{1}{27}\pi\sqrt{\frac{2n}{3}}\right). \quad (3.15)$$

Taking into account the asymptotic series expansion for large arguments [10]

$$K_\nu(z) \propto \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{\sqrt{z}} \left[1 + \mathcal{O}\left(\frac{1}{z}\right)\right] \quad (3.16)$$

we immediately arrive at the leading asymptotic behavior given by equation (2.2).

#### 4. Asymptotic behavior of plane partitions

We consider the energy spectrum of a 2-dimensional system in the following form:

$$\varepsilon(j_1, j_2) = j_1 + j_2. \quad (4.1)$$

To obtain  $\Gamma(E)$ , we repeat the derivation presented in the previous section following [7], see also [11]. Thus, the entropy  $S(\beta)$  is equal to

$$S(\beta) = \beta E + \ln Z(\beta), \quad (4.2)$$

where for energy spectrum (4.1), the partition function is

$$Z(\beta) = \prod_{\text{all the energies}} (1 - e^{-\beta\varepsilon})^{-1} = \prod_{j_1=1}^{\infty} \prod_{j_2=1}^{\infty} [1 - e^{-\beta(j_1+j_2)}]^{-1}. \quad (4.3)$$

For the logarithm of the partition function  $Z(\beta)$ , one has:

$$\ln Z(\beta) = - \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \ln [1 - e^{-\beta(j_1+j_2)}] = - \sum_{j=1}^{\infty} g_j \ln [1 - e^{-\beta j}], \quad (4.4)$$

where the  $j$ th level degeneracy is equal to

$$g_j = j + 1 \simeq j, \quad (4.5)$$

therefore,

$$\ln Z(\beta) = - \sum_{j=1}^{\infty} j \ln [1 - e^{-\beta j}]. \quad (4.6)$$

After applying the Euler–Maclaurin summation formula, the entropy can be expressed in such a form:

$$S(\beta) = \beta E + \ln Z(\beta) = \beta E + \frac{\zeta(3)}{\beta^2} + \frac{1}{12} \ln \beta - \frac{1}{6}. \quad (4.7)$$

The standard saddle-point method, for  $\Gamma(E)$  from equation (3.1), yields:

$$\Gamma(E) = \frac{\exp[S(\beta_0)]}{\sqrt{2\pi S''(\beta_0)}}. \quad (4.8)$$

As in the previous section, for the stationary point  $\beta_0$ , so that  $S'(\beta_0) = 0$ , one obtains

$$\beta_0 = \left[ \frac{2\zeta(3)}{E} \right]^{1/3} \quad (4.9)$$

and

$$S''(\beta_0) = \frac{3}{[2\zeta(3)]^{1/3}} E^{4/3}. \quad (4.10)$$

Thus, the number of microstates is

$$\Gamma(E) = \frac{1}{\sqrt{6\pi}} [2\zeta(3)]^{7/36} E^{-25/36} \exp \left\{ \frac{3}{2} [2\zeta(3)]^{1/3} E^{2/3} - \frac{1}{6} \right\}. \quad (4.11)$$

Substituting energy  $E$  with integer  $n$ , we immediately obtain the result for plane partitions in the following way:

$$p_2(n) = \frac{1}{\sqrt{6\pi}} [2\zeta(3)]^{7/36} n^{-25/36} \exp \left\{ \frac{3}{2} [2\zeta(3)]^{1/3} n^{2/3} - \frac{1}{6} \right\}. \quad (4.12)$$

Our estimation differs from the result of Wright [12]

$$p_2^W(n) = \frac{1}{\sqrt{6\pi}} [2\zeta(3)]^{7/36} n^{-25/36} \exp \left\{ \frac{3}{2} [2\zeta(3)]^{1/3} n^{2/3} + c \right\} \quad (4.13)$$

by a constant factor:

$$c = \zeta'(-1) = -0.165421\dots \quad \text{versus} \quad -\frac{1}{6} = -0.166666\dots$$

Note that Wright's asymptotics was later confirmed by Nanda [5] with a much more sophisticated analysis of the oscillator system as compared to our approach.

## 5. Results and discussion

We performed calculations for the number of ordinary partitions  $p(n)$  and plane partitions  $p_2(n)$  using the expressions (3.15) and (4.12) obtained in this work.

The comparison with exact values and the leading asymptotics by Hardy and Ramanujan are given in table 1 and in figure 1. In the figure, the relative errors are plotted. For  $n = 32$  and  $n > 33$ , our correction becomes negative. For  $n > 20$ , the error is less than one per cent, and it never exceeds seven per cent except for  $n = 1$ , where  $p(n) = 1$ , and our formula yields 2, and  $n = 5$  with our value of 8 versus  $p(n) = 5$ , which leads to about 15 per cent error. Beyond this domain, the maximum relative deviation of 0.0083285 is found for  $n = 186$ .

