KAPITSA PENDULUM AND PRINCIPLE OF WHIRLIGIG

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A comparison of the mechanism of stabilization of unstable states of physical systems in a rapidly oscillating external field (on example the Kapitsa's pendulum) with a whirligig type stabilization mechanism is presented in this work. It is shown that the whirligig mechanism is more efficient than the stabilization in rapidly changing fields for stabilizing unstable states if or when it can be actually used.

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INTRODUCTION

At present, the mechanism of stabilization of unstable states in a variety of physical systems in rapidly changing external fields has been well studied and widely used. Kapitsa works [1, 2] initiated a whole field of investigations – the vibrational mechanics. Numerous publications dedicated to the description of various physical effects observed in this case. Sufficient example is the publications [1 - 6]. The most known example of stabilization of unstable states with the rapid change in the system's parameters is the stabilization of the upper unstable location of a mathematical pendulum by rapidly changing the position of its suspension point known as the Kapitsa pendulum. The same mechanism is the basis of high-frequency radio field pressure forces (Gaponov-Miller forces).

Works [7 - 10] suggest a mechanism to stabilize a wide range of unstable physical systems. This mechanism allows to stabilize both classical and quantum systems. In particular, the possibility of stabilization of a radiation flow in plasma and a flow of charged particles in plasma has been suggested. The possibility to suppress the local instability has been demonstrated using the example of the Lorenz system. For quantum systems this mechanism is reminiscent of the quantum Zeno effect. The main feature of this mechanism is the introduction to a system under stabilization an additional degree of freedom. The interaction of this degree of freedom with one of the degrees of freedom of an unstable system under certain conditions, allows you stabilize the system. These conditions are conveniently described using a visual image of a children's toy – whirligig. Let us imagine that we have a real whirligig which does not rotate. Its vertical position is unstable. The whirligig will fall down during a characteristic time T. If we now rotate the whirligig, and the period of its rotation is significantly smaller than T, the vertical position of the whirligig is stable. This simple visual image contains two basic characteristics required to implement the stabilization mechanisms. Indeed, time of the whirligig fall (when it is not running) is convenient to associate with the lifetime of the unstable state of the physical system under investigation. To make this state stable it is necessary to introduce a mechanism similar to the rotation of the whirligig. Such a mechanism can be the introduction of an additional degree of freedom, which effectively interacts with one of the unstable physical system's own degree of freedom.

At first glance, whirligig mechanism looks equivalent to the mechanism of stabilization in rapidly oscillating external fields. Indeed, for the whirligig-type stabilization one needs to introduce rapidly changing processes (the analogue to whirligig rotation). However, as one can see below, in all cases where the principle of whirligig can be used, it is more efficient than the stabilization using the rapidly changing external fields. This work proves this fact. Here in the second section we transform the Kapitsa equations to the form which allows the comparison of the whirligig stabilization mechanism with the stabilization mechanism using external rapidly changing fields.

1. GENERAL CONSIDERATIONS

At first, let us briefly express some general considerations that allow us to understand the whirligig mechanism of stabilization. In most cases, the unstable states of dynamic systems are locally characterized by singular points of "saddle" type. The unstable nodes and focuses occur much less frequently. Here is the example of transforming an unstable point of "saddle" type into an elliptical point (to the point of "center" type). Phase portraits of the neighborhood of the saddle point are shown in Figs. 1,a,b,c.

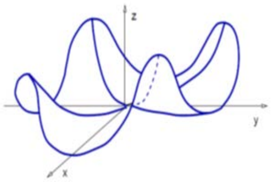

Fig. 1,a. Phase portrait of a singular point of the "saddle" type

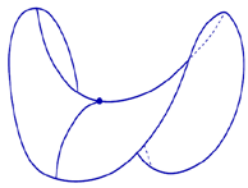

Fig. 1,b. Phase portrait of a singular point of the "saddle" type

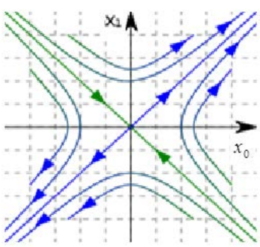

Fig. 1,c. Singular point of the "saddle" type on the phase plane

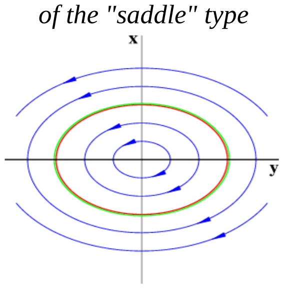

Fig. 2. Phase portrait of a singular point of "center" type

The equations in the phase plane, which describe the dynamics of phase trajectories in the neighborhood of the saddle point, have the form:

$$\dot{x}_0 = \gamma \cdot x_1 \;\; \dot{x}_1 = \gamma x_0 \,. \qquad (1)$$

In most real cases, each of the dependent variables in equations (1) is some characteristic a degree of freedom of the system under investigation. For example, this may be complex amplitudes of interacting nonlinear waves. Therefore, in this paper we assume that each of these first-order equations describes one of the degrees of freedom of the system under investigation. Suppose now that we want to transform a neighborhood of a singular saddle point into the neighborhood, which corresponds to a stable singular point, for example, elliptical point (Fig. 2).

