

Some novel effects in superconducting nanojunctions

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In this paper we address several new developments in the theory of dc Josephson effect in superconducting weak links. We analyze an interplay between quantum interference effects and Andreev reflection in SNS nanojunctions with insulating barriers and demonstrate that these effects may qualitatively modify the Josephson current in such structures. We also investigate an impact of the parity effect on persistent currents in superconducting nanorings interrupted by a quantum point contact (QPC). In the limit of zero temperature and for the odd number of electrons in the ring we predict complete suppression of the supercurrent across QPC with one conducting mode. In nanorings with SNS junctions a π -state can occur for the odd number of electrons. Changing this number from even to odd yields *spontaneous* supercurrent in the ground state of such rings without any externally applied magnetic flux.

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In 1926 Albert Einstein posed a remarkable question [1]: «Of particular interest is the question whether a link between two superconductors also turns superconducting». The answer to this question was provided by Brian Josephson in 1962 [2]. It was predicted by Josephson that dissipationless flow of Cooper pairs between two different superconductors separated by an insulating barrier is possible provided this supercurrent I_s does not exceed some critical value I_c . Furthermore, the dependence of this current on the phases of macroscopically coherent wave functions of Cooper pairs was established in a very simple form [2]

$$I_s = I_c \sin \phi, \quad (1)$$

where ϕ is the difference between the phases of the BCS order parameters of two superconductors. Equation (1) represents the *dc Josephson effect*. The answer to the Einstein's question [1] turned out to be positive.

What if the total current I flowing through the barrier is larger than I_c ? In this case a part of the net current across the barrier is transferred by normal electrons (quasiparticles) and the rest of it is carried by Cooper pairs. While the second contribution, I_s , remains dissipationless and is again described by

Eq. (1), the first — quasiparticle — contribution to the current is dissipative and, hence, causes a nonzero voltage drop V across the insulating barrier. In the presence of this voltage the coherent phase difference ϕ acquires a time dependence described by another famous Josephson relation

$$\frac{\partial \phi}{\partial t} = \frac{2eV}{\hbar}. \quad (2)$$

Combining Eqs. (1) and (2) one immediately arrives at the conclusion that for any nonzero V the supercurrent I_s changes in time. In the case of time-independent voltages one has $\phi = 2eVt/\hbar$ and, hence, the Josephson current (1) will oscillate in time with the fundamental frequency proportional to the voltage V . Equation (2) and related to it oscillations of the supercurrent represent the essence of the *ac Josephson effect*.

Soon after these Josephson's predictions the microscopic theory of both dc [3] and ac [4,5] was constructed and these effects have been observed experimentally [6,7]. Huge number of publications as well as several monographs are devoted to various aspects of these effects. It turned out that physics encoded in these phenomena is very rich and important for understanding of basic properties of superconductivity it-

self. More than forty years after its discovery the Josephson effect still attracts attention of many researchers and keeps providing us with new interesting physics.

In this paper we will discuss several new phenomena which theoretical understanding was achieved only very recently. In the next section we very briefly review already well known and established results which concern dc Josephson effect in various types of superconducting weak links. Sections 2 and 3 are devoted to possible new effects [8,9] which emerge and gain importance as one decreases the size of a weak link eventually turning it to a nanostructure with only few conducting channels. Fabrication of such quantum point contacts (QPC) – unthinkable at the time of discovery of the Josephson effects – is now becoming a routine procedure. Hence, the new effects discussed here can be directly observed and investigated in a modern experiment.

1. Instead of introduction

Relatively soon after the Josephson's discovery it was understood that nondissipative transport of Cooper pairs between two superconductors is possible not only through a (usually very thin) insulating barrier, but also in various other situations. One of such situations is realized in the so-called SNS structures, i.e., if a piece of a normal metal is placed in-between two superconductors. In contrast to tunnel junctions, in SNS systems at sufficiently low temperatures appreciable supercurrent can flow even though a normal layer can be as thick as few microns. This is because the wave function of Cooper pairs or, more precisely, the anomalous Green function, penetrates into the normal metal from a superconductor at the length $\sim v_F/T$ for ballistic and $\sim \sqrt{D/T}$ for diffusive metals (here and below $D = v_F l/3$ and l are the diffusion coefficient and the elastic mean free path, respectively). Clearly, at temperatures much lower than the critical temperature T_c of a superconductor this length becomes large (as compared, e.g., to the superconducting coherence length), and macroscopic quantum coherence is established between two superconducting banks separated by a normal metal.

Further studies revealed an interesting mechanism of Cooper pair transfer in such systems. It turned out that the supercurrent flow is directly related to another fundamentally important phenomenon: Andreev reflection [10]. Suffering Andreev reflections at both SN interfaces, quasiparticles with energies below the

superconducting gap are effectively «trapped» inside the N layer and form a discrete set of levels [10]. It was demonstrated [11] that in the presence of the phase difference ϕ across the SNS junction these levels acquire a shift proportional to this phase difference. Thus, on one hand, the position of the quasiparticle energy levels in such systems can be tuned by passing the supercurrent and, on the other hand, the magnitude of this supercurrent can be established by taking the derivative of the quasiparticle energy with respect to ϕ with subsequent summation over the whole energy spectrum. The microscopic theory [11,12] leads to the following expression for the current density through clean SNS systems:

$$j = \frac{e^2 p_F^2 v_F}{6\pi^2 d} \phi, \quad -\pi < \phi < \pi. \quad (3)$$

This expression is valid at $T \rightarrow 0$ and for N-metal layers with thickness $d \gg \xi_0 \sim v_F/\Delta$, where Δ is superconducting order parameter. The most important features of this result are (i) the strongly nonsinusoidal current-phase relation, cf. Eqs. (1) and (3), and (ii) the linear dependence of the current on the gap in the quasiparticle spectrum $\epsilon_{qp} \sim v_F/d$ in the direction normal to NS interfaces.

It is interesting that qualitatively both features (i) and (ii) survive not only for ballistic but also for diffusive SNS junctions even though in the latter case discrete Andreev levels are washed out due to elastic scattering of quasiparticles on impurities in the N metal. It was demonstrated microscopically [13–15] that at low temperatures $T \ll D/d^2$ the current-phase relation in diffusive SNS junctions also deviates from the sinusoidal one* and the critical Josephson current is again proportional to the gap in the quasiparticle spectrum, in this case the Thouless energy $\epsilon_{qp} = D/d^2$. The exact value of the critical Josephson current in long diffusive SNS junctions can be established only numerically. One finds [15]

$$I_c = 10.82 \frac{\epsilon_{qp}}{eR_N} \quad (4)$$

where R_N is the junction normal state resistance.

The above results – both for ballistic and diffusive limits – are valid for sufficiently long junctions. One can also decrease the thickness of the normal metal d and gradually crossover to the limit of short superconducting constrictions. A microscopic description of the dc Josephson effect of such type of weak links was developed by Kulik and Omel'yanchouk [16]. Also in

* This deviation is not as strong as in the ballistic case: The supercurrent in long diffusive junctions reaches its maximum at the value $\phi \simeq 1.27\pi/2$ [15].

such systems at low temperatures the current-phase dependence deviates from $\sin\phi$ and the critical current $I_c(T \rightarrow 0)$ is again proportional to the combination ϵ_{qp}/eR_N , where now $\epsilon_{qp} = \Delta$. A crossover between the two limits of long SNS junctions and short superconducting weak links can also be described microscopically. In the clean case this task can be trivially accomplished by solving the Eilenberger equations [17–19], while in the dirty limit one should make use of the Usadel equations [20] which can be solved only numerically. The latter task has recently been carried out in Ref. 15.

