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Properties of Modified Riemannian Extensions

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Let M be an n-dimensional differentiable manifold with a symmetric connection ∇ and T^*M be its cotangent bundle. In this paper, we study some properties of the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ on T^*M defined by means of a symmetric $(0,2)$-tensor field c on M. We get the conditions under which T^*M endowed with the horizontal lift ${}^HJ$ of an almost complex structure J and with the metric $\widetilde{g}_{\nabla,c}$ is a Kähler–Norden manifold. Also curvature properties of the Levi–Civita connection of the metric $\widetilde{g}_{\nabla,c}$ are presented.

Key words: cotangent bundle, Kähler–Norden manifold, modified Riemannian extension, Riemannian curvature tensors, semi-symmetric manifold.

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1. Introduction

Let M be an n-dimensional differentiable manifold and T^*M be its cotangent bundle. There is a well-known natural construction which yields, for any affine connection ∇ on M, a pseudo-Riemannian metric $\widetilde{g}_\nabla$ on T^*M. The metric $\widetilde{g}_\nabla$ is called the Riemannian extension of ∇. Riemannian extensions were originally defined by Patterson and Walker [15] and further studied by Afifi [2], thus relating pseudo-Riemannian properties of T^*M with the affine structure of the base manifold (M,∇). Moreover, Riemannian extensions were also considered by Garcia-Rio et al. in [8] in relation to Osserman manifolds (see also Derdzinski [5]). Since Riemannian extensions provide a link between affine and pseudo-Riemannian geometries, some properties of the affine connection ∇ can be investigated by means of the corresponding properties of the Riemannian extension $\widetilde{g}_\nabla$. For instance, ∇ is projectively flat if and only if $\widetilde{g}_\nabla$ is locally conformally flat [2]. For Riemannian extensions, also see [1, 7, 9, 11, 12, 17, 19, 21, 22]. In [3, 4], the authors introduced a modification of the usual Riemannian extensions which is called the modified Riemannian extension.

Let M_{2k} be a $2k$-dimensional differentiable manifold endowed with an almost complex structure J and a pseudo-Riemannian metric g of signature (k,k) such that $g(JX,Y) = g(X,JY)$ for arbitrary vector fields X and Y on M_{2k}. Then the metric g is called the Norden metric. Norden metrics are referred to as anti-Hermitian metrics or B-metrics. The study of such manifolds is interesting because there exists a difference between the geometry of a $2k$-dimensional almost complex manifold with Hermitian metric and the geometry of a $2k$-dimensional almost complex manifold with Norden metric. A notable difference between Norden metrics and Hermitian metrics is that $G(X,Y) = g(X,JY)$ is another Norden metric, rather than a differential 2-form. Some authors considered almost complex Norden structures on the cotangent bundle [6, 13, 14].

In this paper, we will use a deformation of the Riemannian extension on the cotangent bundle T^*M over (M,∇) by means of a symmetric tensor field c on M, where ∇ is a symmetric affine connection on M. The metric is the so-called modified Riemannian extenson. In Section 3, in the particular case where ∇ is the Levi–Civita connection on a Riemannian manifold (M,g), we get the conditions under which the triple $(T^*M, {}^HJ, \widetilde{g}_{\nabla,c})$ is a Kähler–Norden manifold, where ${}^HJ$ is the horizontal lift of an almost complex structure J and $\widetilde{g}_{\nabla,c}$ is the modified Riemannian extension. Section 4 deals with curvature properties of the Levi–Civita connection of the modified Riemannian extension $\widetilde{g}_{\nabla,c}$.

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞. Also, we denote by $\Im^p_q(M)$ the set of all tensor fields of type (p,q) on M, and by $\Im^p_q(T^*M)$ the corresponding set on the cotangent bundle T^*M. The Einstein summation convention is used, the range of the indices i,j,s being always $\{1,2,\ldots,n\}$.

2. Preliminaries

2.1. The cotangent bundle

Let M be an n-dimensional smooth manifold and denote by $\pi : T^*M \to M$ its cotangent bundle whose fibres are cotangent spaces to M. Then T^*M is a $2n$-dimensional smooth manifold and some local charts induced naturally from local charts on M can be used. Namely, a system of local coordinates $\left(U, x^i\right)$, $i = 1,\ldots,n$ in M induces on T^*M a system of local coordinates $\left(\pi^{-1}(U),\ x^i,\ x^{\bar{i}} = p_i\right)$, $\bar{i} = n+i = n+1,\ldots,2n$, where $x^{\bar{i}} = p_i$ are the components of covectors p in each cotangent space $T_x^*M,\ x \in U$ with respect to the natural coframe $\{dx^i\}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions in U of a vector field X and a covector (1-form) field ω on M, respectively. Then the vertical lift ${}^V\omega$ of ω, the horizontal lift ${}^HX$ and the complete lift ${}^CX$ of X are given, with respect to the induced coordinates, by

$$ {}^V\omega = \omega_i \partial_{\bar{i}}, \tag{2.1} $$

$$ {}^HX = X^i \partial_i + p_h \Gamma^h_{ij} X^j \partial_{\bar{i}} \tag{2.2} $$

and

$$ {}^CX = X^i \partial_i - p_h \partial_i X^h \partial_{\bar{i}}, $$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial x^{\bar{i}}}$ and Γ^h_{ij} are the coefficients of a symmetric (torsion-free) affine connection ∇ in M.

The Lie bracket operation of vertical and horizontal vector fields on T^*M is given by the formulas

$$ \begin{cases} \left[{}^HX, {}^HY\right] = {}^H[X,Y] + {}^V(p \circ R(X,Y)) \\ \left[{}^HX, {}^V\omega\right] = {}^V(\nabla_X \omega) \\ \left[{}^V\theta, {}^V\omega\right] = 0 \end{cases} \tag{2.3} $$

for any X, $Y \in \Im^1_0(M)$ and θ, $\omega \in \Im^0_1(M)$, where R is the curvature tensor of the symmetric connection ∇ defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ (for details, see [24]).

