ALL TYPES OF COHERENT RADIATIONS, WHICH ARISE IN THE COLLISION OF RELATIVISTIC ELECTRONS WITH CRYSTAL

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The general quantum theory, that describes all types of X-and-γ-rays coherent radiations, which arise in the collision of relativistic electrons with a crystal target, is described briefly. Such types of coherent radiations as: ordinary coherent bremsstrahlung and channeling radiation, emitted by a relativistic electron; different types of coherent polarization radiations, emitted by the crystal atoms and nucleus; the interference of these two types of coherent radiations can be described within the framework of the present theoretical model.

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1. INTRODUCTION

Let us consider the total X-and-γ-rays radiation, which arises in the collision of relativistic electron with a single crystal target. On the whole we will concentrate our attention upon the case of high electron energies $E \gg mc^2$ and photon energies $\hbar\omega \ll E$.

For the most applications the interaction of the relativistic electrons with the radiation field can be treated as a small perturbation. On the other hand, the interaction of the electron with crystal atoms, in principle, can be handled rigorously by including it in the unperturbed Hamiltonian. The interaction of the electromagnetic field with crystal atoms and nuclei can be included in the unperturbed Hamiltonian for electromagnetic field in the crystal media too.

So it is convenient to use the quantum electrodynamics in Furry notation [1].

2. GENERAL CROSS SECTION

2.1. The total Hamiltonian for the system of relativistic electron, crystal and electromagnetic radiation

Classical Hamiltonian of the system of relativistic electron, crystal and electromagnetic radiation has the form:

$$H = \underbrace{H_e + H_{e-c}} + \underbrace{H_r + H_{r-c}} + H_{e-r}\,, \qquad (1)$$

where H_e is the Hamiltonian of free relativistic electron, H_{e-c} is the Hamiltonian of interaction between relativistic electron and crystal, H_r is the Hamiltonian of free electromagnetic radiation, H_{r-c} is the Hamiltonian of interaction between radiation and crystal, H_{e-r} is the Hamiltonian of interaction between relativistic electron and radiation. The internal state of crystal is conserved, therefore the Hamiltonian of the crystal is not included.

Let us choose the eigenfunctions $\psi_n(\vec r)e^{-iE_nt/\hbar}$ and $\vec A_\omega(\vec r)e^{-i\omega t}$ of corresponding operators: $\hat H'_e = \hat H_e + \hat H_{e-c}$ and $\hat H'_r = \hat H_r + \hat H_{r-c}$ as the generalized coordinates of the fields.

Thus in our case the unperturbed states for the relativistic electron satisfy the Dirac equation:

$$\left(-ic\hbar\hat{\vec\alpha}\vec\nabla + mc^2\hat\beta - i\hbar\frac{\partial}{\partial t}\right)\psi(\vec r,\, t) = -U_{e-c}(\vec r)\psi(\vec r,\, t)\,, \qquad (2)$$

where $H_{e-c} = U_{e-c}(\vec r)$, and $U_{e-c}(\vec r)$ is the potential energy of interaction the relativistic electron with static crystal potential.

The unperturbed states of the electromagnetic field in the crystal media $\vec A(\vec r,\, t)$ satisfy the Maxwell equation:

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec A(\vec r,\, t) = -\frac{4\pi}{c}\vec j(\vec r,\, t)\,, \qquad (3)$$

where $\vec j(\vec r,\, t)$ is the current density inside the crystal matter in the presence of the electromagnetic field, which means as a result of field scattering on the crystal electrons and nuclei, i.e. $\vec j(\vec r,\, t) = \vec j^{(el)}(\vec r,\, t) + \vec j^{(nucl)}(\vec r,\, t)$. Besides, the Coulomb gauge $\vec\nabla \cdot \vec A = 0$ should be used.

Finally, the Hamiltonian of interaction between relativistic electron and radiation can be presented in the form:

$$H_{e-r} = \frac{1}{c}\int \vec J(\vec r,\, t)\cdot\vec A(\vec r,\, t)d^3r\,, \qquad (4)$$

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where: $\vec{J} = -ec\psi^{+}\hat{\vec{\alpha}}\psi$ is the current of the relativistic electron; $\psi(\vec{r},\, t) = \sum_n C_n e^{-iE_n t/\hbar}\psi_n(r)$ is the complete wave function of the incoming relativistic electron; $e^{-iE_n t/\hbar}\psi_n(\vec{r})$ are the electron wave functions, which characterize relativistic electron states in the crystal media and satisfy the equation (2); wave amplitudes C_n are obtained by "sewing together" complete electron wave function on the boundary crystal-vacuum, ($|C_n|^2$ is the probability of the population the state $\psi_n(\vec{r},t)$); $\vec{A}(\vec{r},\, t)$ is the vector-potential of the electromagnetic field in the crystal media, which satisfies the equation (3) with taking into account the boundary conditions on the boundary crystal-vacuum, as well.