Let us also note that in all the above considerations intermetallic interfaces were assumed to be perfectly transparent. It is also straightforward to generalize the analysis in order to account for electron scattering at the insulating barrier which can be present inside a weak link. For short superconducting junctions containing an insulating barrier with an arbitrary energy independent transmission the corresponding generalization has been worked out by Haberkorn et al. [21]. This analysis yields a general formula for the Josephson current which matches with the Ambegaokar–Baratoff result [3] in the weak tunneling limit and crosses over to the Kulik–Omel'yanchouk's expression [16] for clean constrictions at transmissions approaching unity. It is interesting that the result [16] for diffusive constrictions can also be recovered from the formula of Ref. 21 after its slight generalization. In order to do so one should assume that the transmission is not the same for all conducting channels but rather obeys the Dorokhov's distribution formula. Combining this formula with the expression [21] and summing over all conducting channels one arrives at the result [16] for diffusive weak links.

One can also investigate transport properties of more complicated layered structures which contain both normal metal layers and insulating barriers. For instance, SNS systems with one insulating barrier, such as SINS and SNINS were analyzed by a number of authors [22–28]. For an extended review summarizing various features of dc Josephson effect in different types of superconducting weak links and further references we refer the reader to Refs. 29–31.

Most of the results reviewed above were obtained already long time ago and are by now well established and well understood. One can think that considering dc Josephson effect in even more complicated structures like, for instance, SNS structures with two or three insulating barriers, may at most yield somewhat more cumbersome expressions but would not allow to enclose any new physics beyond what has already been understood in simpler situations. Below we will show that it is not so. Just on the contrary, in the next

section we will demonstrate that qualitatively new effects may occur in SNS junctions with more than one insulating barriers, in particular provided the cross section of such junctions is reduced to be comparable to the square of the Fermi wavelength.

2. Josephson effect and quantum interference of quasiparticles

In this section we will analyze the dc Josephson effect in SNS systems which contain several insulating barriers. In this case electrons scattered at different barriers can interfere inside the junction. We will demonstrate that such interference may lead to qualitatively new effects and cause severe modifications of the supercurrent across the junction. We will see that these modifications can go in both directions, i.e., the Josephson current can be dramatically decreased by *destructive interference* of quasiparticles or, on the contrary, increased as a result of their *constructive interference*. The first situation is realized for sufficiently short junctions, while for longer ones the second effect might become more pronounced.

The phenomenon of quantum interference of quasiparticles is of primary importance for SNS structures with few conducting channels. The interest to such structures grew considerably after several experimental groups have succeeded in connecting a carbon nanotube to two superconductors and performing transport measurements in such systems [32–34]. More conventional SNS structures with many conducting channels and several insulating barriers are also of considerable interest, for instance in relation to possible applications, see, e.g., Ref. 35 and further references therein. We will demonstrate that for such systems quantum interference effects are also important provided there exist more than two scatterers inside the junction.

On a theoretical side a significant difficulty is that the powerful formalism of quasiclassical energy-integrated Eilenberger Green functions [17–19,30] supplemented by the Zaitsev boundary conditions [36] cannot be directly applied to systems containing more than one insulating barrier. An important ingredient of the derivation [36] is the assumption that such barriers are located sufficiently far from each other, so that *interference effects* emerging from electron scattering can be totally neglected. It is also essential that Zaitsev boundary conditions do not depend on the scattering phases. Since here we are just interested in investigating the quantum interference of quasiparticles, we are not in a position to use the quasiclassical Eilenberger formalism for our purposes. One possibility to circumvent this problem is to apply the formalism [37,38] within which the presence of an ar-

bitrary number of barriers in the system can be accounted for by linear boundary conditions. Another – even more straightforward – possibility to analyze the dc Josephson effect in structures with several insulating barriers is to directly solve the exact Gor'kov equations [39]. Here we will follow the second approach.

The results presented in this section were obtained in collaboration with Galaktionov [8]. A similar approach has also been used independently by Brinkman and Golubov [40].

2.1. General formalism

In what follows we will assume that our system is uniform along the directions parallel to the interfaces (coordinates y and z). Performing the Fourier transformation of the normal G and anomalous F^+ Green function with respect to these coordinates

$$G_{\omega_n}(\mathbf{r}, \mathbf{r}') = \int \frac{d^2 \mathbf{k}_{\parallel}}{(2\pi)^2} G_{\omega_n}(x, x', \mathbf{k}_{\parallel}) e^{i\mathbf{k}_{\parallel}(\mathbf{r}_{\parallel} - \mathbf{r}'_{\parallel})}$$

we express the Gor'kov equations in the following standard form

$$\begin{pmatrix} i\omega_n - \hat{H} & \Delta(x) \\ \Delta^*(x) & i\omega_n + \hat{H}_c \end{pmatrix} \begin{pmatrix} G_{\omega_n}(x, x', \mathbf{k}_{\parallel}) \\ F_{\omega_n}^+(x, x', \mathbf{k}_{\parallel}) \end{pmatrix} = \begin{pmatrix} \delta(x - x') \\ 0 \end{pmatrix}. \quad (5)$$

Here $\omega_n = (2n + 1)\pi T$ is the Matsubara frequency, and $\Delta(x)$ is the superconducting order parameter. The Hamiltonian \hat{H} in Eq. (5) reads

$$\hat{H} = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + \frac{\tilde{\mathbf{k}}_{\parallel}^2}{2m} - \varepsilon_F + V(x). \quad (6)$$

Here $\tilde{\mathbf{k}}_{\parallel} = \mathbf{k}_{\parallel} - e/c \mathbf{A}_{\parallel}(x)$, ε_F is Fermi energy, the term $V(x)$ accounts for the external potentials (including the boundary potential), \mathbf{A}_{\parallel} is the vector potential. The Hamiltonian \hat{H}_c is obtained from \hat{H} (6) by inverting the sign of the electron charge e .

As usually, it is convenient to separate fast oscillations of the Green functions $\propto \exp(\pm ik_x x)$ from the envelope of these functions changing at much longer scales as compared to the atomic ones. Then one can construct a particular solution of the Gor'kov equations (5) in the following form

$$\begin{pmatrix} G_{\omega_n}(x, x', \mathbf{k}_{\parallel}) \\ F_{\omega_n}^+(x, x', \mathbf{k}_{\parallel}) \end{pmatrix} = \bar{\phi}_{+1}(x) g_1(x') e^{ik_x(x-x')} + \bar{\phi}_{-2}(x) g_2(x') e^{-ik_x(x-x')} \quad \text{if } x > x' \quad (7)$$

and

$$\begin{pmatrix} G_{\omega_n}(x, x', \mathbf{k}_{\parallel}) \\ F_{\omega_n}^+(x, x', \mathbf{k}_{\parallel}) \end{pmatrix} = \bar{\phi}_{-1}(x) f_1(x') e^{-ik_x(x-x')} + \bar{\phi}_{+2}(x) f_2(x') e^{ik_x(x-x')} \quad \text{if } x < x'. \quad (8)$$

These functions satisfy Gor'kov equations at $x \neq x'$. Here $\bar{\phi}_{\pm}$ are two linearly independent solutions of the equation

$$\begin{pmatrix} i\omega_n - \hat{H}_{\pm}^a & \Delta(x) \\ \Delta^*(x) & i\omega_n + \hat{H}_{\pm c}^a \end{pmatrix} \bar{\phi}_{\pm} = 0. \quad (9)$$

The solution $\bar{\phi}_{+1}$ does not diverge at $x \rightarrow +\infty$, while $\bar{\phi}_{+2}$ is well-behaved at $x \rightarrow -\infty$. Similarly, two linearly independent solutions $\bar{\phi}_{-1,2}$ do not diverge, respectively, at $x \rightarrow -\infty$ and $x \rightarrow +\infty$.