To achieve this, one can do as follows. We introduce into the system an additional degree of freedom connected to the one of the degrees of freedom of the unstable system. The simplest model that describes the dynamics of the system in the vicinity of the saddle point after such modification differs from the equation (1) only by additional equation:

$$\dot{x}_0 = \gamma \cdot x_1 + \delta \cdot x_2 \, ; \quad \dot{x}_1 = \gamma x_0 \, ; \quad \dot{x}_2 = -\delta \cdot x_0 \, . \quad (2)$$

This new degree of freedom is connected to one of the degrees of freedom of the original system by the coupling coefficient δ . The system of equations (2) is equivalent to the equation of the linear pendulum:

$$\ddot{x}_0 + \left(\delta^2 - \gamma^2\right) x_0 = 0 \, . \quad (3)$$

From equation (3) is immediately obvious that as soon as the coefficient that describes the connection between the degrees of freedom is greater than the instability increment ($\delta > \gamma$), an unstable saddle point turns into an elliptical point. The phase space, shown on Fig. 1,c, became the phase space which is shown on Fig. 2. This simple algorithm of transforming the unstable saddle point into an elliptical point well characterizes the whirligig principle. Indeed, if we do not introduce an additional degree of freedom, our system is unstable (whirligig falls). Moreover, the characteristic instability time ($T \sim 1/\gamma$) can be associated with the whirligig fall time. The inclusion of an additional degree of freedom which stabilizes our system is similar to the presence of the whirligig rotation. Moreover, the whirligig stabilization has not only qualitative but also quantitative analogy. Indeed, for the vertical position of a whirligig to be stable, it is necessary that the rotation period is shorter than the time of falling. In our model (see equation (3)), the system becomes stable if the coupling coefficient becomes greater than the instability increment ($\delta > \gamma$). Moreover, if the increment of the instability is zero, then the system of equations (2) describes oscillations with frequency $\delta = 2\pi / T_{rot}$. Thus, there are qualitative and quantitative similarities between the considered mechanism and a mechanism of stabilizing a whirligig's vertical position.

Let us make the following remark. We are accustomed to the fact that the increase in the number of degrees of freedom of a dynamical system leads to tougher conditions for the realization of its steady state. Indeed assume our physical system is described by the following system of equations:

$$\dot{Z}_n = F_n(\vec{Z}, t) \, . \quad (4)$$

The nature of the stability of this system at the selected point of the phase space is described by a linear system of equations that describes the dynamics of small deviations $\vec{Z} = \vec{Z}_0 + \vec{x}$:

$$\dot{\vec{x}} = \hat{A}\vec{x}. \quad (5)$$

In many cases, the coefficients of the matrix can be considered constant. Then, to determine the type of the singular point of the system (5), we must find the roots of the characteristic equation:

$$\alpha_0 \lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + ... + \alpha_{n-1} \lambda + \alpha_n = 0. \quad (6)$$

Rause-Gurvitz criterion states that the higher order of the equations makes it more difficult to satisfy the conditions of stability for this system. In the example above we have increased the number of degrees of freedom, but have achieved the opposite result. It may look like an exceptional case. However, it is not true. [7 - 11] contain numerous examples showing how the introduction of an additional degree of freedom to complex systems such as those that describe the stabilization of radiation fluxes in the plasma can also lead to a stabilization of unstable states. This result echoes the result provided in Haken's book [12] about the role of the order parameters. Thus, the introduction of an additional degree of freedom is an analogue of the Haken's parameter of order.