Hamiltonian of interaction H_{e-r} (see Eq.4) is treated as a small perturbation.

2.2. Cross section of all types of coherent radiations of relativistic electrons in crystals

Quantum transitions occur between electron states and between photon states, which are described by the eigenfunctions of the Hamiltonian operators $\hat{H}'_e$ and $\hat{H}'_r$, for electron and electromagnetic field inside the crystal matter.

Solution of our task can be obtained within the framework of the first-order theory of perturbation. The radiation process describes by the first order term of the $\hat{S}$-matrix. Corresponding Feynman diagram is shown in Fig.1.

Solid lines in Fig.1. represent the unperturbed states of the charged particle, i.e. electron in the initial state $|i\rangle$ and in the final state $|f\rangle$, which are the solutions of the Dirac equation for electron in the crystal potential (see Eq.2); and wavy line represents the unperturbed photon states described by functions $A_\mu^\lambda(\vec{r},t)$, which are the solutions of the wave equation (see Eq.3).

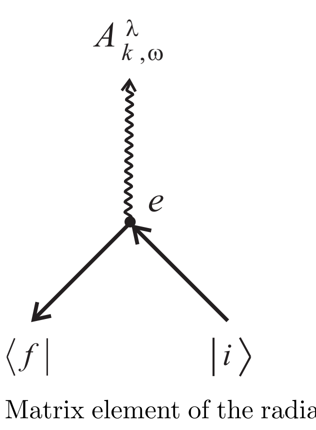

Fig.1. *Feynman diagram describes all types of radiation, which arise in the collision of relativistic electron with crystal*

Matrix element of the radiation process has the form:

$$S^{(1)}_{i\to f} = \frac{e}{\hbar c}\int_{-\infty}^{\infty} d^3r \int_{-\infty}^{\infty} dt\psi_f^{+}(\vec{r},t)\hat{\vec{\alpha}}\cdot\vec{A}^{*}_{\vec{k}\lambda}(\vec{r},t)\psi_i(\vec{r},t)\,. \tag{5}$$

Expression for electron wave function is the following:

$$\psi_n(\vec{r},t) = \sqrt{\frac{E_n+mc^2}{2E_nV}}\begin{pmatrix}\vec{u}_n\\ \frac{c\hat{\vec{\sigma}}\hat{\vec{p}}}{E_n+mc^2}\vec{u}_n\end{pmatrix}\phi_n(\vec{r})e^{-iE_n t/\hbar}\,, \tag{6}$$

where $\vec{u}_n$ are two-component unit spinors. The photon field $\vec{A}(\vec{r},t)$ is given in the form of the Bloch waves:

$$\vec{A}(\vec{r},t) = \sqrt{\frac{2\pi\hbar c}{kV}}\sum_{\vec{g},\lambda}\left[\vec{e}_{\vec{g},\lambda}a_{\vec{g},\lambda}e^{i[(\vec{k}+\vec{g})\vec{r}-\omega t]} + H.c.\right]\,, \tag{7}$$

where vectors $\vec{g}$ are the crystal reciprocal lattice vectors. In the expression (5) must be used one of the single item of the sum (7).

After integrating matrix element (5) over the time we obtain:

$$S^{(1)}_{i\to f} = 2\pi M_{i\to f}\cdot\delta\left(\frac{E_i-E_f}{\hbar}-\omega\right)\,, \tag{8}$$

where matrix element $M_{i\to f}$ has the form:

$$M_{i\to f} = \frac{e}{\hbar c}\sqrt{\frac{2\pi\hbar c}{kV}}\int d^3r\psi_f^{+}(\vec{r})\hat{\vec{\alpha}}\cdot\vec{e}_{\vec{k}\lambda}e^{-i\vec{k}\vec{r}}\psi_i(\vec{r})\,. \tag{9}$$

The number of the final states in the interval of the particle momenta d^3p_f is $Vd^3p_f/(2\pi\hbar)^3$. The number of the final states in the interval of the photon momenta $\hbar^3d^3k$ is $Vd^3k/(2\pi)^3$. Therefore the probability of the transitions into these states, which relates to the one particle and one photon, is $|S_{i\to f}|^2\frac{Vd^3p_f}{(2\pi\hbar)^3}\frac{Vd^3k}{(2\pi)^3}$. The square of δ-function is:

$$[2\pi\delta(E_i-E_f)]^2 = 2\pi T\delta(E_i-E_f)\,. \tag{10}$$

Dividing the probability of the transitions into final states on the time T, we obtain that the number of transitions per unit time into the interval of momenta $d^3p_f\hbar^3d^3k$ is equal:

$$dP = \frac{1}{T}\left|S_{i\to f}\right|^2\frac{Vd^3p_f}{(2\pi\hbar)^3}\frac{Vd^3k}{(2\pi)^3}\,. \tag{11}$$