In Eq. (9) we defined

$$\hat{H}_{\pm}^a = \mp i v_x \partial_x - \frac{e}{c} \mathbf{A}_{\parallel}(x) \mathbf{v}_{\parallel} + \frac{e^2}{2mc^2} \mathbf{A}_{\parallel}^2(x) + \tilde{V}(x). \quad (10)$$

Here $k_x = m v_x = \sqrt{k_F^2 - k_{\parallel}^2}$, $\tilde{V}(x)$ represents a slowly varying part of the potential which *does not* include fast variations possibly occurring at metallic interfaces. The latter will be accounted for by the boundary conditions to be considered below.

The functions $f_{1,2}(x)$ and $g_{1,2}(x)$ are determined with the aid of the continuity condition for the Green functions at $x = x'$ and the condition resulting from the integration of $\delta(x - x')$ in Eq. (5).

A general solution of the Gor'kov equations has the form

$$\begin{pmatrix} G_{\omega_n}(x, x') \\ F_{\omega_n}^+(x, x') \end{pmatrix} = \begin{pmatrix} G_{\omega_n}(x, x') \\ F_{\omega_n}^+(x, x') \end{pmatrix}_{\text{part}} + [l_1(x') \bar{\phi}_{+1}(x) + l_2(x') \bar{\phi}_{+2}(x)] e^{ik_x x} + [l_3(x') \bar{\phi}_{-1}(x) + l_4(x') \bar{\phi}_{-2}(x)] e^{-ik_x x}. \quad (11)$$

For systems which consist of several metallic layers the particular solution is obtained with the aid of the procedure outlined above provided both coordinates x and x' belong to the same layer. Should x and x' belong to different layers, the particular solution is zero because in that case the δ -function in Eq. (5) fails. The functions $l_{1,2,3,4}(x')$ in each layer should be derived from the proper boundary conditions. These are just the matching conditions for the wave functions on the left and on the right side of a potential barrier $A_1 \exp(ik_{1x}x) + B_1 \exp(-ik_{1x}x)$ and $A_2 \exp(ik_{2x}x) + B_2 \exp(-ik_{2x}x)$, respectively. These conditions have the standard form (see, e.g., [41]):

$$A_2 = \alpha A_1 + \beta B_1, \quad B_2 = \beta^* A_1 + \alpha^* B_1, \quad |\alpha|^2 - |\beta|^2 = \frac{k_{1x}}{k_{2x}}. \quad (12)$$

The equations

$$R = \left| \frac{\beta}{\alpha} \right|^2, \quad D = 1 - R = \frac{k_{1x}}{k_{2x} |\alpha|^2} \quad (13)$$

define, respectively, the reflection and transmission coefficients of the barrier. Applying these boundary conditions at each insulating barrier one uniquely determines all the unknown functions in Eq. (11) and thereby completes the construction of the Green functions for our problem. For further details we refer the reader to Ref. 8.

We are now in a position to specify the general expression for the Josephson current across ballistic SNS junctions which contain an arbitrary number of insulating barriers. In what follows we will assume that a thin specularly reflecting insulating barriers (I) are situated at both SN interfaces. Additional such barriers can also be present inside the N metal. Transmissions of these barriers may take any value from zero to one. We also assume that electrons propagate ballistically between any two adjacent barriers and that no electron-electron or electron-phonon interactions are present in the normal metal. For simplicity we will restrict our attention to the case of identical superconducting electrodes with singlet isotropic pairing and neglect suppression of the superconducting order parameter Δ in the electrodes close to the SN interface. The phase of the order parameter is set to be $-\phi/2$ ($+\phi/2$) in the left (right) electrode. As before, the thickness of the normal layer will be denoted by d .

Employing the standard formula for the current density

$$J = \frac{ie}{m} T \sum_{\omega_n} \int \frac{d^2 k_{\parallel}}{(2\pi)^2} (\nabla_{x'} - \nabla_x)_{x' \rightarrow x} G_{\omega_n}(x, x', \mathbf{k}_{\parallel}) \quad (14)$$

and making use of the expressions for the Green functions, one arrives at the following result

$$J = 4eT \sum_{\omega_n > 0} \int_0^{k_F} \frac{k_x dk_x}{2\pi} \frac{\sin \phi}{\cos \phi + W}, \quad (15)$$

where the function W depends on the number of insulating barriers. This function will be specified below for the case of two and three barriers.

Note that the integral over k_x in Eq. (15) can be replaced by a sum over independent conducting channels

$$\frac{\mathcal{A}}{2\pi} \int_0^{k_F} k_x dk_x (\dots) \rightarrow \sum_m^N (\dots), \quad (16)$$

where \mathcal{A} is the junction cross section. In this case $D_{1,2}$ and $R_{1,2}$ may also depend on the channel index m .

2.2. SINI'S junctions with few conducting channels

Let us first consider SNS junctions with two insulating barriers, one at each NS interface. In this case the function W in (15) takes the form

$$W = \frac{4\sqrt{R_1 R_2}}{D_1 D_2} \frac{\Omega_n^2}{\Delta^2} \cos \chi + \frac{\Omega_n^2 (1 + R_1)(1 + R_2) + \omega_n^2 D_1 D_2}{D_1 D_2 \Delta^2} \cosh \frac{2\omega_n d}{v_x} + \frac{2(1 - R_1 R_2)}{D_1 D_2} \frac{\Omega_n \omega_n}{\Delta^2} \sinh \frac{2\omega_n d}{v_x}. \quad (17)$$

Here $\chi = 2k_x d + \phi$ is the phase of the product $\alpha_2^* \beta_2 \alpha_1^* \beta_1$. Equations (15), (17) provide a general expression for the dc Josephson current in SINI'S structures valid for arbitrary transmissions D_1 and D_2 .

Let us first analyze the above result for the case of one conducting channel $N = 1$. We observe that the first term in Eq. (17) contains $\cos(2k_x d + \phi)$ which oscillates at distances of the order of the Fermi wavelength. Provided at least one of the barriers is highly transparent and/or (for sufficiently long junctions $d \gtrsim \xi_0$) the temperature is high $T \gg v_F/d$ this oscillating term is unimportant and can be neglected. However, at lower transmissions of both barriers and for relatively short junctions $d \lesssim v_F/T$ this term turns out to be of the same order as the other contributions to W (17). In this case the supercurrent is sensitive to the exact positions of the discrete energy levels inside the junction which can in turn vary considerably if d changes at the atomic scales $\sim 1/k_F$. Hence, one can expect sufficiently strong sample-to-sample fluctuations of the Josephson current even for junctions with nearly identical parameters.

Let us first consider the limit of relatively short SINI'S junctions in which case we obtain

$$I = \frac{e\Delta}{2} \frac{\mathcal{T} \sin \phi}{\mathcal{D}} \tanh \left[\frac{\mathcal{D}\Delta}{2T} \right], \quad (18)$$

where we defined

$$\mathcal{D}(\phi) = \sqrt{1 - \mathcal{T} \sin^2(\phi/2)} \quad (19)$$

and an effective normal transmission of the junction

$$\mathcal{T} = \frac{D_1 D_2}{1 - R_1 R_2 + 2\sqrt{R_1 R_2} \cos \chi}. \quad (20)$$

Equation (18) has exactly the same functional form as the result derived by Haberkorn et al. [21] for SIS junctions with an arbitrary transmission of the insulating barrier. This result is recovered from our Eqs. (18), (20) if we assume, e.g., $D_1 \ll D_2$ in which case the total transmission (20) reduces to $\mathcal{T} \simeq D_1$.

