Advances in Astronomy and Space Physics, **3**, 98-101 (2013)

Application of the full reduction technique for solution of equations with vector form non-linearity

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We consider making use of the full reduction algorithm for solving the equations with a vector non-linearity. The solutions of such the equations describe the planetary scale non-linear vortex structures of the Earth atmosphere, ionosphere and magnetosphere. We present the modification of full reduction technique for Charney-Obukhov equation with periodic boundary conditions. This technique allows to reduce significantly calculation time and to apply much more detailed spatial grid for studying non-linear processes in the near-Earth space.

Key words: experimental and mathematical techniques, numerical simulation studies

INTRODUCTION

In many cases, after an analytical study of the non-linear system of differential equations, there is a need to be sure in the correctness of the selected small parameters and obtained partial solutions. Numerical integration of differential equations allows to study the influence of small perturbations in the system. For the description of large-scale wave structures of Rossby type that are observed in the atmosphere-ionosphere of the Earth, one uses the Charney-Obukhov equation [1, 2, 4, 6, 7, 8, 9]. Also in a strong magnetic field of plasma a vortex can exist, similar to Rossby vortices observed in fluids in rotating systems [11, 12, 13]. This analogy comes from the fact that the form of Hasegawa-Mima equation for non-linear drift waves in a plasma fully coincides with the Charney-Obukhov equation (Charney-Obukhov equation are written for the stream functions, and the Hasegawa-Mima equation for the perturbed plasma potential). These equations contain a vector non-linearity in the form of Poisson brackets. Let us consider the numerical scheme to study the dynamics of initial perturbations in the physical system described by Charney-Obukhov equation:

$$\frac{\partial\left(\Delta\psi-\psi\right)}{\partial t}+\beta\frac{\partial\psi}{\partial y}+\psi\frac{\partial\psi}{\partial y}+\{\psi,\Delta\psi\}=0.$$

We consider the vector non-linearity impact onto evolution of the vortex structures in the form of Poisson brackets. The main problem for numerical integration of such a system is numerical instability in the calculation of the finite difference approximations of the main differential equations. That causes the explosive growth of the kinetic energy of the system. The effects of this instability are discussed in detail in [5]. To avoid these effects we use the conservative form of non-linear vector operators.

APPLICATION OF FULL REDUCTION TECHNIQUE

In order to avoid the influence of the estimated effects of instability in long-term integration we have applied numerical methods based on the approximate representation of vector operations in the finite approximation derived from the condition of momentum, kinetic energy, and vorticity conservation in the system [5]. The equation is considered in a coordinate system where axes x, y with constant step h denote distance in space, and z-axis is the amplitude perturbation. Differential equation is re-written in a form of the system of differential equations with lower order. The calculation has been done in two steps: numerical integration of next level of Z function making use of Runge-Kutta 4th order schema ($Z=\Delta\psi-\psi$), and after obtaining of Z_{n+1} calculated on a grid ψ_{n+1}. For the calculation of Z_{n+1} we use the Arakawa approximation of the second-order accuracy [5]:

$$\begin{aligned}\{\vec{a},\vec{b}\}=-\,&[(b_{i,j-1}+b_{i+1,j-1}-b_{i,j+1}-b_{i+1,j+1})\times\\&(a_{i+1,j}+a_{i,j})+\\&+(b_{i-1,j-1}+b_{i,j-1}-b_{i-1,j+1}-b_{i,j+1})(a_{i,j}+a_{i-1,j})+\\&+(b_{i+1,j-1}+b_{i+1,j+1}-b_{i-1,j}-b_{i-1,j+1})(a_{i,j+1}+a_{i,j})+\\&+(b_{i+1,j-1}+b_{i+1,j}-b_{i-1,j-1}-b_{i-1,j})(a_{i,j}+a_{i,j-1})+\\&+(b_{i+1,j}-b_{i,j+1})(a_{i+1,j+1}-a_{i,j})+\\&+(b_{i+1,j}-b_{i,j+1})(a_{i+1,j+1}-a_{i,j})+\\&+(b_{i+1,j}-b_{i,j+1})(a_{i+1,j+1}-a_{i,j})+\\&+(b_{i+1,j}-b_{i,j+1})(a_{i+1,j+1}-a_{i,j})]\,/\,\left[12h^2\right],\end{aligned}$$