The cross section is defined as a ratio of velocity of transition dP and the flux of the initial particles $|J_{inc}|$, which is defined as $|J_{inc}| = |\vec{v}_i|/V$, where: $\vec{v}_i = c^2\vec{p}_i/E_i$ is the velocity of the initial particle, V is the volume which was used for normalization. Using such calculations one can obtain the following cross section:

$$d^3\sigma = \alpha\left(\frac{\hbar}{mc}\right)^2\frac{E_imc}{p_i\hbar\omega}\left|M'_{i\to f}\right|^2\times$$
$$\times\delta(E_i-E_f-\hbar\omega)\frac{\hbar^3d^3kd^3p_f}{(2\pi mc)^6}\,, \tag{12}$$

where

$$M'_{i\to f} = \int d^3r\psi_f^{+}(\vec{r})\hat{\vec{\alpha}}\cdot\vec{e}_{\vec{k}\lambda}e^{-i\vec{k}\vec{r}}\psi_i(\vec{r})\,. \tag{13}$$

In the case $\hbar\omega \ll E$, we can assume $E_i \approx E_f = E$ and simplify the expression for matrix element $M'_{i\to f}$. Let us calculate the product of matrices in the expression (13), using the properties of Pauli spin-matrices

$\hat{\sigma}_j$:

$$\begin{pmatrix} \vec{u}_f^{+} \\ \vec{u}_f^{+} \frac{c\hat{\vec{\sigma}}\hat{\vec{p}}}{E_f+mc^2} \end{pmatrix} \hat{\alpha}_j \begin{pmatrix} \vec{u}_i \\ \frac{c\hat{\vec{\sigma}}\hat{\vec{p}}}{E_i+mc^2}\vec{u}_i \end{pmatrix} =$$
$$\vec{u}_f^{+} \frac{c}{E_i+mc^2}\hat{\sigma}_j(\hat{\vec{\sigma}}\cdot\hat{\vec{p}})\vec{u}_i + \vec{u}_f^{+} \frac{c}{E_f+mc^2}(\hat{\vec{\sigma}}\cdot\hat{\vec{p}})\hat{\sigma}_j\vec{u}_i \simeq$$
$$\frac{c}{E+mc^2}\vec{u}_f^{+}\left\{\hat{\sigma}_j(\hat{\vec{\sigma}}\cdot\hat{\vec{p}}) + (\hat{\vec{\sigma}}\cdot\hat{\vec{p}})\hat{\sigma}_j\right\}\vec{u}_i \simeq \frac{2}{mc\gamma}\hat{p}_j\vec{u}_f^{+}\hat{1}\vec{u}_i\,. \tag{14}$$

Thus we obtain a rather simple expression for matrix element $M'_{i\to f}$:

$$M'_{i\to f} = \frac{\vec{u}_f^{+}\vec{u}_i}{mc^2\gamma}\int d^3r\phi_f^*(\vec{r})\hat{\vec{p}}\cdot\vec{A}^*(\vec{r})\phi_i(\vec{r})\,. \tag{15}$$

Usually we are not interested in the electron spin states. Therefore we must perform summation in the cross section over the final electron spin states and averaging over the initial electron spin states, which lead to:

$$\left\langle \sum_f |\vec{u}_f^{+}\vec{u}_i|^2 \right\rangle_i = \frac{1}{2}\sum_{f,i}|\vec{u}_f^{+}\vec{u}_i|^2 = 1. \tag{16}$$

3. SOLUTIONS OF THE WAVE EQUATIONS (2) AND (3)

For the simplicity it is convenient to choose a crystal with cubic lattice and use the Cartesian coordinate system connected with cubic edges. We suppose that crystal lattice has finite numbers of periods $-N_1/2 \le n_1 \le N_1/2$ in the direction, which creates a small angle with momentum of incoming relativistic electron, and infinite numbers of periods for other two directions $-\infty \le n_i \le \infty$ $(i = 2, 3)$. Within the framework of the present model the main task is in the searching solutions of the wave equations (2) and (3), which satisfy boundary conditions. Outside the crystal equations (2) and (3) describe free relativistic electron and free electromagnetic field, respectively. On the boundary crystal-vacuum the wave functions, which represent solutions of (2) and (3) inside the crystal, must be "sewn together" with wave functions, which represent solutions of (2) and (3) outside the crystal.