As we have already discussed the total transmission \mathcal{T} and, hence, the Josephson current fluctuate depending on the exact position of the bound states inside the junction. The resonant transmission is achieved for $2k_x d + \varphi = \pm\pi$, in which case we get

$$\mathcal{T}_{\text{res}} = \frac{D_1 D_2}{(1 - \sqrt{R_1 R_2})^2}. \quad (21)$$

This equation demonstrates that for symmetric junctions $D_1 = D_2$ at resonance the Josephson current does not depend on the barrier transmission at all. In this case $\mathcal{T}_{\text{res}} = 1$ and our result (18) coincides with the formula derived by Kulik and Omel'yanchouk [16] for ballistic constrictions. In the limit of low transmissions $D_{1,2} \ll 1$ we recover the standard Breit–Wigner formula $\mathcal{T}_{\text{res}} = 4D_1 D_2 / (D_1 + D_2)^2$ and reproduce the result obtained by Glazman and Matveev [42] for the problem of resonant tunneling through a single Anderson impurity between two superconductors.

Note that our results (18)–(20) also support the conclusion reached by Beenakker [43] that the Josephson current across sufficiently short junctions has a universal form and depends only on the total scattering matrix of the weak link which can be evaluated in the normal state. Although this conclusion is certainly correct in the limit $d \rightarrow 0$, its applicability range depends significantly on the physical nature of the scattering region. From Eqs. (15), (17) we observe that the result (18), (19) applies at $d \ll \xi_0$ not very close to the resonance. On the other hand, at resonance the above result is valid only under a more stringent condition $d \ll \xi_0 D_{\text{max}}$, where we define $D_{\text{max}} = \max(D_1, D_2)$.

Now let us briefly analyze the opposite limit of sufficiently long junctions $d \gg \xi_0$. Here we will restrict ourselves to the most interesting case $T = 0$. From Eqs. (15), (17) we obtain

$$I = \frac{ev_x \sin \phi}{\pi d z_1} \left[\frac{\arctan \sqrt{z_2/z_1}}{\sqrt{z_2/z_1}} \right],$$

$$z_{1,2} = \cos^2 \frac{\phi}{2} + \frac{1}{D_1 D_2} (R_+ \pm 2\sqrt{R_1 R_2} \cos \chi), \quad (22)$$

where $R_+ = R_1 + R_2$. For a fully transparent channel $D_1 = D_2 = 1$ the above expression reduces to the well known Ishii–Kulik result [11,12]

$$I = \frac{ev_x \phi}{\pi d}, \quad -\pi < \phi < \pi, \quad (23)$$

whereas if one transmission is small $D_1 \ll 1$ and $D_2 \approx 1$ we reproduce the result [22]

$$I = \frac{ev_x D_1 \sin \phi}{2d}. \quad (24)$$

Provided the transmissions of both NS interfaces are low $D_{1,2} \ll 1$ we obtain in the off-resonant region

$$I = \frac{ev_x}{4\pi d} D_1 D_2 \sin \phi \Upsilon[\chi], \quad (25)$$

where $\Upsilon[\chi]$ is a 2π -periodic function defined as

$$\Upsilon[\chi] = \frac{\chi}{\sin \chi}, \quad -\pi < \chi < \pi. \quad (26)$$

In the vicinity of the resonance $||\chi| - \pi| \lesssim D_{\text{max}}$ the above result does not hold anymore. Exactly at resonance $\chi = \pm\pi$ we get

$$I = \frac{ev_x \sqrt{D_1 D_2} \sin \phi}{4d \left\{ \cos^2 \frac{\phi}{2} + \frac{1}{4} \left(\sqrt{\frac{D_1}{D_2}} - \sqrt{\frac{D_2}{D_1}} \right)^2 \right\}^{1/2}}. \quad (27)$$

For a symmetric junction $D_{1,2} = D$ this formula yields

$$I = \frac{ev_x D \sin \phi / 2}{2d}, \quad -\pi < \phi < \pi, \quad (28)$$

while in a strongly asymmetric case $D_1 \ll D_2$ we again arrive at the expression (24). This implies that at resonance the barrier with higher transmission D_2 becomes effectively transparent even if $D_2 \ll 1$. We conclude that for $D_{1,2} \ll 1$ the maximum Josephson current is proportional to the product of transmissions $D_1 D_2$ off resonance, whereas exactly at resonance it is proportional to the lowest of two transmissions D_1 or D_2 .

We observe that both for short and long SINI'S junctions interference effects may enhance the Josephson effect or partially suppress it depending on the exact positions of the bound states inside the junction. We also note that in order to evaluate the supercurrent across SINI'S junctions it is in general *not* sufficient to derive the transmission probability for the corresponding NINI'N structure. Although the normal transmission of the above structure is given by Eq. (20) for *all* values of d , the correct expression for the Josephson current can be recovered by combining Eq. (20) with the results [21,43] in the limit of short junctions $d \ll D\xi_0$ only. In this case one can neglect suppression of the anomalous Green functions inside the normal layer and, hence, the information about the normal transmission turns out to be sufficient. On

the contrary, for longer junctions the decay of Cooper pair amplitudes inside the N layer cannot be anymore disregarded. In this case the supercurrent will deviate from the form (18) even though the normal transmission of the junction (20) will remain unchanged. This deviation becomes particularly pronounced for long junctions, i.e., for $d \gg \xi_0$ out of resonance and for $d \gg D\xi_0$ at resonance.

Generalization of the above results to the case of an arbitrary number of independent conducting channels $N > 1$ is trivial: The supercurrent is simply given by the sum of the contributions from all the channels. These contributions are in general not equal because the phase factors $\chi = 2k_x d + \varphi$ change randomly for different channels. Hence, mesoscopic fluctuations of the supercurrent should become smaller with increasing number of channels and eventually disappear in the limit of large N .

In the latter limit the Josephson current is obtained by averaging over all values of the phase χ . This limit was already studied in details [8,40] and will not be considered here. We will only point out that – as it was demonstrated in Ref. 8 – in the limit $N \rightarrow \infty$ interference effects are effectively averaged out and exactly the same result can be reproduced by means of the Eilenberger formalism supplemented by Zaitsev boundary conditions. It is also worthwhile to emphasize that the latter statement applies only to the junctions with two insulating barriers. Below we will show that for systems with more than two barriers quasiparticle interference effects turn out to be even more significant, and the correct result for the current cannot be recovered with the aid of Zaitsev boundary conditions even in the limit $N \rightarrow \infty$.

2.3. Josephson current in SINI'NI'S junctions

Let us now turn to SNS structures with three insulating barriers. As before, two of them are located at SN interfaces, and the third barrier is inside the N layer at a distance d_1 and d_2 , respectively, from the left and right SN interfaces. The transmission and reflection coefficients of this intermediate barrier are denoted as D_0 and $R_0 = 1 - D_0$, whereas the left and the right barriers are characterized by $D_1 = 1 - R_1$ and $D_2 = 1 - R_2$, respectively.

The supercurrent is calculated along the same lines as it was done for the case of two barriers. The final result is again expressed by Eq. (15), where the function W is now defined by a substantially more cumbersome expression than one for the two barriers case. This expression was evaluated in Ref. 8 and will not be presented here. We will go over to the final results.

2.3.1. One channel limit. Let us first discuss the case of one conducting channel. In the limit of short

junctions $d \ll \xi_0 D_{\max}$ we again reproduce the result (18) where the total effective transmission of the normal structure with three barriers takes the form

$$\mathcal{T} = \frac{2t_1 t_0 t_2}{1 + t_1 t_0 t_2 + C(\phi_{1,2}, t_{0,1,2})}, \quad (29)$$

where

$$C = \cos \chi_1 \sqrt{(1 - t_0^2)(1 - t_1^2)} + \cos \chi_2 \sqrt{(1 - t_0^2)(1 - t_2^2)} + (\cos \chi_1 \cos \chi_2 - t_0 \sin \chi_1 \sin \chi_2) \sqrt{(1 - t_1^2)(1 - t_2^2)}. \quad (30)$$

Here we define $t_{0,1,2} = D_{0,1,2}/(1 + R_{0,1,2})$ and $\chi_{1,2} = 2k_x d_{1,2} + \varphi_{1,2}$. For later purposes let us also perform averaging of this transmission over the phases $\chi_{1,2}$. We obtain

$$\langle \mathcal{T} \rangle = \frac{2t_1 t_0 t_2}{\sqrt{2t_1 t_0 t_2 + t_1^2 t_0^2 + t_1^2 t_2^2 + t_0^2 t_2^2 - t_1^2 t_0^2 t_2^2}}. \quad (31)$$

In particular, in the case of similar barriers with small transparencies $D_{0,1,2} \approx D \ll 1$ the average normal transmission of our structure is $\langle \mathcal{T} \rangle \sim D^{3/2}$. Suppression of the average transmission below the value $\sim D$ is a result of destructive interference and indicates the tendency of the system towards localization.