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$$\{\vec{a},\vec{b}\} = -\left[(b_{i,j-1}+b_{i+1,j-1}-b_{i,j+1}-b_{i+1,j+1})\times\right.$$
$$\times(a_{i+1,j}+a_{i,j})+$$
$$+(b_{i-1,j-1}+b_{i,j-1}-b_{i-1,j+1}-b_{i,j+1})(a_{i,j}+a_{i-1,j})+$$
$$+(b_{i+1,j-1}+b_{i+1,j+1}-b_{i-1,j}-b_{i-1,j+1})(a_{i,j+1}+a_{i,j})+$$
$$+(b_{i+1,j-1}+b_{i+1,j}-b_{i-1,j-1}-b_{i-1,j})(a_{i,j}+a_{i,j-1})+$$
$$+(b_{i+1,j}-b_{i,j+1})(a_{i+1,j+1}+a_{i,j})+$$
$$+(b_{i+1,j}-b_{i,j+1})(a_{i+1,j+1}+a_{i,j})+$$
$$+(b_{i+1,j}-b_{i,j+1})(a_{i+1,j+1}+a_{i,j})+$$
$$\left.+(b_{i+1,j}-b_{i,j+1})(a_{i+1,j+1}+a_{i,j})\right]/\left[12h^2\right].$$

Superscript shows the number of the temporary layer, subscript i shows the x- and y- coordinates. The system of equations is modified for periodic boundary conditions at the edges of the grid.

Here is the system of equations $\Delta\psi-\psi=Z(x)$ on a rectangular grid $\bar{\omega}=\{x_{ij}=(ih_1,jh_2)\in G,\ 0\le i\le M,\ 0\le j\le N,\ l_1=Mh_1,\ l_2=Nh_2\}$, with boundary γ, introduced in rectangle $G=\{0\le x_\alpha\le l\alpha,\ \alpha=1,2\}$, that can be presented as the system of vector equations of a special form [10]. In the operator form on the grid, the system has a form:

$$\begin{cases}\psi_{\bar{x}_1x_2}+\psi_{x_1\bar{x}_2}-\psi=-Z(x),\ x\in\omega,\\ \psi(x)=g(x),\ x\in\gamma,\end{cases}\qquad(1)$$

where

$$\psi_{\bar{x}_1x_2}=\frac{1}{h_1^2}\left[\psi(i+1,j)-2\psi(i,j)+\psi(i-1,j)\right],$$

$$\psi_{x_1\bar{x}_2}=\frac{1}{h_1^2}\left[\psi(i,j+1)-2\psi(i,j)+\psi(i,j-1)\right],$$

$$\psi(x_{ij})=\psi(i,j).$$

Now we need to turn the scheme (1). To do this, we multiply (1) on $(-h_2^2)$ and write out the difference derivative of $\psi_{\bar{x}_1x_2}$ by points. Now, let $\vec{\Psi}_j$ be the vector of $M-1$ dimensions, components of which are the values of the function $\psi(i,j)$ on the internal nodes of the grid $\bar{\omega}$ on j-th line,

$$\vec{\Psi}_j=\{\psi(1,j),\psi(2,j),\ldots,\psi(M-1,j)\},\ 0\le j\le N,$$

and $\vec{F}_j$ is a vector with $M-1$ dimension:

$$\vec{F}_j=\{h_2^2\bar{\varphi}(1,j),h_2^2\varphi(2,j),\ldots,h_2^2\varphi(M-2,j),$$
$$h_2^2\bar{\varphi}(M-1,j)\},\quad 0\le j\le N,$$

$$\vec{F}_j=\{g(1,j),g(2,j),\ldots,g(M-1,j)\},\quad j=0,N.$$

Let us define the matrix $\hat{C}$ of $(M-1)\times(M-1)$ dimensions as follows:

$$\hat{C}\vec{V}=\{\Lambda v(1),\Lambda v(2),\ldots,\Lambda v(M-1)\}\vec{V}=$$
$$=\{v(1),v(2),\ldots,v(M-1)\},$$

where differential operator Λ is introduced in a form:

$$\Lambda v(i)=2v(i)-h_2^2v_{\bar{x}_1x_2}(i),\quad 1\le i\le M-1,$$

$$v(0)=v(M)=0.$$

Matrix $\hat{C}$ has the form:

$$\hat{C}=\begin{vmatrix}2(1+\alpha) & -\alpha & 0 & \cdots & 0\\ -\alpha & 2(1+\alpha) & -\alpha & \cdots & \\ 0 & -\alpha & 2(1+\alpha) & \cdots & 0\\ & \cdots & -\alpha & 2(1+\alpha) & -\alpha\\ 0 & 0 & \cdots & -\alpha & 2(1+\alpha)\end{vmatrix},$$

where $\alpha=\dfrac{h_2^2}{h_1^2}$. In the case of $1+\alpha>\alpha$, the matrix $\hat{C}$ is non-degenerate.