2.2. Expressions in the adapted frame

We insert the adapted frame which allows the tensor calculus to be efficiently done in T^*M. With the symmetric affine connection ∇ in M, we can introduce the adapted frames on each induced coordinate neighborhood $\pi^{-1}(U)$ of T^*M. In each local chart $U \subset M$, we write $X_{(j)} = \dfrac{\partial}{\partial x^j}$, $\theta^{(j)} = dx^j$, $j = 1, \ldots, n$. Then from (2.1) and (2.2), we can see that these vector fields have, respectively, the local expressions

$$ {}^HX_{(j)} = \partial_j + p_a \Gamma^a_{hj} \partial_{\bar{h}}, $$

$$ {}^V\theta^{(j)} = \partial_{\bar{j}} $$

with respect to the natural frame $\left\{\partial_j, \partial_{\bar{j}}\right\}$. These $2n$-vector fields are linearly independent and they generate the horizontal distribution of ∇ and the vertical distribution of T^*M, respectively. The set $\left\{{}^H X_{(j)}, {}^V \theta^{(j)}\right\}$ is called the frame adapted to the connection ∇ in $\pi^{-1}(U) \subset T^*M$. By putting

$$\begin{array}{rcl} E_j & = & {}^H X_{(j)}, \\ E_{\bar{j}} & = & {}^V \theta^{(j)}, \end{array} \tag{2.4}$$

we can write the adapted frame as $\{E_\alpha\} = \left\{E_j, E_{\bar{j}}\right\}$. The indices $\alpha, \beta, \gamma, \ldots = 1, \ldots, 2n$ indicate the indices with respect to the adapted frame.

Using (2.1), (2.2) and (2.4), we have

$${}^V \omega = \left(\begin{array}{c} 0 \\ \omega_j \end{array} \right) \tag{2.5}$$

and

$${}^H X = \left(\begin{array}{c} X^j \\ 0 \end{array} \right) \tag{2.6}$$

with respect to the adapted frame $\{E_\alpha\}$ (for details, see [24]). By the straightforward calculations, we have the lemma below.

Lemma 1. *The Lie brackets of the adapted frame of T^*M satisfy the following identities:*

$$\begin{array}{rcl} [E_i, E_j] & = & p_s R_{ijl}{}^s E_{\bar{l}}, \\ \left[E_i, E_{\bar{j}}\right] & = & -\Gamma^j_{il} E_{\bar{l}}, \\ \left[E_{\bar{i}}, E_{\bar{j}}\right] & = & 0, \end{array}$$

where $R_{ijl}{}^s$ denote the components of the curvature tensor of the symmetric connection ∇ on M.

3. Kähler–Norden Structures on the Cotangent Bundle

We first give the definition of pure tensor fields with respect to a $(1,1)$-tensor field J.

Definition 1. *For a $(1,1)$-tensor field J, the $(0,s)$-tensor field t is called pure with respect to J if*

$$t(JX_1, X_2, \ldots, X_s) = t(X_1, JX_2, \ldots, X_s) = \ldots = t(X_1, X_2, \ldots, JX_s)$$

for any $X_1, X_2, \ldots, X_s \in \Im^1_0(M)$. For more information about the pure tensor, see [16, 20, 23].

An almost complex Norden manifold (M, J, g) is a real $2k$-dimensional differentiable manifold M with an almost complex structure J and a pseudo-Riemannian metric g of neutral signature (k, k) such that

$$g(JX, Y) = g(X, JY)$$

for all $X, Y \in \Im_0^1(M)$, i.e., g is pure with respect to J. A Kähler–Norden (anti-Kähler) manifold can be defined as a triple (M, J, g) which consists of a smooth manifold M endowed with an almost complex structure J and a Norden metric g such that $\nabla J = 0$, where ∇ is the Levi–Civita connection of g. It is well known that the condition $\nabla J = 0$ is equivalent to the C-holomorphicity (analyticity) of the Norden metric g [10], i.e., $\Phi_J g = 0$, where Φ_J is the Tachibana operator [16, 20, 23]: $(\Phi_J g)(X, Y, Z) = (JX)(g(Y, Z)) - X(g(JY, Z)) + g((L_Y J)X, Z) + g(Y, (L_Z\ J)X)$ for all $X, Y, Z \in \Im_0^1(M)$. Also note that $G(Y, Z) = g(JY, Z)$ is the twin Norden metric. Since in dimension 2 a Kähler–Norden manifold is flat, we assume in the sequel that $n = \dim M \geq 4$.

Next, for a given symmetric connection ∇ on an n-dimensional manifold M, the cotangent bundle T^*M can be equipped with a pseudo-Riemannian metric $\widetilde{g}_\nabla$ of signature (n, n): the Riemannian extension of ∇ [15], given by

$$\widetilde{g}_\nabla(^C X, ^C Y) = -\gamma(\nabla_X Y + \nabla_Y X),$$

where $^C X, ^C Y$ denote the complete lifts to T^*M of vector fields X, Y on M. Moreover, for any $Z \in \Im_0^1(M)$, $Z = Z^i \partial_i$, γZ is the function on T^*M defined by $\gamma Z = p_i Z^i$. The Riemannian extension is expressed by

$$\widetilde{g}_\nabla = \begin{pmatrix} -2p_h \Gamma_{ij}^h & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix}$$

with respect to the natural frame.

Now we give a deformation of the Riemannian extension above by means of a symmetric $(0, 2)$-tensor field c on M whose metric is called the modified Riemannian extension. The modified Riemannian extension is expressed by

$$\widetilde{g}_{\nabla,c} = g_\nabla + \pi^* c = \begin{pmatrix} -2p_h \Gamma_{ij}^h + c_{ij} & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix} \tag{3.1}$$

with respect to the natural frame. It follows that the signature of $\widetilde{g}_{\nabla,c}$ is (n, n).

Denote by ∇ the Levi–Civita connection of a semi-Riemannian metric g. In this section, we will consider T^*M equipped with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ over a pseudo-Riemannian manifold (M, g). Since the vector fields $^H X$ and $^V \omega$ span the module of vector fields on T^*M, any tensor field is determined on T^*M by their actions on ${}^HX$ and ${}^V\omega$. The modified Riemannian extension $\widetilde{g}_{\nabla,c}$ has the following properties:

$$\begin{aligned} \widetilde{g}_{\nabla,c}({}^HX,{}^HY) &= c(X,Y), \\ \widetilde{g}_{\nabla,c}({}^HX,{}^V\omega) &= g_{\nabla,c}({}^V\omega,{}^HX) = \omega(X), \\ \widetilde{g}_{\nabla,c}({}^V\omega,{}^V\theta) &= 0 \end{aligned} \tag{3.2}$$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, which characterize $\widetilde{g}_{\nabla,c}$.