Let us now proceed to the limit of a long junction $d_{1,2} \gg \xi_0$ and $T = 0$. In the off-resonant region for $d_1 = d_2$ we find

$$I = \frac{ev_x D_1 D_0 D_2 \sin \phi}{8\pi d_1} \frac{\Upsilon[\chi_1] - \Upsilon[\chi_2]}{\cos \chi_2 - \cos \chi_1}. \quad (32)$$

This expression diverges at resonance (i.e., at $\chi_1 \simeq \pi$ or $\chi_2 \simeq \pi$) where it becomes inapplicable. In the resonant region $\chi_2 \simeq \pi$ we obtain

$$I = \frac{ev_x \sqrt{D_1 D_0 D_2} \sin \phi}{4d \sqrt{2(1 + \cos \chi_1)[\mathcal{T}^{-1} - \sin^2(\phi/2)]}}. \quad (33)$$

2.3.2. Many channel junctions. As it was already discussed, in the many channel limit it is appropriate to average the current over the scattering phases. Practically in any realistic physical realization the widths d_1 and d_2 fluctuate independently on the atomic scale. In this case averaging over χ_1 and χ_2 should also be performed independently. If d_1 and d_2 do not change on the atomic scale but are incommensurate, independent averaging over the two phases is to be performed as well. Independent averaging cannot be fulfilled only in the (physically irrelevant) case of strictly commensurate d_1 and d_2 which will not be considered below.

Technically independent averaging over the scattering phases $\chi_1 = x$ and $\chi_2 = \lambda x$ amounts to evaluating the integral of the expression

$1/[t + \cos x \cos(\lambda x)]$ from $x = 0$ to some large value $x = L$. At $\lambda = 1$ the result of this integration is $L/\sqrt{t(1+t)}$. However, if λ is irrational, the integral approaches the value $2LK(1/t^2)/\pi t$, where $K(h) = F(\pi/2, h)$ is the complete elliptic integral.

Let us assume that the transparencies of all three interfaces are small as compared to one. After averaging over the two scattering phases we arrive at the final expression for the current

$$J = \frac{ek_F^2}{\pi^2} D_{\text{eff}} \sin(\phi T) \sum_{\omega_n > 0} \frac{\Delta^2}{\Omega_n^2} K \left[\frac{\Delta^2 \sin^2(\phi/2)}{\Omega_n^2} \right], \quad (34)$$

where we define the effective transmission

$$D_{\text{eff}} = \int_0^1 d\mu \mu \sqrt{D_0 D_1 D_2}. \quad (35)$$

Hence, for similar barriers we obtain the dependence $J \propto D^{3/2}$ rather than $J \propto D$ (as it would be the case for independent barriers). The latter dependence would follow from the calculation based on Zaitsev boundary conditions for the Eilenberger propagators. We observe, therefore, that quantum interference effects *decrease* the Josephson current in systems with three insulating barriers. This is essentially quantum effect which cannot be recovered from Zaitsev boundary conditions even in the multichannel limit. This effect has exactly the same origin as a quantum suppression of the average normal transmission $\langle \mathcal{T} \rangle$ due to localization effects. Further limiting expressions for short junctions can be directly recovered from Eq. (31).

We also note that the current-phase relation (34) deviates from a pure sinusoidal dependence even though all three transmissions are small $D_{0,1,2} \ll 1$. At $T = 0$ the critical Josephson current is reached at $\phi \simeq 1.7$ which is slightly higher than $\pi/2$. Although this deviation is quantitatively not very significant, it is nevertheless important as yet one more indication of quantum interference of electrons inside the junction.

Finally, let us turn to the limit of long junctions $d_{1,2} \gg \xi_0$. We again restrict ourselves to the case of low transparent interfaces. At high temperatures $T \gg v_F/2\pi d_{1,2}$ we get $J \propto D_0 D_1 D_2 \exp(-d/\xi(T))$, where $d = d_1 + d_2$ and $\xi(T) = v_F/(2\pi T)$. In this case the anomalous Green function strongly decays deep in the normal layer. Hence, interference effects are not important and the interfaces can be considered as independent from each other. In the opposite limit $T \ll Dv_F/d$, however, interference effects become important, and the current becomes proportional to

$D^{5/2}$ rather than to D^3 . Explicitly, at $T \rightarrow 0$ with the logarithmic accuracy we get

$$J = \frac{ek_F^2 v_F \sin \phi}{16\pi^2 \sqrt{d_1 d_2}} \int_0^1 d\mu \mu^2 D_1 D_2 \sqrt{D_0} \ln D_0^{-1}. \quad (36)$$

We see that, in contrast to short junctions, in the limit of thick normal layers interference effects *increase* the Josephson current as compared to the case of independent barriers. The result (36), as well as one of Eqs. (34), (35) cannot be obtained from the Eilenberger approach supplemented by Zaitsev boundary conditions.

2.4. Some conclusions

By directly solving the Gor'kov equations we evaluated the dc Josephson current in SNS junctions containing two and three insulating barriers with arbitrary transmissions SINI'S and SINI'NI''S junctions, respectively. Our results can be directly applied both to the junctions with few conducting channels (such as, e.g., superconductor-carbon nanotube-superconductor junctions [32–34]) and to more conventional SNS structures in the many channel limit. We have demonstrated that an interplay between the proximity effect and quantum interference of quasiparticles may play a crucial role in such systems causing strong modifications of the Josephson current.

For the system with two barriers and few conducting channels we found strong fluctuations of the Josephson critical current depending on the exact position of the resonant level inside the junction. For short junctions $d \ll \xi_0 D$ at resonance the Josephson current does not depend on the barrier transmission D and is given by the standard Kulik – Omel'yanchouk formula [16] derived for ballistic weak links. In the limit of long SNS junctions $d \gg \xi_0$ resonant effects may also lead to strong enhancement of the supercurrent, in this case at $T \rightarrow 0$ and at resonance the Josephson current is proportional to D and not to D^2 as it would be in the absence of interference effects.

While the above results for few conducting channels cannot be obtained by means of the approach employing Zaitsev boundary conditions, in the many channel limit and for junctions with two barriers the latter approach *does* allow to recover correct results. This is because the contributions sensitive to the scattering phase are effectively averaged out during summation over conducting channels.

Quantum interference effects turn out to be even more important in the proximity systems which contain three insulating barriers. In this case the quasiclassical approach based on Zaitsev boundary

conditions fails even in the limit of many conducting channels. In that limit the Josephson current is *decreased* for short junctions ($J \propto D^{3/2}$) as compared to the case of independent barriers ($J \propto D$). This effect is caused by destructive interference of electrons reflected from different barriers and indicates the tendency of the system towards localization. In contrast, for long SNS junctions with three barriers an interplay between quantum interference and proximity effect leads to enhancement of the Josephson current at $T \rightarrow 0$: We obtained the dependence $J \propto D^{5/2}$ instead of $J \propto D^3$ for independent barriers.