Now we can proceed to the system of vector equations of a special form with constant coefficients:

$$\begin{cases}-\vec{\Psi}_{j-1}+\hat{C}\vec{\Psi}_j-\vec{\Psi}_{j+1}=\vec{F}_j,\ 1\le j\le N-1\\ \vec{\Psi}_0=\vec{F}_0,\\ \vec{\Psi}_N=\vec{F}_N.\end{cases}$$

Regular methods for the solution of such a system are given in detail in [10]. Making use of the results for the particular case of equations with constant coefficients we find the solution in a rectangular grid. The idea of full reduction technique is to exclude the unknown $\vec{\Psi}_j$ with odd j from the equations, then with j, multiples of two, then four, etc. Each new step of the exclusion process reduces the number of unknown variables, and if N is the degree of two, i. e. $N=2^n$, then as a result the values of $\vec{\Psi}_{N/2}$ are obtained. The reverse run of algorithm provides calculation of the values $\vec{\Psi}_j$ with numbers j, multiple of $N/4$, then $N/8$, $N/16$, etc. As a result, we obtain the following equations for calculating $\vec{\Psi}_j$ through the vectors $\vec{p}$ and $\vec{S}$:

$$S_j^{(k-1)}=\prod_{l=1}^{2^{k-1}}\alpha_{l,k-1}C_{l,k-1}^{-1}\left(\vec{p}_{j-2^{k-1}}^{(k-1)}+\vec{p}_{j+2^{k-1}}^{(k-1)}\right),$$

$$\vec{p}_j^{(k-1)}=0,5\left(\vec{p}_j^{(k-1)}+\vec{S}_j^{(k-1)}\right),\quad \vec{p}_j^{(0)}\equiv\vec{F}_j,$$

$$\left(j=2^k,2\cdot2^k,3\cdot2^k\ldots N-2^k,\ k=1,2,\ldots,n-1\right);$$

$$\vec{\Psi}_j=\sum_{l=1}^{2^{k-1}}C_{l,k-1}^{-1}\left[\vec{p}_j^{(k-1)}+\right.$$
$$\left.+\alpha_{l,k-1}\left(\vec{\Psi}_{j-2^{k-1}}^{(k-1)}+\vec{\Psi}_{j+2^{k-1}}^{(k-1)}\right)\right],$$

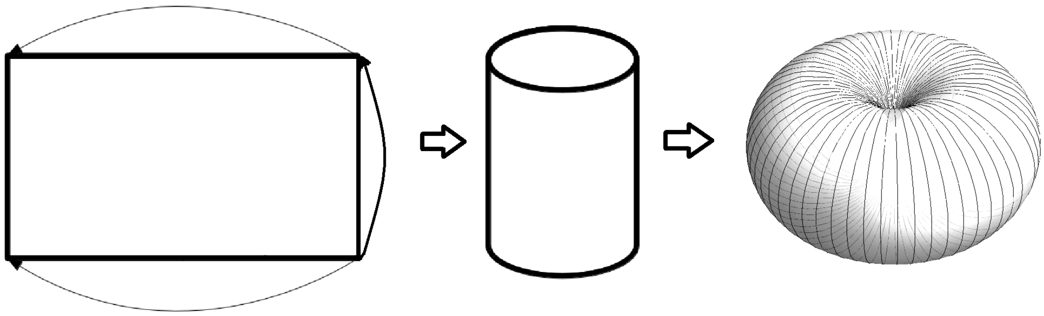

Fig. 1: The scheme of the transformation of a rectangular grid into the torus with periodic boundary conditions.

$$\vec{\Psi}_0 = \vec{F}_0, \quad \vec{\Psi}_N = \vec{F}_N,$$

$$\left(j = 2^k, 3 \cdot 2^k, 5 \cdot 2^k..N - 2^k,\ k = n, n-1, \dots, 1\right),$$

where

$$C^{-1}_{l,k-1} = C - 2\cos\left[\frac{(2l-1)\pi}{2^k}\right] E\,\alpha_{l,k-1} = \\ = \frac{(-1)^{l+1}}{2^{k-1}} \sin\left[\frac{(2l-1)\pi}{2^k}\right].$$

The obtained formulae describe the application of the full reduction technique with just vector addition, vector multiplication by number and inversion of matrices. If $\hat{C}$ is a three-diagonal matrix, then any $C_{l,k-1}$ will be also three-diagonal as well.