The horizontal lift of a $(1,1)$-tensor field J to T^*M is defined by

$$\begin{aligned} {}^HJ({}^HX) &= {}^H(JX), \\ {}^HJ({}^V\omega) &= {}^V(\omega \circ J) \end{aligned} \tag{3.3}$$

for any $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$. Moreover, it is well known that ıf J is an almost complex structure on (M,g), then its horizontal lift ${}^HJ$ is an almost complex structure on T^*M [24]. Now we prove the following theorem.

Theorem 1. *Let (M,J,g) be a Kähler–Norden manifold. Then T^*M is a Kähler-Norden manifold equipped with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ and the almost complex structure ${}^HJ$ if and only if the symmetric $(0,2)$-tensor field c on M is a holomorphic tensor field with respect to the almost complex structure J.*

P r o o f. Let (M,J,g) be a Kähler–Norden manifold. Put

$$A\left(\widetilde{X},\widetilde{Y}\right) = \widetilde{g}_{\nabla,c}\left({}^HJ\widetilde{X},\widetilde{Y}\right) - \widetilde{g}_{\nabla,c}\left(\widetilde{X},{}^HJ\widetilde{Y}\right)$$

for any $\widetilde{X},\widetilde{Y} \in \Im_0^1(T^*M)$. For all vector fields $\widetilde{X}$ and $\widetilde{Y}$, which are of the form ${}^V\omega$, ${}^V\theta$ or ${}^HX$, ${}^HY$, from (3.2) and (3.3), we have

$$\begin{aligned} A\left({}^HX,{}^HY\right) &= \widetilde{g}_{\nabla,c}\left({}^HJ({}^HX),{}^HY\right) - \widetilde{g}_{\nabla,c}\left({}^HX,{}^HJ({}^HY)\right) \\ &= \widetilde{g}_{\nabla,c}\left({}^H(JX),{}^HY\right) - \widetilde{g}_{\nabla,c}\left({}^HX,{}^H(JY)\right) \\ &= c(JX,Y) - c(X,JY), \\ A\left({}^HX,{}^V\theta\right) &= \widetilde{g}_{\nabla,c}\left({}^HJ({}^HX),{}^V\theta\right) - \widetilde{g}_{\nabla,c}\left({}^HX,{}^HJ({}^V\theta)\right) \\ &= \widetilde{g}_{\nabla,c}\left({}^H(JX),{}^V\theta\right) - \widetilde{g}_{\nabla,c}\left({}^HX,{}^V(\theta\circ J)\right) \\ &= \theta(JX) - (\theta\circ J)(X), \\ A\left({}^V\omega,{}^HY\right) &= \widetilde{g}_{\nabla,c}\left({}^HJ({}^V\omega),{}^HY\right) - \widetilde{g}_{\nabla,c}\left({}^V\omega,{}^HJ({}^HY)\right) \\ &= (\omega\circ J)(Y) - \omega(JY), \\ A\left({}^V\omega,{}^V\theta\right) &= \widetilde{g}_{\nabla,c}\left({}^HJ({}^V\omega),{}^V\theta\right) - \widetilde{g}_{\nabla,c}\left({}^V\omega,{}^HJ({}^V\theta)\right) \\ &= \widetilde{g}_{\nabla,c}\left({}^HJ({}^V\omega),{}^V\theta\right) - \widetilde{g}_{\nabla,c}\left({}^V\omega,{}^HJ({}^V\theta)\right) \\ &= \widetilde{g}_{\nabla,c}\left({}^V(\omega\circ J),{}^V\theta\right) - \widetilde{g}_{\nabla,c}\left({}^V\omega,{}^V(\theta\circ J)\right) \\ &= 0. \end{aligned}$$

From the last equations, if the symmetric $(0,2)-$tensor field c is pure with respect to J, we say that $A\left(\tilde{X},\tilde{Y}\right)=0$, i.e., the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ is pure with respect to ${}^{H}J$.

Now we are interested in the holomorphy property of the modified Riemannian extension $g_{\nabla,c}$ with respect to ${}^{H}J$. We calculate

$$\begin{aligned}(\Phi_{{}^{H}J}\widetilde{g}_{\nabla,c})(\tilde{X},\tilde{Y},\tilde{Z}) &= ({}^{H}J\tilde{X})(\widetilde{g}_{\nabla,c}(\tilde{Y},\tilde{Z}))-\tilde{X}(\widetilde{g}_{\nabla,c}({}^{H}J\tilde{Y},\tilde{Z}))\\ &+ \widetilde{g}_{\nabla,c}((L_{\tilde{Y}}\,{}^{H}J)\tilde{X},\tilde{Z})+\widetilde{g}_{\nabla,c}(\tilde{Y},(L_{\tilde{Z}}\,{}^{H}J)\tilde{X})\end{aligned}$$

for all $\tilde{X},\tilde{Y},\tilde{Z}\in\Im_0^1(T^*M)$. Then we obtain the following equations:

$$\begin{aligned}(\Phi_{{}^{H}J}\widetilde{g}_{\nabla,c})({}^{V}\omega,{}^{V}\theta,{}^{H}Z) &= 0,\\ (\Phi_{{}^{H}J}\widetilde{g}_{\nabla,c})({}^{V}\omega,{}^{V}\theta,{}^{V}\sigma) &= 0,\\ (\Phi_{{}^{H}J}\widetilde{g}_{\nabla,c})({}^{V}\omega,{}^{H}Y,{}^{V}\sigma) &= 0,\\ (\Phi_{{}^{H}J}\widetilde{g}_{\nabla,c})({}^{V}\omega,{}^{H}Y,{}^{H}Z) &= (\omega\circ\nabla_Y J)(Z)+(\omega\circ\nabla_Z J)(Y),\\ (\Phi_{{}^{H}J}\widetilde{g}_{\nabla,c})({}^{H}X,{}^{V}\omega,{}^{H}Z) &= (\Phi_J g)(X,\widetilde{\omega},Z)-g((\nabla_{\widetilde{\omega}}J)X,Z),\\ (\Phi_{{}^{H}J}\widetilde{g}_{\nabla,c})({}^{H}X,{}^{V}\omega,{}^{V}\sigma) &= 0,\\ (\Phi_{{}^{H}J}\widetilde{g}_{\nabla,c})({}^{H}X,{}^{H}Y,{}^{H}Z) &= (\Phi_J c)(X,Y,Z))\\ &\quad+(p\circ R(Y,JX)-p\circ R(Y,X)J)(Z)\\ &\quad+(p\circ R(Z,JX)-p\circ R(Z,X)J)(Y),\\ (\Phi_{{}^{H}J}\widetilde{g}_{\nabla,c})({}^{H}X,{}^{H}Y,{}^{V}\sigma) &= (\Phi_J g)(X,Y,\widetilde{\sigma})-g(Y,(\nabla_{\widetilde{\sigma}}J)X),\end{aligned}$$

where $\widetilde{\omega}=g^{-1}\circ\omega=g^{ij}\omega_j$ is the associated vector field of ω. On the other hand, the Riemannian curvature R of Kähler–Norden manifolds is pure [10], that is,

$$R(JX,Y)=R(X,JY)=R(X,Y)J=JR(X,Y).$$

Hence, from the equations above, it follows that $\Phi_{{}^{H}J}\widetilde{g}_{\nabla,c}=0$ if and only if $\Phi_J c=0$, which completes the proof. ■

4. Curvature Properties of the Levi–Civita Connection of the Modified Riemannian Extension $\widetilde{g}_{\nabla,c}$

In this section, we give the conditions under which the cotangent bundle T^*M equipped with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ is respectively locally flat, locally symmetric, conformally flat, projectively flat, semi-symmetric and Ricci semi-symmetric.

Let us consider T^*M equipped with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ for a given symmetric connection ∇ on M. By virtue of (2.5) and (2.6), the modified Riemannian extension $(\widetilde{g}_{\nabla,c})_{\beta\gamma}$ and its inverse $(\overline{g}_{\nabla,c})^{\beta\gamma}$ have the following components with respect to the adapted frame $\{E_\alpha\}$:

$$(\widetilde{g}_{\nabla,c})_{\beta\gamma} = \begin{pmatrix} c_{ij} & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix}, \tag{4.1}$$

$$(\overline{g}_{\nabla,c})^{\beta\gamma} = \begin{pmatrix} 0 & \delta_j^i \\ \delta_i^j & -c_{ij} \end{pmatrix}. \tag{4.2}$$

Theorem 2. *Let ∇ be a symmetric connection on M and T^*M be the cotangent bundle with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ over (M,∇). Then*

i) $(T^*M, \widetilde{g}_{\nabla,c})$ is locally flat if and only if (M,∇) is locally flat and the components c_{ij} of c satisfy the condition

$$\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) = 0; \tag{4.3}$$

ii) $(T^*M, \widetilde{g}_{\nabla,c})$ is locally symmetric if and only if (M,∇) is locally symmetric and the components c_{ij} of c satisfy the condition

$$\begin{aligned} &\nabla_l\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_l\nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) \\ &-R_{ijk}{}^m(\nabla_l c_{mh}) - R_{ijh}{}^m(\nabla_l c_{km}) = 0. \end{aligned} \tag{4.4}$$

P r o o f. The Levi–Civita connection $\widetilde{\nabla}$ of $\widetilde{g}_{\nabla,c}$ is characterized by the Koszul formula

$$\begin{aligned} 2\widetilde{g}_{\nabla,c}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},\widetilde{Z}) &= \widetilde{X}(\widetilde{g}_{\nabla,c}(\widetilde{Y},\widetilde{Z})) + \widetilde{Y}(\widetilde{g}_{\nabla,c}(\widetilde{Z},\widetilde{X})) - \widetilde{Z}(\widetilde{g}_{\nabla,c}(\widetilde{X},\widetilde{Y})) \\ -\widetilde{g}_{\nabla,c}(\widetilde{X},[\widetilde{Y},\widetilde{Z}]) &+ \widetilde{g}_{\nabla,c}(\widetilde{Y},[\widetilde{Z},\widetilde{X}]) + \widetilde{g}_{\nabla,c}(\widetilde{Z},[\widetilde{X},\widetilde{Y}]) \end{aligned}$$

for all $\widetilde{X},\widetilde{Y}$ and $\widetilde{Z} \in \Im_0^1(T^*M)$. Using (4.1), (4.2) and Lemma 1, the following formulas can be checked by a straightforward computation:

$$\begin{aligned} \widetilde{\nabla}_{E_{\overline{i}}}E_{\overline{j}} &= 0,\ \widetilde{\nabla}_{E_{\overline{i}}}E_j = 0, \\ \widetilde{\nabla}_{E_i}E_{\overline{j}} &= -\Gamma_{ih}^j E_{\overline{h}}, \\ \widetilde{\nabla}_{E_i}E_j &= \Gamma_{ij}^h E_h + \{p_s R_{hji}{}^s + \frac{1}{2}(\nabla_i c_{jh} + \nabla_j c_{ih} - \nabla_h c_{ij})\}E_{\overline{h}}, \end{aligned} \tag{4.5}$$

where $R_{hji}{}^s$ are the components of the curvature tensor field R of the symmetric connection ∇ on M.