3. Parity affected Josephson current

Let us now turn to a different issue which – to the best of our knowledge – has not yet attracted much attention in the literature. Namely, we will discuss an interplay between the parity effect and the dc Josephson current in superconducting weak links. The results presented in this section have been obtained in collaboration with Sharov [9].

It is well known that thermodynamic properties of isolated superconducting systems are sensitive to the parity of the total number of electrons [44,45] even though this number \mathcal{N} is macroscopically large. This parity effect is a direct consequence of the fundamental property of a superconducting ground state described by the condensate of Cooper pairs. The number of electrons forming this condensate is necessarily even, hence, for odd \mathcal{N} at least one electron always remains unpaired having an extra energy equal to the superconducting energy gap Δ . At sufficiently low temperatures a clear difference between the superconducting states with even and odd \mathcal{N} was demonstrated experimentally [45,46].

Can the supercurrent be affected by this parity effect? At the first sight the answer to this question should be negative because of the fundamental uncertainty relation $\delta\mathcal{N}\delta\phi \gtrsim 1$. Should the electron number \mathcal{N} be fixed, fluctuations of the superconducting phase ϕ become large disrupting the supercurrent in the system. On the other hand, suppressing fluctuations of the phase ϕ will destroy the parity effect because of large fluctuations of \mathcal{N} .

Despite that, below we will demonstrate that in certain superconducting structures the parity effect can coexist with the nonvanishing supercurrent. Consider a superconducting system which can support circular persistent currents (PC). An example is provided by an isolated superconducting ring pierced by the magnetic flux Φ in which case circulating PC is induced in the ring. In accordance with the number-phase uncertainty relation the global superconducting phase of the ring fluctuates strongly in this

case, however these fluctuations are decoupled from the supercurrent and therefore can be integrated out without any influence on the latter. In what follows we will show that the parity effect may substantially modify PC in superconducting nanorings, in particular for odd number of electrons.

3.1. Parity projection formalism

In order to investigate systematically the influence of the electron parity number on persistent currents in superconducting nanorings we will employ the well known parity projection formalism [48–50]. Recapitulating the key points of this approach we will closely follow Ref. 49.

The grand canonical partition function $Z(T, \mu) = \text{Tr} \exp [-\beta(\mathcal{H} - \mu\mathcal{N})]$ is connected to the canonical one $Z(T, \mathcal{N})$ by means of the following equation

$$Z(T, \mu) = \sum_{\mathcal{N}=0}^{\infty} Z(T, \mathcal{N}) \exp\left(\frac{\mu\mathcal{N}}{T}\right). \quad (37)$$

Here and below \mathcal{H} is the system Hamiltonian, \mathcal{N} is the total number of electrons and $\beta \equiv 1/T$. Inverting this relation and defining the canonical partition functions Z_e and Z_o , respectively, for even ($\mathcal{N} \equiv \mathcal{N}_e$) and odd ($\mathcal{N} \equiv \mathcal{N}_o$) ensembles, one gets

$$Z_{e/o}(T) = \frac{1}{2\pi} \int_{-\pi}^{\pi} du e^{-i\mathcal{N}_{e/o}u} Z_{e/o}(T, iTu), \quad (38)$$

where

$$\begin{aligned} Z_{e/o}(T, \mu) &= \frac{1}{2} \text{Tr} \{ [1 \pm (-1)^{\mathcal{N}}] e^{-\beta(\mathcal{H} - \mu\mathcal{N})} \} = \\ &= \frac{1}{2} [Z(T, \mu) \pm Z(T, \mu + i\pi T)] \end{aligned} \quad (39)$$

are the parity projected grand canonical partition functions. For $\mathcal{N} \gg 1$ it is sufficient to evaluate the integral in (38) within the saddle point approximation which yields

$$Z_{e/o}(T) \sim \exp [-\beta(\Omega_{e/o} - \mu_{e/o}\mathcal{N}_{e/o})], \quad (40)$$

where $\Omega_{e/o} = -T \ln Z_{e/o}(T, \mu)$ are the parity projected thermodynamic potentials. They can be presented in the form

$$\Omega_{e/o} = \Omega_f - T \ln \left[\frac{1}{2} (1 \pm e^{-\beta(\Omega_b - \Omega_f)}) \right], \quad (41)$$

where

$$\Omega_{f/b} = -T \ln [\text{Tr} \{ (\pm 1)^{\mathcal{N}} e^{-\beta(\mathcal{H} - \mu\mathcal{N})} \}]. \quad (42)$$

Chemical potentials $\mu_{e/o}$ are defined by the saddle point condition $N_{e/o} = -\partial\Omega_{e/o}(T, \mu_{e/o})/\partial\mu_{e/o}$.

The main advantage of the above analysis is that it allows to express the canonical partition functions and thermodynamical potentials in terms of the parity projected grand canonical ones thereby enormously simplifying the whole calculation. We further note that Ω_f is just the standard grand canonical thermodynamic potential and Ω_b represents the corresponding potential linked to the partition function $Z(T, \mu + i\pi T)$. It is easy to see [49] that in order to recover this function one can evaluate the true grand canonical partition function $Z(T, \mu)$, express the result as a sum over the Fermi Matsubara frequencies $\omega_f = 2\pi T(m + 1/2)$ and then substitute the Bose Matsubara frequencies $\omega_b = 2\pi Tm$ instead of the Fermi ones. This procedure will automatically yield the correct expression for $Z(T, \mu + i\pi T)$ and, hence, for Ω_b .

Having found the thermodynamic potentials for the even and odd ensembles one can easily determine the equilibrium current I . Here we will be interested in describing the currents flowing in isolated superconducting rings pierced by the external magnetic flux Φ_x . Then in the case of even/odd total number of electrons one obtains

$$I_{e/o} = I_f \pm \frac{I_b - I_f}{e^{\beta(\Omega_b - \Omega_f)} \pm 1}, \quad (43)$$

where the upper/lower sign corresponds to the even/odd ensemble and we have defined

$$I_{e/o} = -c \left(\frac{\partial\Omega_{e/o}}{\partial\Phi_x} \right)_{\mu(\Phi_x)}, \quad I_{f/b} = -c \left(\frac{\partial\Omega_{f/b}}{\partial\Phi_x} \right)_{\mu(\Phi_x)}.$$

3.2. Parity effect in nanorings and blocking of the supercurrent

Let us now make use of the above general expressions and investigate the influence of the parity effect on PC in superconducting nanorings with quantum point contacts (QPC). Before turning to concrete calculations we shall specify the model for our system. We shall consider mesoscopic superconducting rings with cross section s and perimeter $L = 2\pi R$. The rings will be assumed sufficiently thin, i.e. $\sqrt{s} \ll \lambda_L$, where λ_L is the London penetration length. Superconductivity will be described within the (parity projected) mean field BCS theory. At sufficiently low temperatures this description is justified provided quantum phase slips (QPS) [51–53] in nanorings can be neglected. This requirement in turn implies that the ring cross section should be sufficiently large. With the aid of the results [51] one concludes that the QPS tunnel-

ing amplitude remains exponentially small provided the condition $s \gg \lambda_F^2 \sqrt{\xi_0/l}$ is satisfied. Here λ_F is the Fermi wavelength, $\xi_0 \sim v_F/\Delta$ is the coherence length and l is the electron elastic mean free path which is assumed to be shorter than ξ_0 . For generic systems QPS effects can usually be neglected provided the transversal size of the wire/ring \sqrt{s} exceeds ~ 10 nm. Hence, the total number of conducting channels in the ring $N_r \sim s/\lambda_F^2$ should inevitably be large $N_r \gg 1$. In addition, the ring perimeter L should not be too large, so that one could disregard the QPS-induced reduction of the PC amplitude [53]. Finally, we will neglect the difference between the mean field values of the BCS order parameter for the even and odd ensembles [48,49]. This is legitimate provided the volume of a superconducting ring is large enough, $\mathcal{V} = Ls \gg 1/v\Delta$, where v is the density of states at the Fermi level and Δ is the BCS order parameter for a bulk superconductor at $T = 0$. All these requirements can easily be met in a modern experiment.