3.1. Wave equations for the waves in the media with periodical heterogeneity

3.1.1. *Equation for charged particles*

We can search for the solution of the Dirac equation (2) in the form of (6) and transform it into the Pauli equation [2]. Then we can see that for relativistic electrons, if $E \gg mc^2$, the wave functions $\phi(\vec{r})$ approximately satisfy Schrödinger equation for particle with mass $M = E/c^2 = m\gamma$ and energy $\epsilon = (E^2 - m^2c^4)/2E$. The accuracy of such approximation is order of $\sim 1/\gamma = mc^2/E$. Introducing variables: $K^2 = \vec{p}^2/\hbar^2 = 2M\epsilon/\hbar^2$ and $V_{cr}(\vec{r}) = 2MU_{e-c}(\vec{r})/\hbar^2$, we can rewrite this equation in the form:

$$\left[\nabla^2 + K^2\right]\phi(\vec{r}) = -V_{cr}(\vec{r})\phi(\vec{r})\,. \tag{17}$$

Potential function $V_{cr}(\vec{r})$ is the following sum of atomic potential functions $V_{at}(\vec{r} + \vec{R}_{\vec{L},j})$:

$$V_{cr}(\vec{r}) = \sum_{\vec{L},j} V_{at}(\vec{r} + \vec{R}_{\vec{L},j})\,, \tag{18}$$

where $\vec{R}_{\vec{L},j} = \vec{L} + \vec{r}_j + \vec{u}_{\vec{L},j}$ are coordinates of crystal atoms; $\vec{L} = \vec{a}_1n_1 + \vec{a}_2n_2 + \vec{a}_3n_3$ are lattice vectors ($n_{1,2,3}$ – integer numbers, $\vec{a}_{1,2,3}$ – periods of the crystal lattice); $\vec{r}_j$ are structure vectors; $\vec{u}_{\vec{L},j}$ are thermal displacements of crystal atoms from their equilibrium positions.

Equation (17) describes the scattering of the plane waves on the potential deepening. Therefore, on the whole, the Schrödinger equation, which corresponds to (17), can have discrete energy levels $\epsilon_n < 0$ and space-localized wave functions $\phi_n(\vec{r})$, corresponding to them.

3.1.2. *Equation for electromagnetic field*

Now let us consider the equation (3) in the trivial case of the spreading the electromagnetic field in the crystal matter with frequency, which is larger than the atomic and nuclear resonance frequencies $\omega > \omega_R$. In that case the propagation of the electromagnetic field inside the crystal can be described as a process of purely elastic Waller scattering of electromagnetic waves in the crystal medium with periodically distributed electron density $\varrho_{cr}(\vec{r}) = \phi_s^+(\vec{r})\phi_s(\vec{r})$, where $\phi_s(\vec{r})$ is the wave function of the crystal electron subsystem. The quantum mechanical state of the crystal remains unchanged during the scattering process. Therefore one can suppose that the current density induced by mono-harmonic component of the electromagnetic field $\vec{A}_\omega(\vec{r})$ looks like:

$$\vec{j}_\omega^{(el)}(\vec{r}) = \frac{e^2}{mc}\varrho_{cr}(\vec{r})\vec{A}_\omega(\vec{r})\,, \tag{19}$$

Evidently that the mono-harmonic components of the vector-potential of the electromagnetic field in a crystal matter satisfy the wave equation:

$$\left[\vec{\nabla}^2 + \frac{\omega^2}{c^2}\right]\vec{A}_\omega(\vec{r}) = \frac{4\pi e^2}{mc^2}\varrho_{cr}(\vec{r})\vec{A}_\omega(\vec{r})\,. \tag{20}$$

One can see, that components of vector-function $\vec{A}_\omega(\vec{r}) = \sum_\lambda \vec{e}_\lambda \cdot a_{\omega,\lambda}(\vec{r})$, which satisfy equation (20), are independent. If we introduce the following designations: $\omega^2/c^2 = {k'}^2$ and $(4\pi e^2/mc^2)\varrho_{cr}(\vec{r}) = V'_{cr}(\vec{r})$, we obtain the wave equation for scalar function $a_{\omega,\lambda}(\vec{r})$ which is analogous to (17).

In spite of such similarity of (17) and (20), they have vital difference: periodical functions $V_{cr}(\vec{r})$ and $V'_{cr}(\vec{r})$ in these equations have different signs. Equation (17) describes the scattering of the plane waves on the potential deepening. Equation (20) describes the scattering of the plane waves on the potential barrier. Therefore equation (20) has only continuous spectrum of frequencies $\omega > 0$, and all wave functions $a_{\omega,\lambda}(\vec{r})$ are the plane waves.

Further we will concentrate our efforts to the methods of obtaining solutions of stationary wave equations (17) and (20) with periodical potential functions $+V_{cr}(\vec{r})$ and $-V'_{cr}(\vec{r})$, respectively [3].

3.2. Born approximations

If we assume $V_{cr}(\vec{r}) = 0$ in the equation (17) (or $V'_{cr}(\vec{r}) = 0$ in the equation (20)), then the equation has solutions, which are plane waves $e^{i\vec{k}\vec{r}}$. We have such solutions in the emptiness outside the crystal target.