2. DYNAMICS OF KAPITSA PENDULUM

Let's briefly provide the main results that characterize the dynamics of the Kapitsa pendulum. Such pendulum is described by a mathematical pendulum with periodically changed parameters:

$$\ddot{x} + \Omega^2 \left(1 + \varepsilon \cos\left(\omega \cdot t\right)\right) \sin x = 0 \, . \quad (7)$$

For small values of the angle ($x \equiv \varphi << 1$), this equation is the Mathieu equation. The solution of equation (7) is studied in all areas of the parameters. We are interested in the area of non-resonant parameters. We assume that the frequency of parametric excitation is large ($\omega >> \Omega$), and the parameter ε has no constrains. The stability of the upper position of a mathematical pendulum can be achieved in this parameters area. It is well known (e.g., [3]) that the dynamics of such nonlinear oscillator is determined by the effective potential:

$$U(x) = -\cos(x) + \alpha \cdot \sin^2(x) \quad , \quad (8)$$

where – $\alpha = \left(\frac{a}{l}\right)^2 \cdot \left(\frac{\omega}{2\Omega}\right)^2$; l – length of the pendulum; $l >> a$ – oscillation amplitude of the point of suspension; $\Omega = \sqrt{l/g}$ – the frequency of the linear oscillation of the pendulum; ω – the frequency of the oscillation of the point of suspension.

Stable position corresponds to the minimum of the potential (8). Fig. 3 shows the potential in the case when the conditions of the top position stability of the pendulum are not satisfied. One can see that the stable position is the lowest position of the pendulum. However, if the oscillation frequency of the suspension point or its oscillation amplitude increases, the upper position ($x \equiv \varphi = \pi$) may become stable too. This case is shown on Fig. 4. One can see that the upper position of the pendulum ($x \equiv \varphi = 3.14$) gets local minima. The location of the pendulum is stable at these points. These results are well known before (see, e.g., [1 - 5]).

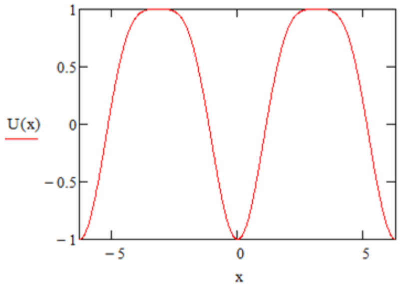

Fig. 3. The effective potential at $\alpha = 0.5$

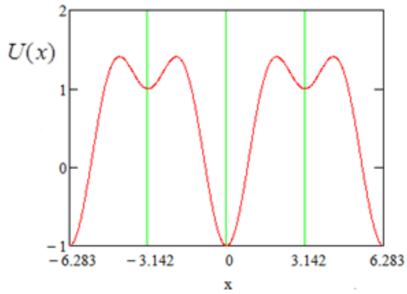

Fig. 4. The effective potential at $\alpha = 1.2$

The mathematical model of the Kapitsa pendulum inconvenient for comparison with other stabilization mechanisms in the form presented above. Therefore, for convenience of comparison, we somewhat simplified the model (7). Namely, we are interested in the dynamics in the vicinity of the unstable position of the pendulum only, i.e. in the vicinity of the point $x = \pi$. The dynamics of Kapitsa pendulum in this region of angles can be described by the system of equations:

$$\ddot{x}_0 - \Omega^2 x_0 = \Omega^2 \varepsilon \cos(\omega \cdot t) x_0 = x_1 \cdot x_0, \quad (9)$$

$$\ddot{x}_1 + \omega^2 x_1 = 0, \quad x_1(0) = \Omega^2 \varepsilon; \ \dot{x}_1(0) = 0.$$

The first equation of the system (9) describes the dynamics of the Kapitsa pendulum in the vicinity of an unstable equilibrium point. However, we have added to this equation the equation of a linear oscillator, which oscillations together with initial conditions describe the vibrations of the suspension point of the pendulum. Such modification is useful for comparison of the known mechanism of the Kapitsa pendulum stabilization with the mechanism, which uses the principle of whirligig. The system of equations (9) describes the movement in constant (in time) potential and in the external rapidly oscillating field. It is convenient to rewrite it as:

$$\ddot{x} - x = \varepsilon \cos(\omega_N \cdot \tau) x \equiv f(x, \tau), \quad (10)$$

here $U(x) = -x^2/2$, $\tau = \Omega \cdot t$;

$\dot{x} \equiv dx/d\tau \ \varepsilon \cos(\omega \cdot t) x \equiv f(x,t)$;

$\omega_N = \omega/\Omega >> 1$.