The task at hand is now to evaluate the thermodynamic potentials $\Omega_{f/b}$. Within the mean field treatment these quantities can be expressed in terms of the excitation energies ε_k and the superconducting order parameter $\Delta(\mathbf{r})$. One finds [49]

$$\Omega_f = \tilde{\Omega} - 2T \sum_k \ln \left(2 \cosh \frac{\varepsilon_k}{2T} \right), \quad (44)$$

$$\Omega_b = \tilde{\Omega} - 2T \sum_k \ln \left(2 \sinh \frac{\varepsilon_k}{2T} \right), \quad (45)$$

where $\tilde{\Omega} = \int d^3\mathbf{r} |\Delta(\mathbf{r})|^2 / g + \text{Tr} \{ \hat{\xi} \}$, g is the BCS coupling constant and $\hat{\xi}$ is the single-particle energy operator:

$$\hat{\xi} = \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial \mathbf{r}} - \frac{e}{c} \mathbf{A}(\mathbf{r}) \right)^2 + U(\mathbf{r}) - \mu, \quad (46)$$

$\mathbf{A}(\mathbf{r})$ is the vector potential and $U(\mathbf{r})$ describes the potential profile due to disorder and interfaces.

The excitation spectrum ε_k has the form

$$\varepsilon_k = \varepsilon(\mathbf{p}) = \mathbf{p}\mathbf{v}_s + \sqrt{\xi^2 + \Delta^2}, \quad (47)$$

where \mathbf{p} is the quasiparticle momentum, $\xi = (p^2 - \tilde{\mu})/2m$, and $\tilde{\mu} = \mu(\Phi_x) - mv_s^2/2$. The superconducting velocity vector \mathbf{v}_s is oriented in the direction along the ring and is defined by the well known expression

$$v_s = \frac{\hbar}{2mR} \min_n \left(n - \frac{\Phi_x}{\Phi_0} \right). \quad (48)$$

This expression as well as the excitation spectrum (47) are the periodic functions of the flux Φ_x with the period equal to the superconducting flux quantum $\Phi_0 = hc/2e$.

Consider the most interesting case $T \rightarrow 0$. Making use of the above expressions one easily finds

$$I_e = ev_s \rho_e s, \quad \mu_e = \mu(\rho_e) + mv_s^2/2 \quad (49)$$

for the even ensemble and

$$I_o = ev_s \rho_o s - e \frac{v_F}{L} \text{sgn}(v_s), \quad \rho_o = \rho + \frac{1}{\mathcal{V}} \frac{|v_s|}{v_\mu} \quad (50)$$

for the odd one. Here $\rho_{e/o} = \mathcal{N}_{e/o}/\mathcal{V}$ are the electron densities for the even and odd ensembles, ρ is the grand canonical electron density at $T=0$, $v_\mu = \sqrt{2\mu/m}$ and v_s is assumed to be small as compared to the critical velocity $v_c = \Delta/p_F$. We also note that the second Eq. (50) is an implicit equation for the chemical potential μ_o .

Equation (49) – being combined with (48) – coincides with that obtained for the grand canonical ensemble. In particular, the current I_e represents the well known «saw tooth» dependence on magnetic flux. In contrast, for odd ensembles there exists an additional flux-dependent contribution to PC (50) which cannot be viewed just as a renormalization of ρ_o .

Unfortunately this parity effect is rather small in multichannel rings as we mentioned above. Estimating the leading contribution to $I_{e/o}$ as $I \sim ev_F N_r/L$, we find

$$(I_e - I_o)/I \sim 1/N_r \ll 1.$$

The results (49), (50) hold as long as $T \ll \hbar v_F/L$. At higher temperatures the parity effect gets even smaller and eventually disappears at temperatures exceeding the parameter [45] $T^* \approx \Delta/\ln(v\mathcal{V}\sqrt{\Delta T^*})$. The corresponding expressions are readily obtained within our formalism, but we will not consider them here.

Rather we turn to a somewhat different system – a superconducting ring interrupted by QPC – in which the parity effect turns out to play a much more important role. In this case the thermodynamic potential of the system Ω consists of two different contributions*

$$\Omega = \Omega^{(r)}(\mu, T, \Phi_x, \phi) + \Omega^{(c)}(\mu, T, \phi) \quad (51)$$

from the bulk part of the ring and from QPC, respectively. The optimal value of the phase difference ϕ across QPC is fixed by the condition $\partial\Omega/\partial\phi = 0$ which reads

$$-c \frac{\partial\Omega^{(r)}}{\partial\Phi_x} = -\frac{2e}{\hbar} \frac{\partial\Omega^{(c)}}{\partial\phi}. \quad (52)$$

Here we made use of the fact that the thermodynamic potentials of the ring depend both on Φ_x and ϕ only via the superfluid velocity $v_s = (1/4\pi mR) \times (\phi - 2\pi\Phi_x/\Phi_0)$, in which case one can put $\partial/\partial\Phi_x = -(2e/\hbar c)(\partial/\partial\phi)$. The left-hand side of Eq. (52) represents the current flowing inside the superconducting ring $I^{(r)} = -c \partial\Omega^{(r)}/\partial\Phi_x \simeq (ev_F N_r/L)(\phi - 2\pi\Phi_x/\Phi_0)$. This value should be equal to the current across QPC which is given by the right-hand side of Eq. (52). Estimating the maximum value of the latter for a single channel QPC as $2e\mathcal{T}\Delta/\hbar$, we obtain

$$\phi \simeq 2\pi \frac{\Phi_x}{\Phi_0}, \quad \text{if } L \ll L^*, \quad (53)$$

$$\phi \simeq 2\pi n, \quad \text{if } L \gg L^*, \quad (54)$$

where $L^* = \xi_0 N_r/\mathcal{T} \gg \xi_0$. In a more general case of QPC with N conducting channels in the expression for L^* one should set

$$\mathcal{T} \rightarrow \sum_n^N \mathcal{T}_n.$$

In what follows we will consider the most interesting limit $N \ll N_r$ and $L \ll L^*$. Due to Eq. (53) in this case the dependence $I_{e/o}(\Phi_x)$ is fully determined by the current-phase relation for QPS which can be found by means of Eq. (43) with $I_{f/b} = -(2e/\hbar)\partial\Omega_{f/b}^{(c)}/\partial\phi$. It is convenient to employ the formula [8]

$$I_{f/b} = \frac{2e}{\hbar} \sum_{n=1}^N \mathcal{T} \sum_{\omega_{f/b}} \frac{\sin\phi}{\cos\phi + W_n(\omega_{f/b})}. \quad (55)$$

In the case of short QPS one has

$$W_n(\omega) = (2/\mathcal{T}_n)(1 + \omega^2/\Delta^2) - 1,$$

where \mathcal{T}_n is the transmission of the n th conducting channel. Substituting this function into (55) and

* The relation (51) is strictly applicable only for grand canonical ensembles. However, at this point the difference between canonical and grand canonical ensembles is unimportant and can be disregarded.

summing over ω_f one recovers the standard result [16,21]

$$I_f(\phi) = -\frac{2e}{\hbar} \sum_{n=1}^N \frac{\partial \varepsilon_n(\phi)}{\partial \phi} \tanh \frac{\varepsilon_n(\phi)}{2T}, \quad (56)$$

where

$$\varepsilon_n(\phi) = \Delta \sqrt{1 - \mathcal{T}_n \sin^2(\phi/2)}. \quad (57)$$