We supposed (see Eq.14), that inside the crystal target $V_{cr}(\vec{r}) \ll K^2$ (or $V'_{cr}(\vec{r}) \ll k'^2$). Then the scattered wave is much less than initial incoming plane wave, and $V_{cr}(\vec{r})$ in the equation (17) (or $V'_{cr}(\vec{r})$ in the equation (20)) can be considered as a perturbation. So, the solution of (17) can be obtained in the form of rather quickly convergent series:

$$\phi(\vec{r}) = \phi^{(0)}(\vec{r}) + \phi^{(1)}(\vec{r}) + \phi^{(2)}(\vec{r}) + \ldots , \qquad (21)$$

where $|\phi^{(n+1)}(\vec{r})| < |\phi^{(n)}(\vec{r})|$ (for (20) we can write analogous series).

It is convenient to obtain such solutions using the method of Green functions. With the help of the Green function method the solution of differential problem like (17) (or (20)) can be converted into the integral problem:

$$\phi(\vec{r}) = \int G(\vec{r}, \vec{r}')V_{cr}(\vec{r}')\phi(\vec{r}')d^3r' , \qquad (22)$$

and then using (21), the iterative method can be applied.

Green function of operator $(\nabla^2 + K^2)$ can be defined as the function $G(\vec{r}, \vec{r}')$, that satisfies the equation:

$$(\nabla^2 + K^2)\, G(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}') . \qquad (23)$$

Representing Green function by its Fourier integral

$$G(\vec{r}, \vec{r}') = \frac{1}{(2\pi)^3} \int d^3k\, e^{-i\vec{k}\cdot\vec{r}} G(\vec{k}, \vec{r}') \qquad (24)$$

and substituting it into equation (22), we obtain equation for Fourier transformation of Green function:

$$G(\vec{k}, \vec{r}') = -\frac{e^{i\vec{k}\cdot\vec{r}'}}{k^2 - K^2} . \qquad (25)$$

Then using (24) and (25) we can write:

$$G(\vec{r}, \vec{r}') = -\frac{1}{(2\pi)^3} \int d^3k \frac{e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')}}{k^2 - K^2} =$$
$$-\frac{1}{(2\pi)^3} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{e^{ik|\vec{r}-\vec{r}'|\cos\theta}}{k^2 - K^2} k^2 dk \sin\theta d\theta d\varphi =$$
$$-\frac{1}{2\pi^2|\vec{r} - \vec{r}'|} \int_0^\infty \frac{\sin(k|\vec{r} - \vec{r}'|)}{k^2 - K^2} k dk . \qquad (26)$$

3.2.1. *The whole Born series*

Using the explicit expressions of the Green function (26) and initial plane wave function we can obtain the Born series:

$$\phi(\vec{r}) = \phi_0(\vec{r}) +$$
$$+ \sum_{n=1}^{\infty} \int \ldots \int G(\vec{r} - \vec{r}_n)V(\vec{r}_n)G(\vec{r}_n - \vec{r}_{n-1})V(\vec{r}_{n-1}) \times \ldots$$
$$\times G(\vec{r}_3 - \vec{r}_2)V(\vec{r}_2)G(\vec{r}_2 - \vec{r}_1)V(\vec{r}_1)\psi_0(\vec{r}_1)d^3r_1 \ldots d^3r_n . \qquad (27)$$

In the whole Born series (27) we can use the representation of the Green functions in the form of their Fourier integrals (24), and after integration over the space coordinates we can obtain the Born series in the momentum representation:

$$\phi(\vec{r}) \to e^{i\vec{k}_i\vec{r}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2\pi)^{3n}} \int \ldots \int \frac{e^{-i\vec{q}_1\vec{r}} F(\vec{q}_1)}{(\vec{k}_i - \vec{q}_1)^2 - K^2} \times \right.$$
$$\times \frac{e^{-i\vec{q}_2\vec{r}} F(\vec{q}_2)}{(\vec{k}_i - \vec{q}_1 - \vec{q}_2)^2 - K^2} \ldots \times$$
$$\left. \times \frac{e^{-i\vec{q}_n\vec{r}} F(\vec{q}_n)}{(\vec{k}_i - \vec{q}_1 - \ldots - \vec{q}_n)^2 - K^2} d^3q_1 \ldots d^3q_n \right\} , \qquad (28)$$

where: $F(\vec{q}_n) \equiv F_{cr}(\vec{q})$ are the Fourier transformations of the potential function $V(\vec{r}) \equiv V_{cr}(\vec{r})$ ($V_{cr}(\vec{r})$ is defined by Eq.18), and vectors $\vec{q}_n$ denote the momentums, which are passed to the scatterer (atom or crystal) during the n-multiple scattering, i.e. $\vec{q}_n = \vec{k}_n - \vec{k}_{n-1}$.