Then, using standard averaging procedure (see, for example, [3]), we find the following expression for the effective potential, which determines the dynamics of the pendulum in the vicinity of its upper unstable position:

$$U_{eff}(x) = -\frac{x^2}{2} + \frac{1}{2\omega_N{}^2}\langle f^2 \rangle;$$

$$U_{eff}(x) = -\frac{x^2}{2} + \frac{\varepsilon^2 \cdot x^2}{4\omega_N{}^2} = -\frac{x^2}{2}\left[1 - \frac{\varepsilon^2}{2\omega_N{}^2}\right]. \quad (11)$$

The steady state corresponds to the minimum of this potential. The stability condition has the form:

$$\varepsilon^2 > 2\omega_N{}^2 >> 1. \quad (12)$$

When expressed in familiar variables it transforms to well-known expression: $\omega > \sqrt{2} \cdot \Omega \cdot \frac{l}{a}$; $\Omega = \sqrt{l/g}$. By definition $\omega_N = \omega/\Omega >> 1$ is a large parameter.

Thus, we obtained known Kapitsa result on stabilization of the upper unstable position of the pendulum. The stabilization was achieved by applying external influence to the parameters of the pendulum. At the same time, if we consider the system (9), we can say that stabilization mechanism, in some sense is a nonlinear mechanism. Indeed, if we have kept the reverse influence of the pendulum dynamics on the oscillator, which causes fluctuations in the point of suspension, then it would be a non-linear system. However, we ignored this bond.

As a result, the first equation (9) remains linear equation with parameters subjected to external high-frequency disturbances. Note also that equation (10) shows the parameters under which the external high-frequency perturbations can change the character of the unstable saddle point. Indeed, if we drop the right side of the system (10), the remaining part of this equation describes the dynamics in a neighborhood of a singular unstable point of "saddle" type. The presence of external high-frequency perturbations (right-hand side of equation (10)) with the parameters defined by the inequality (12) transforms an unstable singular point of "saddle" type to an elliptic point. Below we show that there is another opportunity to change the nature of an unstable point turning it into a stable singular point. To achieve this let's rewrite equation (10) without the right side:

$$\dot{x}_0 = \Omega \cdot x_1 \ \ \dot{x}_1 = \Omega \cdot x_0. \quad (13)$$

These equations describe the unstable state of "saddle" type. Let's modify this system of equations assuming that the variable of the first equation in (13) is linearly connected to some other (additional) variable x_2:

$$\dot{x}_0 = \Omega \cdot x_1 + \delta \cdot x_2,$$

$$\dot{x}_1 = \Omega \cdot x_0, \quad \dot{x}_2 = -\delta \cdot x_0, \quad (14)$$

where δ – is the coefficient of the connection.

The system (14) is equivalent to the equation of the linear pendulum:

$$\ddot{x}_0 + (\delta^2 - \Omega^2) x_0 = 0. \quad (15)$$

We have already dealt with such equation (see equation (2) and (3)). From (15) follows that, in order to stabilize the unstable upper position of the pendulum, it is necessary to satisfy the condition $\delta > \Omega$ only.

This stabilization condition is much weaker than the stabilization as the result of rapidly changing fields (12).

CONCLUSIONS

Is was shown above, that the usage of the whirligig principle allows to stabilize an unstable state in much simple and easier way than it can be achieved using external rapidly oscillating fields. However, it should be understood that the whirligig principle is not always possible to realize. For example, one can hardly imagine a mathematical Kapitza pendulum, oscillation of which somehow connected with some other degree of freedom and the frequency of these oscillations is large than the fundamental frequency of the pendulum. However, in many other cases (see, for example, [7-11]) it would be easy to achieve. Let us formulate the main conclusions of this work:

1. The principle of whirligig stabilization of unstable states cannot be reduced to the stabilization mechanism, based on usage an external rapidly oscillating fields.
2. In all cases where the whirligig principle can be used for stabilization of unstable states this mechanism is more effective than other stabilization mechanisms.

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МАЯТНИК КАПИЦЫ И ПРИНЦИП ЮЛЫ

В.А. Буц

Дано сравнение механизма стабилизации неустойчивых состояний физических систем во внешнем быстроосциллирующем поле (на примере маятника Капицы) с механизмом стабилизации юлы. Показано, что в тех случаях, когда для стабилизации неустойчивых состояний может быть использован принцип юлы, он оказывается более эффективным, чем механизм стабилизации в быстроменяющихся полях.

МАЯТНИК КАПИЦІ ТА ПРИНЦИП ДЗИҐИ

В.О. Буц

Дано порівняння механізму стабілізації нестійких станів фізичних систем у зовнішньому швидкоосцилюючому полі (на прикладі маятника Капиці) з механізмом стабілізації дзиги. Показано, що в тих випадках, коли для стабілізації може бути використаний принцип дзиґи, він є більш ефективним, ніж механізм стабілізації в швидкозмінних полях.