The Riemannian curvature tensor $\widetilde{R}$ of T^*M with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ is obtained from the well-known formula

$$\widetilde{R}\left(\widetilde{X},\widetilde{Y}\right)\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im_0^1(T^*M)$. Then from Lemma 1 and (4.5), after standard computations, the Riemannian curvature tensor $\widetilde{R}$ is obtained as follows:

$$\begin{aligned}
\widetilde{R}(E_i, E_j)E_k &= R_{ijk}{}^h E_h \\
&\quad +\{p_s(\nabla_i R_{hkj}{}^s - \nabla_j R_{hki}{}^s) \\
&\quad +\frac{1}{2}\{\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}), \\
&\quad -R_{ijk}{}^m c_{mh} - R_{ijh}{}^m c_{km}\}\}E_{\overline{h}}, \\
\widetilde{R}(E_i, E_j)E_{\overline{k}} &= R_{jih}{}^k E_{\overline{h}}, \\
\widetilde{R}(E_i, E_{\overline{j}})E_k &= -R_{hki}{}^j E_{\overline{h}}, \\
\widetilde{R}(E_{\overline{i}}, E_j)E_k &= R_{hkj}{}^i E_{\overline{h}}, \\
\widetilde{R}(E_{\overline{i}}, E_j)E_{\overline{k}} &= 0, \ \widetilde{R}(E_i, E_{\overline{j}})E_{\overline{k}} = 0, \\
\widetilde{R}(E_{\overline{i}}, E_{\overline{j}})E_k &= 0, \ \widetilde{R}(E_{\overline{i}}, E_{\overline{j}})E_{\overline{k}} = 0
\end{aligned} \tag{4.6}$$

with respect to the adapted frame $\{E_\alpha\}$.

i) We now assume that $R = 0$ and equation (4.3) holds, then from the equations in (4.6) it follows that $\widetilde{R} = 0$. Conversely, under the assumption that $\widetilde{R} = 0$, we evaluate the first equation in (4.6) at an arbitrary point $(x^i, p_i) = (x^i, 0)$ in the zero section of T^*M and we have

$$\begin{aligned}
0 &= [\widetilde{R}(E_i, E_j)E_k]_{(x^i,0)} = R_{ijk}{}^h E_h + \{\frac{1}{2}\{\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) \\
&\quad -R_{ijk}{}^m c_{mh} - R_{ijh}{}^m c_{km}\}\}E_{\overline{h}}
\end{aligned}$$

from which we get $R = 0$ and $\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) = 0$.

ii) We consider the components of $\widetilde{\nabla}\widetilde{R}$. Using (4.5) and (4.6), by a direct computation, we obtain the following relations:

$$\begin{aligned}
\widetilde{\nabla}_l \widetilde{R}_{ijk}{}^h &= \nabla_l R_{ijk}{}^h, \\
\widetilde{\nabla}_l \widetilde{R}_{ijk}{}^{\overline{h}} &= p_s(\nabla_l \nabla_i R_{hkj}{}^s - \nabla_l \nabla_j R_{hki}{}^s) + \frac{1}{2}\{\nabla_l \nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) \\
&\quad -\nabla_l \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) - (\nabla_l R_{ijk}{}^m)c_{mh} - R_{ijk}{}^m(\nabla_l c_{mh}) \\
&\quad -(\nabla_l R_{ijh}{}^m)c_{km} - R_{ijh}{}^m(\nabla_l c_{km})\}, \\
\widetilde{\nabla}_l \widetilde{R}_{ij\overline{k}}{}^{\overline{h}} &= \nabla_l R_{jih}{}^k, \\
\widetilde{\nabla}_l \widetilde{R}_{i\overline{j}k}{}^{\overline{h}} &= -\nabla_l R_{hki}{}^j, \\
\widetilde{\nabla}_l \widetilde{R}_{\overline{i}jk}{}^{\overline{h}} &= \nabla_l R_{hkj}{}^i, \\
\widetilde{\nabla}_{\overline{l}} \widetilde{R}_{ijk}{}^{\overline{h}} &= \nabla_i R_{hkj}{}^l - \nabla_j R_{hki}{}^l,
\end{aligned}$$

all the others being zero with respect to the adapted frame $\{E_\alpha\}$. With the same method as i), the proof follows from the above equations. ■

We turn our attention to the Ricci tensor of the modified Riemannian extension $\widetilde{g}_{\nabla,c}$. Let $\widetilde{R}_{\alpha\beta} = \widetilde{R}_{\sigma\alpha\beta}{}^{\sigma}$ denote the Ricci tensor of the modified Riemannian extension $\widetilde{g}_{\nabla,c}$. From (4.6), the components of the Ricci tensor $R_{\alpha\beta}$ are characterized by

$$\begin{aligned} \widetilde{R}_{jk} &= R_{jk} + R_{kj} \\ \widetilde{R}_{\bar{j}k} &= 0, \\ \widetilde{R}_{j\bar{k}} &= 0, \\ \widetilde{R}_{\bar{j}\bar{k}} &= 0, \end{aligned} \tag{4.7}$$

with respect to the adapted frame $\{E_\alpha\}$.

Theorem 3. *Let ∇ be a symmetric connection on M and T^*M be the cotangent bundle with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ over (M, ∇). Then $(T^*M, \widetilde{g}_{\nabla,c})$ is Ricci flat if and only if the Ricci tensor of ∇ is skew symmetric (for the Riemannian extension, see* [15]*).*

P r o o f. The proof follows from (4.7). ■

Theorem 4. *Let ∇ be a symmetric connection on M and T^*M be the cotangent bundle with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ over (M, ∇), then $(T^*M, \widetilde{g}_{\nabla,c})$ is a space of constant scalar curvature 0.*

P r o o f. The scalar curvature of the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ is defined by $\widetilde{r} = (\widetilde{g}_{\nabla,c})^{\alpha\beta}\, \widetilde{R}_{\alpha\beta}$. Using (4.2) and (4.7), we get

$$\widetilde{r} = (\widetilde{g}_{\nabla,c})^{ij}\widetilde{R}_{ij} + (\widetilde{g}_{\nabla,c})^{\bar{i}j}\widetilde{R}_{\bar{i}j} + (\widetilde{g}_{\nabla,c})^{i\bar{j}}\widetilde{R}_{i\bar{j}} + (\widetilde{g}_{\nabla,c})^{\bar{i}\bar{j}}\widetilde{R}_{\bar{i}\bar{j}} = 0. \quad ■$$