The Fourier transformation of the crystal potential (18) can be written as a product:

$$F_{cr}(\vec{q}) = \sum_{\vec{L},j} \exp[-i\vec{q}(\vec{L} + \vec{r}_j + \vec{u}_{\vec{L},j})]F_{at}(\vec{q}) =$$
$$= S(\vec{q})e^{-W(\vec{q})}D(\vec{q})F_{at}(\vec{q}) , \qquad (29)$$

where $S(\vec{q}) = \sum_j e^{-i\vec{q}\vec{r}_j}$ is a structure factor, $e^{-W(\vec{q})}$ is a Debye factor, $D(\vec{q})$ is a diffraction factor, and $F_{at}(\vec{q})$ is the Fourier transformation of the atomic potential. One can write the diffraction factor for the crystal with cubic lattice, which has finite number of periods $-N_1/2 \leq n_1 \leq N_1/2$ and infinite numbers of periods for other two directions $-\infty < n_i < \infty$ ($i = 2, 3$) in the form:

$$D(\vec{q}) = \frac{\sin(q_z a N_1/2)}{\sin(q_z a/2)}\, (2\pi/a)^2 \sum_{\vec{g}_t} \delta(\vec{q} - \vec{g})_t , \qquad (30)$$

where $\vec{g}_t$ are the transverse components of the reciprocal lattice vectors.

The Fourier transformation of the atomic potential is represented by the formula:

$$F_{at}(\vec{q}) = \frac{-4\pi e^2}{q^2} \left[Z - F'_{at}(\vec{q}) \right] , \qquad (31)$$

where $F'(\vec{q})$ is the atomic form-factor.

Thus for obtaining wave function $\phi(\vec{r})$, which is the solution of wave equation (17) in the form of Born series, it's necessary to substitute complete expression of $F_{cr}(\vec{q})$ in the form of (29) into (28) and to integrate every term of the sum.

Completely analogous result can be obtained for searching the amplitudes of electromagnetic waves with definite polarization $a_{\vec{k},\lambda}$, which are solutions of wave equation (20). For this aim we must replace $K \to \omega/c$ and $V_{cr}(\vec{r}) \to -V'_{cr}(\vec{r}) \cdot p^{\lambda\lambda'}$ in Eq.28, where polarization factor is $p^{\lambda\lambda'} = \vec{e}^{\lambda}_{h} \cdot \vec{e}^{\lambda'}_{0}$.

3.3. High energy approximation of Bethe equations

From formulae (28), (29) and (30) it follows, that the solution of wave equation type (17) can be represented as a superposition of the plane waves with wave vectors $\vec{k}_i + \vec{g}_t$, where $\vec{g}_t$ are transverse components of the reciprocal lattice vectors, i.e.

$$\phi(\vec{r}) = \sum_{\vec{g}_t} A_{\vec{g}_t}(z) e^{i(\vec{k}_i+\vec{g}_t)\vec{r}}\,, \tag{32}$$

which is the consequence of the Bloch theorem [4]. Besides, we must take into account that the amplitudes of plane waves $A_{\vec{g}_t}(z)$ are slowly-changing functions of the longitudinal coordinate z. Amplitude $A_{\vec{g}_t}(z)$ has a visible changing on the lengths of $l_{\vec{g}_t}$, which are more larger than crystal period a_1, i,e, $l_{\vec{g}_t} \gg a_1$. It means, that the wave $\phi(\vec{r})$ spreads under the small angle respectively the crystal plane and therefore $\vec{k}_i$ has large z-component, which is parallel to the crystal plane. (Such condition is the same, that is necessary for application of the eikonal method.)

Substituting (32) into wave equation (2), multiplying it by $e^{-i(\vec{k}_i+\vec{g}'_t)\vec{r}}$ and averaging over the volume of the crystal cell, and then using the condition $(dA_{\vec{g}_t}/dz))/A_{\vec{g}_t} \sim a_1/l_{\vec{g}_t} \ll 1$, we obtain the system of equations:

$$\begin{aligned} 2ik_z \frac{d}{dz} A_{\vec{g}_t}(z) &= \left[(\vec{k}_t+\vec{g}_t)^2 - \vec{k}_t^2\right] A_{\vec{g}_t}(z) + \\ &+ \sum_{\vec{g}'_t} F(\vec{g}_t - \vec{g}'_t) A_{\vec{g}'_t}(z)\,, \end{aligned} \tag{33}$$

where $F(\vec{g}_t - \vec{g}'_t)$ is the Fourier transformation of the potential function $\langle V(\vec{r})\rangle_z$ averaged along z-coordinate. This system of equations is the high energy approximation of known Bethe equations [4]. They describe the multiple scattering of waves in the crystal.