The same summation over Bose Matsubara frequencies ω_b yields

$$I_b = -\frac{2e}{\hbar} \sum_{n=1}^N \frac{\partial \varepsilon_n(\phi)}{\partial \phi} \coth \frac{\varepsilon_n(\phi)}{2T}. \quad (58)$$

Finally, the difference $\Omega_b - \Omega_f \equiv \Omega_{bf}$ is evaluated as a sum of the ring ($\Omega_{bf}^{(r)}$) and QPS ($\Omega_{bf}^{(c)}$) contributions. The latter is found by integrating $I_{f/b}(\phi)$ over the phase difference ϕ :

$$\Omega_{bf}^{(c)} = 2T \sum_{n=1}^N \ln \coth \left(\frac{\varepsilon_n(\phi)}{2T} \right), \quad (59)$$

while the former is defined by the standard expression [49]

$$\beta \Omega_{bf}^{(r)} = 2\nu \int \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3} \ln \left(\coth \frac{\varepsilon(\mathbf{p})}{2T} \right) \simeq \nu \nu \sqrt{\Delta T} e^{-\Delta/T}.$$

Combining all these results with Eq. (43) we get

$$I_{e/o} = -\frac{2e}{\hbar} \sum_{n=1}^N \frac{\partial \varepsilon_n(\phi)}{\partial \phi} \tanh \frac{\varepsilon_n(\phi)}{2T} \times \left[1 \pm \frac{\left(\coth \frac{\varepsilon_n(\phi)}{2T} \right)^2 - 1}{e^{\beta \Omega_{bf}^{(r)}} \prod_{i=1}^N \left(\coth \frac{\varepsilon_i(\phi)}{2T} \right)^2 \pm 1} \right]. \quad (60)$$

Equation (60) represents the central result of this section. Together with Eq. (53) it establishes the complete dependence of PC on the magnetic flux Φ in isolated superconducting nanorings with QPC.

Consider the most interesting limit $T \rightarrow 0$. In this case for the even number of electrons in the ring PC is given by the expression (56) which coincides identically with that for grand canonical ensembles [16,21]. On the other hand, for the odd number of electrons PC will acquire an additional contribution which turns out to be most important for the case of single channel QPS $N = 1$. In that case the expression in the square brackets of Eq. (60) reduces to zero, i.e., PC

will be totally blocked by the odd electron. Thus, we predict a *novel mesoscopic effect* – parity affected blocking of PC in superconducting nanorings with QPC.

This result has a transparent physical interpretation. Indeed, it is well known [47] that the result (56) can be expressed via the contributions of discrete Andreev levels $E_{\pm}(\phi) = \pm \Delta \mathcal{D}(\phi)$ inside QPS as

$$I(\phi) = \frac{2e}{\hbar} \left[\frac{\partial E_-}{\partial \phi} f_-(E_-) + \frac{\partial E_+}{\partial \phi} f_+(E_+) \right], \quad (61)$$

where $\mathcal{D}(\phi)$ is defined in Eq. (19). Using the Fermi filling factors for these levels $f_{\pm}(E_{\pm}) = [1 + \exp(E_{\pm}(\phi)/T)]^{-1}$ one arrives at Eq. (56). If we now fix the number of electrons inside the ring and consider the limit $T \rightarrow 0$ the filling factors will be modified as follows. For the even number \mathcal{N} all electrons are paired occupying states with energies below the Fermi level. In this case one has $f_-(E_-) = 1$, $f_+(E_+) = 0$, the current is entirely determined by the contribution of the quasiparticle state E_- and Eq. (61) yields the same result as one for the grand canonical ensemble. By contrast, in the case of odd number of electrons one electron always remains unpaired and occupies the lowest available energy state – in our case E_+ – above the Fermi level. Hence, for odd \mathcal{N} one has $f_{\pm}(E_{\pm}) = 1$, the contributions of two quasiparticle energy states in Eq. (61) exactly cancel each other, and the current across QPS remains zero for any ϕ or the magnetic flux Φ_x . This is just the blocking effect which we have already obtained above from a more formal consideration.

For $N > 1$ and/or at nonzero temperatures this parity-affected blocking of PC becomes incomplete. But also in this case the parity effect remains essential at temperatures $T < T^*$ substantially affecting, e.g., the current-phase relation for QPC. For $T > 0$ this relation will deviate from the grand canonical one both for even and odd ensembles [9].

Finally, we consider superconducting rings containing a piece of a normal metal. Here we only restrict our attention to transparent SNS junctions with length of the normal metal $d \gg \xi_0$. In this case for $\omega \ll \Delta$ one has $W_i(\omega) = \cosh(2\omega d/v_F)$. Substituting this function into (55) and repeating the whole calculation as above, in the limit $T \rightarrow 0$ we obtain

$$I_e = \frac{ev_F N}{\pi d} \phi, \quad I_o = \frac{ev_F N}{\pi d} \left(\phi - \frac{\pi \operatorname{sgn} \phi}{N} \right). \quad (62)$$

These results apply for $-\pi < \phi < \pi$ and should be 2π -periodically continued otherwise. We observe that the current I_e again coincides with that for the grand canonical ensembles [11], while for odd \mathcal{N} the current-phase relation is shifted by the value π/N . This

shift has a simple interpretation as being related to the odd electron contribution $(2e/\hbar)\partial E_0/\partial\phi$ from the lowest (above the Fermi level) Andreev state $E_0(\phi)$ inside the SNS junction. Unlike in QPC, this contribution does not compensate for the current from other quasiparticle states. Rather it provides a possibility for a parity-induced π -junction state in our system. Indeed, according to Eq. (62) for single mode SNS junctions the «saw tooth» current-phase relation will be shifted exactly by π . For more than one conducting channel $N > 1$ within the interval $-\pi < \phi < \pi$ there exists a twofold degenerate minimum energy (zero current) state occurring at $\phi = \pm\pi/N$ [54]. In the special case $N = 2$ the current-phase relation $I_o(\phi)$ turns π -periodic.

The well known feature of superconducting rings interrupted by a π -junction is the possibility to develop *spontaneous* supercurrent in the ground state [55]. Although this feature is inherent to any type of π -junctions, in the case of the standard sinusoidal current-phase relation such spontaneous supercurrents can occur only for sufficiently large values of the ring inductance \mathcal{L} [55]. In contrast, in the situation studied here the spontaneous current state is realized for *any* inductance of the ring because of the nonsinusoidal dependence $I_o(\phi)$ (62).

In order to demonstrate that let us assume that no external flux is applied to our system. Then at $T \rightarrow 0$ the energy of an SNS ring with odd number of electrons can be written in the form

$$E_o = \frac{\Phi^2}{2c\mathcal{L}} + \frac{\pi\hbar v_F N}{\Phi_0^2 d} \left(\Phi - \frac{\Phi_0 \operatorname{sgn} \Phi}{2N} \right)^2, \quad (63)$$

where Φ is the flux related to the circular current flowing in the ring. Minimizing this energy with respect to Φ , one easily observes that a nonzero spontaneous current

$$I = \pm \frac{ev_F}{d} \left[1 + \frac{2ev_F N}{d} \frac{\mathcal{L}}{\Phi_0} \right]^{-1} \quad (64)$$

should flow in the ground state of our system. This is yet one more remarkable consequence of the parity effect: Just by changing \mathcal{N} from even to odd one can induce nonzero PC without any external flux Φ_x . In the limit of small inductances $\mathcal{L} \ll \Phi_0 d / ev_F N$ — which is easy to reach in the systems under consideration — the value of I does not depend on the number of channels N and is given by the universal formula $I = \pm ev_F / d$. For generic parameters this value can easily be as large as $I \sim 10$ nA.

In summary, new physical effects emerge from an interplay between the electron parity number and persistent currents in superconducting nanorings. These

effects can be directly tested in modern experiments and possibly used for engineering new types of superconducting flux-charge qubits.

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