NUMERICAL SIMULATIONS

Applying the algorithm of full reduction we developed the modification for periodic boundary conditions. Algorithms of numerical integration have been implemented in the `IDL` programming language on the grid 4096×1000 with a step of 0.1 in time and in space.

The type of solution depends on the ratio of the linear and non-linear terms of equation. In a linear case the initial vortex disturbance solution is unstable and rapidly decays into linear waves. In a non-linear system the vortex solution is stable and weekly attenuates during the drift (Fig. 2).

CONCLUSIONS

We present the modification of the full reduction technique (periodic boundary conditions and additional terms in equation) and its application for study the dynamics of the system with vector non-linearity. The algorithm adapted for a rectangular grid of integration with periodic boundary conditions was developed and implemented. The dynamics of the initial perturbation in a system with vector non-linearity using the methods of numerical integration was studied. The integration was carried out on a grid with a constant step. Acceleration allows to provide the numerical calculations on the grid size 1000×1000 (2-3 hours).

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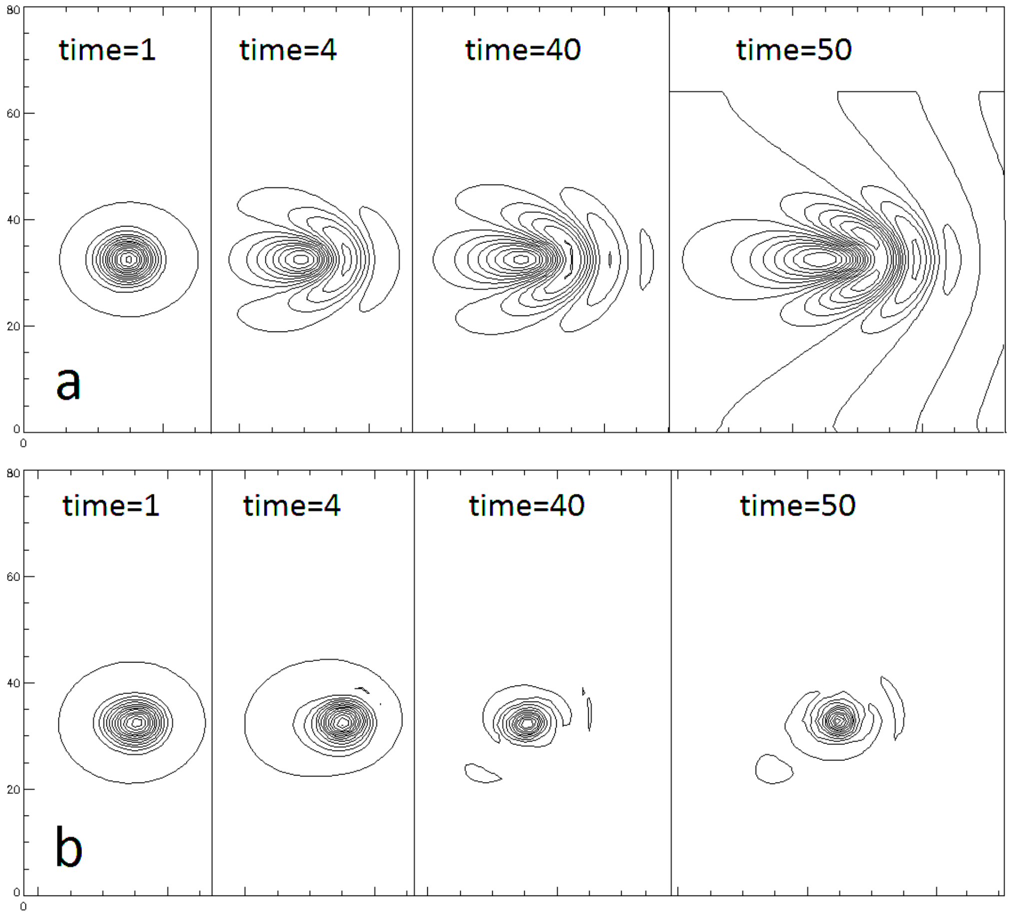

Fig. 2: Dynamics of the initial perturbation in a form of the monopole vortex with the Gaussian potential. a) Dynamics of the linear system (the initial structure decays to linear waves). b) Dynamics of the nonlinear system (nonlinear vortex solution is stable).