If crystal thickness is so small, that a single scattering is predominate, then in the sum over $\vec{g}'_t$ in (33) all amplitudes $A_{\vec{g}'_t}(z)$ are small in comparison with A_0. In this case the system of equations (33) spreads into the system of independent equations, and their solutions lead to the result of Born approximations.

One can search for the general solution of the system Eq.33 in the form:

$$A_{\vec{g}_t}(z) = B_{\vec{g}_t} e^{-i\lambda z}. \tag{34}$$

Substituting (34) into (33) we obtain the system of algebraic equations for amplitudes $B_{\vec{g}_t}$:

$$\left[2k_z\lambda + \vec{k}_t^2 - (\vec{k}_t+\vec{g}_t)^2\right] B_{\vec{g}_t} = +\sum_{\vec{g}'_t} F(\vec{g}_t - \vec{g}'_t) B_{\vec{g}'_t}\,, \tag{35}$$

which has solutions if parameter λ satisfies the "secular" equation:

$$\det\left|\left[2k_z\lambda + \vec{k}_t^2 - (\vec{k}_t+\vec{g}_t)^2\right]\delta_{\vec{g}_t,\vec{g}'_t} - F(\vec{g}_t - \vec{g}'_t)\right| = 0\,. \tag{36}$$

It is easy to see, that parameter $\lambda_n = E_t/(2\hbar v_z k_z)$ has linear correlation with so-called *"energy of transverse motion"* of the relativistic particle E_t in the averaged crystal potential $\langle V(\vec{r})\rangle_z$.

Equation (36) makes it possible to obtain the functional dependence of this parameter from transverse wave vector $\vec{k}_t + \vec{g}_t$, i.e. to obtain the zone structure of the levels of the so-called "transverse energy". The electron states can be subdivided into three groups:

1. *strongly bound (localized) states* which have the separated narrow energy zones, i.e. discrete energy levels;
2. *feebly bound or nearly free states* for which energy zones are wide and covered;
3. *intermediate between 1 and 2 states* which have the separated rather wide energy zones.

Determinant (36) is the two-dimensional analogy to the Hill determinant. For two-dimensional case the method of obtaining the exact solutions is not developed.

For obtaining approximate solutions we need to replace the infinite series of equations (35) by the system of the finite number of equations [4]. The rank of the system of equations can be estimated by the number of the reciprocal lattice points per square of atomic reciprocal screening radius. Crystal potential is changing rather rapidly and therefore this number is order of 100. Modern computers allow us to solve such systems of equations and this method sometimes is applied. But it is good enough for the description of the nearly free states so as the rank of the system in this case is relatively small.

3.4. Bound states or channeling

The representation of the coordinate wave function of relativistic charged particle $\phi(\vec{r})$ as a superposition of plane waves (see Eq.32) is inconvenient for the description of the strongly bound states, which are called "channeling states". For describing such states within the framework of Bethe equations we forced to solve the system of equations (35) with high rank. In order to avoid such difficulties, the wave functions of "channeling states" we can calculate using another, the more convenient method. The complete set of basis functions, which is used for representation searching wave functions, must be created using the set of channeling functions supplemented by the set of plane waves orthogonal to the set of channeling functions. The similar method was developed by C.Herring in 1940 for calculating wave functions of crystal electrons [5].

3.4.1. *Planar channeling wave functions*

Let us suppose, that the conditions of the planar channeling are satisfied. For general case we need to have such basis which includes the wave functions of electron bound (or channeling) states near the crystal planes. (The axial case can be considered analogously).

The system of equations (33) corresponds to the Schrödinger-type equation in the coordinate representation:

$$i\hbar\vec{v}_l\vec{\nabla}_l\phi_{\vec{K}_l}(x,\vec{r}_l) = = \left[-\frac{\hbar^2}{2m\gamma}(\vec{\nabla}_x^2+\vec{K}_x^2)+\langle V(x,\vec{r}_l)\rangle_{y,z}\right]\phi_{\vec{K}_l}(x,\vec{r}_l)\,, \quad (37)$$

where $\langle V(x,\vec{r}_l)\rangle_{y,z}$ is averaged crystal potential $V(\vec{r})$ over the square of the plane $y\times z$. Equation (37) can be solved by the method of variable separation. Its solution looks like $\phi_{\vec{K}_l}(x,\vec{r}_l)=e^{i\vec{K}_l\vec{r}_l}f_{\vec{K}_l}(n,x)$, where the function $f_{\vec{K}_l}(n,x)$ satisfies the Schrödinger-type equation:

$$\left[-\frac{h^2}{2m\gamma}\nabla_x^2+\langle V(x,\vec{r}_l)\rangle_{y,z}-\epsilon^{(n)}_{\vec{K}_l}(\gamma)\right]f_{\vec{K}_l}(n,x)=0\,, \quad (38)$$