The plane partitions are demonstrated in table 2 and in figure 2. Our result for plane partitions provides a better estimation compared to that of Wright up to  $n = 7573$ . At  $n = 2679$ , expression (4.12) starts to underestimate the number of plane partitions while (4.13) asymptotically approaches the real values of  $p_2(n)$  from above for all  $n$ . Interestingly, the approach to the calculation of plain partitions used in this work which takes into account the level degeneracy provides a much better estimate compared to an earlier suggested [15] direct treatment of multidimensional oscillators which failed to produce a correct pre-exponential behavior of the power of  $n$ .

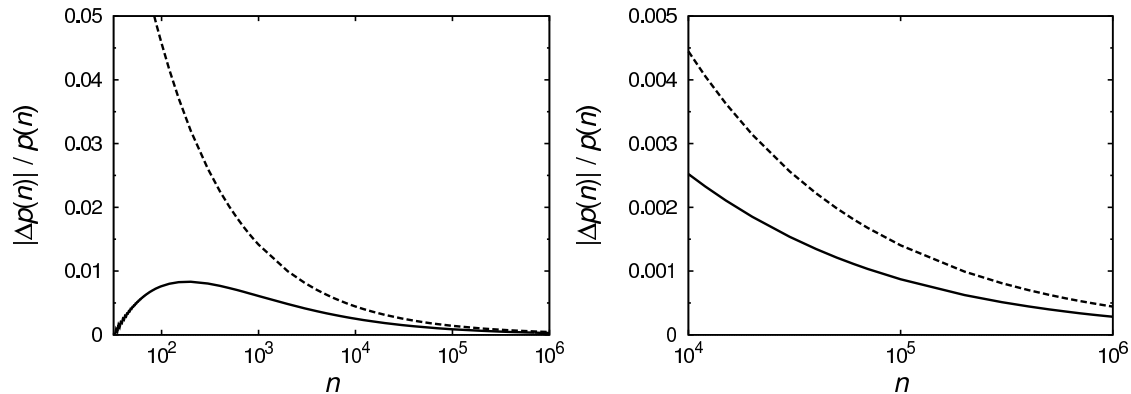
To summarize, the expressions for the number of integer partitions were obtained from the analogy between this number-theoretical problem and the physical problem of calculating the states in a many-boson system within the microcanonical approach. The correction to the main asymptotics for ordinary (one-dimensional) partitions is shown to give a good estimate even for small numbers.

**Table 1.** Number of integer partitions. Real data  $p(n)$  are calculated by wxMaxima 0.8.7. Our result (3.15) is denoted as  $p^{our}(n)$ .

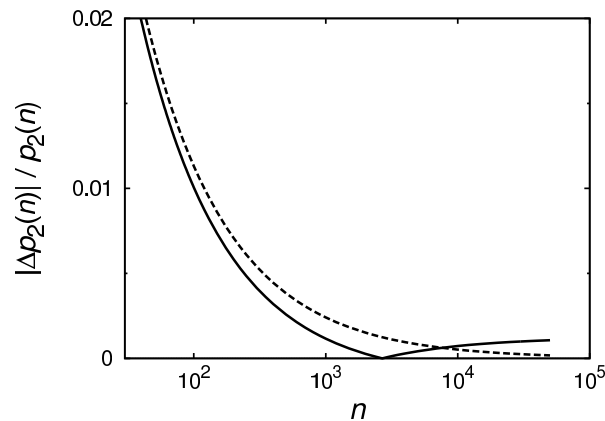
$n$	$p(n)$	$p^{HR}(n)$	$p^{our}(n)$
1	1	2	2
2	2	3	2
3	3	4	3
4	5	6	5
5	7	9	8
6	11	13	11
7	15	18	16
8	22	26	23
9	30	35	31
10	42	48	43
20	627	692	631
30	5604	6080	5607
40	37338	40080	37244
50	204226	217590	203334
60	966467	1024004	961084
70	4087968	4312670	4061899
80	15796476	16606782	15686810
90	56634173	59367760	56218131
100	190569292	199280893	189113660
150	40853235313	42369336269	40515857434
200	3972999029388	4100251432188	3939941762556
300	9253082936723602	9494094811675004	9178612996544433
400	6727090051741041926	6878471626940064454	6675403544180355365
500	2300165032574323995027	2346386625611060168255	2283287889193750426745
600	458004788008144308553622	466396419561000383265349	454786235887791524386094
700	60378285202834474611028659	61401534136286099837866892	59970590732043189238732265
800	5733052172321422504456911979	5823869045997298219672106135	5695741931526069206199862701
900	415873681190459054784114365430	422080911932431823414681187746	413258060326350156090055206821
1000	24061467864032622473692149727991	24401996316802476288263414943062	23914862944527589687173572605174
10000	$3.61673 \times 10^{106}$	$3.63281 \times 10^{106}$	$3.60761 \times 10^{106}$
100000	$2.74935 \times 10^{346}$	$2.75321 \times 10^{346}$	$2.74695 \times 10^{346}$
200000	$1.14215 \times 10^{492}$	$1.14328 \times 10^{492}$	$1.14144 \times 10^{492}$
500000	$1.52473 \times 10^{781}$	$1.52568 \times 10^{781}$	$1.52412 \times 10^{781}$
1000000	$1.47168 \times 10^{1107}$	$1.47234 \times 10^{1107}$	$1.47127 \times 10^{1107}$

**Table 2.** Number of plane integer partitions, real data  $p_2(n)$  are presented according to [13, 14]. Our result is denoted as  $p_2^{\text{our}}(n)$ .