R e m a r k 1. Let ∇ be a symmetric connection on M and T^*M be the cotangent bundle with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ over (M, ∇). The cotangent bundle T^*M with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ is locally conformally flat if and only if its Weyl tensor $\widetilde{W}$ vanishes, where the Weyl tensor is given by

$$\begin{aligned} \widetilde{W}_{\alpha\beta\gamma\sigma} &= \widetilde{R}_{\alpha\beta\gamma\sigma} + \frac{\widetilde{r}}{2(2n-1)(n-1)}\{(\widetilde{g}_{\nabla,c})_{\alpha\gamma}(\widetilde{g}_{\nabla,c})_{\beta\sigma} - (\widetilde{g}_{\nabla,c})_{\alpha\sigma}(\widetilde{g}_{\nabla,c})_{\beta\gamma}\} \\ &\quad - \frac{1}{2(n-1)}((\widetilde{g}_{\nabla,c})_{\beta\sigma}\widetilde{R}_{\alpha\gamma} - (\widetilde{g}_{\nabla,c})_{\alpha\sigma}\widetilde{R}_{\beta\gamma} + (\widetilde{g}_{\nabla,c})_{\alpha\gamma}\widetilde{R}_{\beta\sigma} - (\widetilde{g}_{\nabla,c})_{\beta\gamma}\widetilde{R}_{\alpha\sigma}) \end{aligned}$$

and $\widetilde{R}_{\alpha\beta\gamma\sigma} = \widetilde{R}_{\alpha\beta\gamma}{}^{\lambda}(\widetilde{g}_{\nabla,c})_{\lambda\sigma}$. In [2], it is proved that $(T^*M, \widetilde{g}_{\nabla,c})$ is locally conformally flat if and only if (M, ∇) is projectively flat and the components c_{ij} of c satisfy the condition

$$\nabla_i(\nabla_k c_{jn} - \nabla_n c_{jk}) - \nabla_j(\nabla_k c_{in} - \nabla_n c_{ik}) - R_{ijk}{}^{h} c_{hn} - R_{ijn}{}^{h} c_{kh} = 0. \tag{4.8}$$

Theorem 5. *Let ∇ be a symmetric connection on M and T^*M be the cotangent bundle with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ over (M, ∇). Then $(T^*M, \widetilde{g}_{\nabla,c})$ is projectively flat if and only if (M, ∇) is flat and the components c_{ij} of c satisfy the condition*

$$\nabla_i(\nabla_k c_{jn} - \nabla_n c_{jk}) - \nabla_j(\nabla_k c_{in} - \nabla_n c_{ik}) = 0. \tag{4.9}$$

P r o o f. A manifold is said to be projectively flat if the projective curvature tensor vanishes. The projective curvature tensor is defined by

$$\widetilde{P}_{\alpha\beta\gamma\sigma} = \widetilde{R}_{\alpha\beta\gamma\sigma} - \frac{1}{(2n-1)}((\widetilde{g}_{\nabla,c})_{\alpha\sigma}\widetilde{R}_{\beta\gamma} - (\widetilde{g}_{\nabla,c})_{\beta\sigma}\widetilde{R}_{\alpha\gamma}),$$

where $\widetilde{R}_{\alpha\beta\gamma\sigma} = \widetilde{R}_{\alpha\beta\gamma}{}^{\lambda}(\widetilde{g}_{\nabla,c})_{\lambda\sigma}$.

The non-zero components of projective curvature tensor of the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ are given by

$$\begin{aligned}
\widetilde{P}_{ijkn} \;=\;& R_{ijk}{}^{h} c_{hn} + p_s(\nabla_i R_{nkj}{}^{s} - \nabla_j R_{nki}{}^{s}) \\
&+\frac{1}{2}\{\nabla_i(\nabla_k c_{jn} - \nabla_n c_{jk}) - \nabla_j(\nabla_k c_{in} - \nabla_n c_{ik}) - R_{ijk}{}^{h} c_{hn} - R_{ijn}{}^{h} c_{kh}\} \\
&-\frac{1}{(2n-1)}(c_{in}(R_{jk} + R_{kj}) - c_{jn}(R_{ik} + R_{ki})),
\end{aligned}$$

$$\widetilde{P}_{ijk\bar{n}} = R_{ijk}{}^{n} - \frac{1}{(2n-1)}(\ \delta_i^n(R_{jk} + R_{kj}) - \delta_j^n(R_{ik} + R_{ki}),$$

$$\widetilde{P}_{ij\bar{k}n} = R_{jin}{}^{k},$$

$$\widetilde{P}_{i\bar{j}kn} = R_{kni}{}^{j} + \frac{1}{(2n-1)}\ \delta_n^j(R_{ik} + R_{ki}),$$

$$\widetilde{P}_{\bar{i}jkn} = R_{nkj}{}^{i} - \frac{1}{(2n-1)}\ \delta_n^i(R_{jk} + R_{kj}).$$

The proof follows from the above equations. ■

A semi-Riemannian manifold (M,g), $n=\dim(M)\geq 3$, is said to be semi-symmetric [18] if its curvature tensor R satisfies the condition

$$(R(X,Y)R)(Z,W)U=0, \tag{4.10}$$

and Ricci semi-symmetric if its Ricci tensor satisfies the condition

$$(R(X,Y)Ric)(Z,W)=0 \tag{4.11}$$

for all $X,Y,Z,W,U\in\Im_0^1(M)$, where $R(X,Y)$ acts as a derivation on R and Ric. In local coordinate, conditions (4.10) and (4.11) are respectively written in the following form:

$$\begin{aligned} &((R(X,Y)R)(Z,W)U)_{ijklm}{}^{n}=\nabla_i\nabla_j R_{klm}{}^{n}-\nabla_j\nabla_i R_{klm}{}^{n}\\ =\;& R_{ijp}{}^{n}R_{klm}{}^{p}-R_{ijk}{}^{p}R_{plm}{}^{n}-R_{ijl}{}^{p}R_{kpm}{}^{n}-R_{ijm}{}^{p}R_{klp}{}^{n}\end{aligned}$$

and

$$((R(X,Y)Ric)(Z,W))_{ijkl}=\nabla_i\nabla_j R_{kl}-\nabla_j\nabla_i R_{kl}=R_{ijk}{}^{p}R_{pl}+R_{ijl}{}^{p}R_{kp}.$$

Note that a locally symmetric manifold is obviously semi-symmetric, but in general the converse is not true.