$n$	$p_2(n)$	$p_2^W(n)$	$p_2^{\text{our}}(n)$
1	1		2
2	3		3
3	6		7
4	13		14
5	24		27
6	48		51
7	86		93
8	160		169
9	282		300
10	500		525
15	6879		7165
20	75278		77731
25	696033		715574
30	5668963		5807691
40	281846923		287446950
50	10499640707		10676838029
60	314689799781		319332954078
70	793771067795		8042228226656
80	173781688194937		175851289976638
90	3376508618954817		3413306874910366
100	59206066030052023		59801745303809873
125	52658376905566496345		53104724310582212659
150	30669139297980503425545		30894551767372734822993
175	12797410813969092203145839		1288060682741697065333190
200	4066263490068623016919082185		4089988062550422858471571831
1000	$3.54259 \times 10^{84}$		$3.54670 \times 10^{84}$
10000	$4.50750 \times 10^{401}$		$4.50422 \times 10^{401}$
100000	$1.11796 \times 10^{1876}$		$1.11669 \times 10^{1876}$



**Figure 1.** Comparison of relative errors for the estimations of the number of integer partitions from the real values (calculated by wxMaxima 0.8.7). Solid curve – our result (3.15), dashed curve – the leading asymptotics provided by the Hardy–Ramanujan formula (2.2). Large  $n$  domain in an enlarged view is presented on the right.



**Figure 2.** Comparison of relative errors for the estimations of the number of plane partitions from the real values  $p_2(n)$  [13, 14]. Solid curve – our result (4.12), dashed curve – Wright's formula (4.13).

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## References

1. Leibniz G.W. – In: Sämtliche Schriften und Briefe. Siebente Reihe: Mathematische Schriften, Bd. 1: 1672-1676. Geometrie – Zahlentheorie – Algebra (1. Teil). Akademie-Verlag, Berlin, 1990, p. 740–755.
2. Eulero L., *Novi Commentarii Academiae scientiarum Petropolitanae*, 1753, **3**, 125.
3. Andrews G.E., *The Theory of Partitions*. Addison-Wesley. Reading, MA, 1976.
4. Almkvist G., *Experimental Mathematics*, 1998, **7**, 343; doi:10.1080/10586458.1998.10504380.
5. Nanda V.S., *Math. Proc. Cambridge Philos. Soc.*, 1951, **47**, 591; doi:10.1017/S0305004100026980.
6. Grossmann S., Holthaus M., *Phys. Rev. Lett.*, 1997, **79**, 3557; doi:10.1103/PhysRevLett.79.3557.
7. Tran M.N., Murthy M.V.N., Bhaduri R.J., *Ann. Phys.*, 2004, **311**, 204; doi:10.1016/j.aop.2003.12.004.
8. Comtet A., Leboeuf P., Majumdar S.N., *Phys. Rev. Lett.*, 2007, **98**, 070404; doi:10.1103/PhysRevLett.98.070404.
9. Hardy G.H., Ramanujan S., *Proc. London Math. Soc.*, 1918, **17**, 75; doi:10.1112/plms/s2-17.1.75.
10. Abramowitz M., Stegun I.A., *Handbook of Mathematical Functions*, tenth printing. National Bureau of Standards, 1972.



11. Rovenchak A., Fiz. Nizk. Temp., 2009, **35**, 510 [Low Temp. Phys., 2009, **35**, 400; doi:10.1063/1.3132748].
12. Wright E.M., Q. J. Math., 1931, 177; doi:10.1093/qmath/os-2.1.177.
13. Mutafchiev L., Kamenov E., Compt. Rend. Acad. Bulg. Sci., 2006, **59**, 361.
14. <http://oeis.org/A000219/b000219.txt>
15. Rovenchak A., Computat. Meth. Sci. Technol. (Poznań), 2010, **16**, 187.

## **Асимптотичні формули для розбиттів цілих чисел у підході мікроканонічного ансамблю**

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Розглянуто задачу про розбиття цілих чисел у межах мікроканонічного підходу, який ґрунтується на аналогії між цією задачею з теорії чисел і обчисленням кількості мікростанів багатобозонної системи. Для звичайних (одновимірних) розбиттів отримано поправку до головної асимптотики. Оцінка кількості двовимірних (плоских) розбиттів добре узгоджується з відомими асимптотичними результатами.

**Ключові слова:** *розбиття цілих чисел, плоскі розбиття, бозонні системи*

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