Theorem 6. *Let (M,g) be a semi-Riemannian manifold and T^*M be the cotangent bundle with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ over (M,g). We assume that $\widetilde{R}_{ijk}{}^{\overline{h}}=0$, from which it follows that $\nabla_i R_{hkj}{}^{s}-\nabla_j R_{hki}{}^{s}=0$ and $\nabla_i(\nabla_k c_{jh}-\nabla_h c_{jk})-\nabla_j(\nabla_k c_{ih}-\nabla_h c_{ik})-R_{ijk}{}^{m}c_{mh}-R_{ijh}{}^{m}c_{km}=0$, where R and $\widetilde{R}$ are the curvature tensors of the Levi–Civita connections ∇ and $\widetilde{\nabla}$ of g and $\widetilde{g}_{\nabla,c}$, respectively. Then $(T^*M,\widetilde{g}_{\nabla,c})$ is semi-symmetric if and only if (M,g) is semi-symmetric.*

P r o o f. We consider the condition $(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U}=0$ for all $\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W},\widetilde{U}\in\Im_0^1(T^*M)$. The tensor $(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U}$ has the components

$$\begin{aligned} &((\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{\alpha\beta\gamma\theta\sigma}{}^{\varepsilon}\\ =\;& \widetilde{R}_{\alpha\beta\tau}{}^{\varepsilon}\widetilde{R}_{\gamma\theta\sigma}{}^{\tau}-\widetilde{R}_{\alpha\beta\gamma}{}^{\tau}\widetilde{R}_{\tau\theta\sigma}{}^{\varepsilon}-\widetilde{R}_{\alpha\beta\theta}{}^{\tau}\widetilde{R}_{\gamma\tau\sigma}{}^{\varepsilon}-\widetilde{R}_{\alpha\beta\sigma}{}^{\tau}\widetilde{R}_{\gamma\theta\tau}{}^{\varepsilon}\end{aligned} \tag{4.12}$$

with respect to the adapted frame $\{E_\alpha\}$.

For the case of $\alpha=i,\beta=j,\gamma=k,\theta=l,\sigma=\overline{m},\varepsilon=\overline{n}$ in (4.12), it follows that

$$\begin{aligned} &((\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ijkl\overline{m}}{}^{\overline{n}}\\ =\;& \widetilde{R}_{ijp}{}^{\overline{n}}\widetilde{R}_{kl\overline{m}}{}^{p}+\widetilde{R}_{ij\overline{p}}{}^{\overline{n}}\widetilde{R}_{kl\overline{m}}{}^{\overline{p}}-\widetilde{R}_{ijk}{}^{p}\widetilde{R}_{pl\overline{m}}{}^{\overline{n}}-\widetilde{R}_{ijk}{}^{\overline{p}}\widetilde{R}_{\overline{p}l\overline{m}}{}^{\overline{n}}\\ &-\widetilde{R}_{ijl}{}^{p}\widetilde{R}_{kp\overline{m}}{}^{\overline{n}}-\widetilde{R}_{ijl}{}^{\overline{p}}\widetilde{R}_{k\overline{p}\overline{m}}{}^{\overline{n}}-\widetilde{R}_{ij\overline{m}}{}^{p}\widetilde{R}_{klp}{}^{\overline{n}}-\widetilde{R}_{ij\overline{m}}{}^{\overline{p}}\widetilde{R}_{kl\overline{p}}{}^{\overline{n}}\\ =\;& -R_{ijp}{}^{m}R_{kl}{}_{n}{}^{p}-R_{ijk}{}^{p}R_{pl}{}_{n}{}^{m}-R_{ijl}{}^{p}R_{kpn}{}^{m}-R_{ijn}{}^{p}R_{klp}{}^{m}\\ =\;& -((R(X,Y)R)(Z,W)U)_{ijkl}{}_{n}{}^{m}.\end{aligned} \tag{4.13}$$

For the case of $\alpha = i, \beta = j, \gamma = \overline{k}, \theta = l, \sigma = m, \varepsilon = \overline{n}$ in (4.12), we get

$$\begin{aligned}
&((\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ij\overline{k}lm}{}^{\overline{n}} \\
= \ & \widetilde{R}_{ijp}{}^{\overline{n}}\widetilde{R}_{\overline{k}lm}{}^{p} + \widetilde{R}_{ij\overline{p}}{}^{\overline{n}}\widetilde{R}_{\overline{k}lm}{}^{\overline{p}} - \widetilde{R}_{ij\overline{k}}{}^{p}\widetilde{R}_{plm}{}^{\overline{n}} - \widetilde{R}_{ij\overline{k}}{}^{\overline{p}}\widetilde{R}_{\overline{p}lm}{}^{\overline{n}} \\
& -\widetilde{R}_{ijl}{}^{p}\widetilde{R}_{\overline{k}pm}{}^{\overline{n}} - \widetilde{R}_{ijl}{}^{\overline{p}}\widetilde{R}_{\overline{k}\overline{p}m}{}^{\overline{n}} - \widetilde{R}_{ijm}{}^{p}\widetilde{R}_{\overline{k}lp}{}^{\overline{n}} - \widetilde{R}_{ijm}{}^{\overline{p}}\widetilde{R}_{\overline{k}l\overline{p}}{}^{\overline{n}} \\
= \ & -R_{ijp}{}^{k}R_{nlm}{}^{p} - R_{ijn}{}^{p}R_{plm}{}^{k} - R_{ijl}{}^{p}R_{npm}{}^{k} - R_{ijm}{}^{p}R_{nlp}{}^{k} \\
= \ & -((R(X,Y)R)(Z,W)U)_{ijnlm}{}^{k}. \qquad (4.14)
\end{aligned}$$

For the case of $\alpha = i, \beta = j, \gamma = \overline{k}, \theta = \overline{l}, \sigma = \overline{m}, \varepsilon = \overline{n}$ in (4.12), we have

$$((\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ij\overline{k}\overline{l}\overline{m}}{}^{\overline{n}} = 0. \qquad (4.15)$$

For the case of $\alpha = i, \beta = j, \gamma = k, \theta = l, \sigma = m, \varepsilon = \overline{n}$ in (4.12), we obtain

$$\begin{aligned}
&((\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{ijklm}{}^{\overline{n}} \\
= \ & \widetilde{R}_{ijp}{}^{\overline{n}}\widetilde{R}_{klm}{}^{p} + \widetilde{R}_{ij\overline{p}}{}^{\overline{n}}\widetilde{R}_{klm}{}^{\overline{p}} - \widetilde{R}_{ijk}{}^{p}\widetilde{R}_{plm}{}^{\overline{n}} - \widetilde{R}_{ijk}{}^{\overline{p}}\widetilde{R}_{\overline{p}lm}{}^{\overline{n}} \\
& -\widetilde{R}_{ijl}{}^{p}\widetilde{R}_{kpm}{}^{\overline{n}} - \widetilde{R}_{ijl}{}^{\overline{p}}\widetilde{R}_{k\overline{p}m}{}^{\overline{n}} - \widetilde{R}_{ijm}{}^{p}\widetilde{R}_{klp}{}^{\overline{n}} - \widetilde{R}_{ijm}{}^{\overline{p}}\widetilde{R}_{kl\overline{p}}{}^{\overline{n}}. \qquad (4.16)
\end{aligned}$$

For the case of $\alpha = \bar{i}, \beta = j, \gamma = k, \theta = l, \sigma = m, \varepsilon = \overline{n}$ in (4.12), we obtain

$$\begin{aligned}
&((\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U})_{\bar{i}jklm}{}^{\overline{n}} \\
= \ & \widetilde{R}_{\bar{i}jp}{}^{\overline{n}}\widetilde{R}_{klm}{}^{p} + \widetilde{R}_{\bar{i}j\overline{p}}{}^{\overline{n}}\widetilde{R}_{klm}{}^{\overline{p}} - \widetilde{R}_{\bar{i}jk}{}^{p}\widetilde{R}_{plm}{}^{\overline{n}} - \widetilde{R}_{\bar{i}jk}{}^{\overline{p}}\widetilde{R}_{\overline{p}lm}{}^{\overline{n}} \\
& -\widetilde{R}_{\bar{i}jl}{}^{p}\widetilde{R}_{kpm}{}^{\overline{n}} - \widetilde{R}_{\bar{i}jl}{}^{\overline{p}}\widetilde{R}_{k\overline{p}m}{}^{\overline{n}} - \widetilde{R}_{\bar{i}jm}{}^{p}\widetilde{R}_{klp}{}^{\overline{n}} - \widetilde{R}_{\bar{i}jm}{}^{\overline{p}}\widetilde{R}_{kl\overline{p}}{}^{\overline{n}} \\
= \ & R_{npj}{}^{i}R_{klm}{}^{p} - R_{pkj}{}^{i}R_{nml}{}^{p} + R_{plj}{}^{i}R_{nmk}{}^{p} - R_{pmj}{}^{i}R_{lkn}{}^{p} \\
= \ & ((R(X,Y)R)(Z,W)U)_{nmlkj}{}^{i} - ((R(X,Y)R)(Z,W)U)_{kl\,nmj}{}^{k}. \qquad (4.17)
\end{aligned}$$

The other coefficients of $(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{R})(\widetilde{Z},\widetilde{W})\widetilde{U}$ reduce to one of (4.16), (4.14) or (4.15) by the property of the curvature tensor. The proof follows from (4.13)–(4.17). ■

Theorem 6 immediately gives the following result.

Corollary 1. *Let (M, g) be a semi-Riemannian manifold and T^*M be the cotangent bundle with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ over (M, g). If (M, g) is locally symmetric and the components c_{ij} of c satisfy the condition*

$$\nabla_i(\nabla_k c_{jh} - \nabla_h c_{jk}) - \nabla_j(\nabla_k c_{ih} - \nabla_h c_{ik}) - R_{ijk}{}^{m}c_{mh} - R_{ijh}{}^{m}c_{km} = 0, \qquad (4.18)$$

then $(T^*M, \widetilde{g}_{\nabla,c})$ is semi-symmetric.

Theorem 7. *Let (M,g) be a semi-Riemannian manifold and T^*M be the cotangent bundle with the modified Riemannian extension $\widetilde{g}_{\nabla,c}$ over (M,g). Then $(T^*M,\widetilde{g}_{\nabla,c})$ is Ricci semi-symmetric if and only if (M,g) is Ricci semi-symmetric.*

P r o o f. We study the condition $(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Ric})(\widetilde{Z},\widetilde{W})=0$ for all $\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W} \in \Im_0^1(T^*M)$. The tensor $(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Ric})(\widetilde{Z},\widetilde{W})$ has the components

$$((\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Ric})(\widetilde{Z},\widetilde{W}))_{\alpha\beta\gamma\theta}=\widetilde{R}_{\alpha\beta\gamma}{}^{\varepsilon}\widetilde{R}_{\varepsilon\theta}+\widetilde{R}_{\alpha\beta\theta}{}^{\varepsilon}\widetilde{R}_{\gamma\varepsilon}. \tag{4.19}$$

By putting $\alpha=i,\beta=j,\gamma=k,\theta=l$ in (4.19), it follows that

$$\begin{aligned}((\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Ric})(\widetilde{Z},\widetilde{W}))_{ijkl} &= \widetilde{R}_{ijk}{}^{p}\widetilde{R}_{pl}+\widetilde{R}_{ijl}{}^{p}\widetilde{R}_{kp}\\ &= 2R_{ijk}{}^{p}R_{pl}+2R_{ijl}{}^{p}R_{kp}\\ &= 2((R(X,Y)Ric)(Z,W))_{ijkl},\end{aligned}$$

all the others being zero. This finishes the proof. ■

R e m a r k 2. i) If $c_{ij}=0$, then conditions (4.3), (4.4), (4.8), (4.9) and (4.18) are identically fulfilled.

ii) If c_{ij} is parallel with respect to ∇, then conditions (4.3), (4.4), (4.8), (4.9) and (4.18) are identically fulfilled.

iii) If c_{ij} satisfies the relation $\nabla_i c_{jk}-\nabla_j c_{ik}=\nabla_k \omega_{ij}$, where the components ω_{ij} define a 2–form on M and if (M,∇) is flat, then conditions (4.3), (4.4), (4.8), (4.9) and (4.18) are identically verified.

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