where $\vec{r}_l=\{y,z\}$, and γ is relativistic particle Lorentz-factor. Potential function $\langle V(x,y,z)\rangle_{y,z}$ has rather complicated analytical expression. It can be approximated by the close relative function with simple analytical expression. The Pöschl-Teller potential $V_0/\,ch^2(x/x_0)$ [14] approximates averaged crystal plane potential $\langle V(x,\vec{r}_l)\rangle_{y,z}$ well. And it was shown in [10],[13], that for such plane potential the energies of the crosswise motion are:

$$\epsilon^{(n)}=-\frac{\hbar^2c^2}{2E}\left(\frac{2\nu+n}{x_0}\right)^2\,, \quad (39)$$

where: n is the main quantum number ($n=0,1,2,...$); $\nu=(1-\sqrt{1+16u})/4$; $u=(\gamma/2)(V_0/mc^2)(2\pi x_0/\lambda_c)^2$; V_0 and x_0 are constants, which are obtained by the way of fitting model potential to the real one; E and γ are energy and Lorentz-factor of relativistic electron; λ_c is the Compton length.

$$\begin{aligned} f_1^{(+)}(n,x) &= N_n ch^{2\nu}(x/x_0)\times \quad (40)\\ &\quad G_n(2\nu+n/2,1/2;-sh^2(x/x_0)),\\ f_1^{(-)}(n,x) &= N_n ch^{2\nu}(x/x_0)sh(x/x_0)\times\\ &\quad G_n(2\nu+(n-1)/2,3/2;-sh^2(x/x_0)), \end{aligned}$$

where

$$G_n(p,q;z)\equiv 1+\sum_{k=1}^{n}(-1)^k\begin{pmatrix} n\\ k\end{pmatrix}\times \frac{(p)(p+1)...(p+k-1)}{q(q+1)...(q+k-1)}z^k, \quad (41)$$

are Jacobi polynomials with $p=k+j+1/2$, $q=-(\nu+2+k+j)$, and $\begin{pmatrix} n\\ k\end{pmatrix}$ are binomial coefficients. Factors N_n in (40) are the normalizing factors:

$$N_n=\left[x_0\sum_{k=0}^{n}\sum_{j=0}^{n}c_kc_jB(p,q)\right]^{-1/2}\,, \quad (42)$$

where c_k (and c_j) is the coefficient with z^k (z^j) Jacobi polynomial (41). $B(p,q)=\Gamma(p)\Gamma(q)/\Gamma(p+q)$ are Euler B-functions.

3.5. Complete solution of Dirac equation (2)

Method of the orthogonalized plane waves (OPW) can be applied for calculations which correspond to the wide energy zones. Such method is described in detail in [3], where the set of bound state wave function, was represented by the axial channeling functions $\varphi_\lambda(\vec{\rho})$, where cylindrical coordinate system $\vec{\rho},z$ was used with Z-axis along the atomic string. Such axial channeling functions $\varphi_\lambda(\vec{\rho})$ were obtained by A. Tamura [9] for model atomic-string potential $eW(\vec{\rho})=Ze^2/\rho-C$, where parameters Z and C are defined by the fitting of model potential to the real one. Here we will use the set of bound state wave functions $\varphi_\lambda(\vec{\rho})$ [9] for description the (OPW)-method. (The computations in the planar case are analogous.)

Wave function is presented by superposition of OPW which can be defined by the correlation:

$$OPW_{\vec{k}}\equiv\left|\vec{k}\right\rangle-\sum_{n,\vec{L}}\left|n,\vec{L}\right\rangle\left\langle n,\vec{L}\left|\vec{k}\right.\right\rangle\,, \quad (43)$$

where $|\vec{k}\rangle$ is a plane wave, and $|n,\vec{L}\rangle\equiv\exp(i\vec{k}_t\vec{L})\varphi(\vec{\rho}+\vec{L})$ is normalized function of the bound state that corresponds to $\vec{L}$ crystal cell.

$OPW_{\vec{k}}$ has behavior like plane wave far from the crystal planes or strings and like localized state function $|n,\vec{L}\rangle$ near the atomic strings and planes. These properties of $OPW_{\vec{k}}$ make them the most convenient for calculations the middle states between strongly bound and nearly free state functions. If integral projecting operator $\hat{P}$ is introduced by the correlation

$$\hat{P}=\sum_{n,\vec{L}}\left|n,\vec{L}\right\rangle\left\langle n,\vec{L}\right|\,, \quad (44)$$

then $OPW_{\vec{k}}$ can be represented in the form:

$$OPW_{\vec{k}}=(1-\hat{P})\left|\vec{k}\right\rangle\,. \quad (45)$$

Solution of the equation (37) can be represented in the form which satisfies the Block theorem:

$$\varphi_{\vec{k}}=(1-\hat{P})\sum_{\vec{g}_t}A_{\vec{g}_t}\left|\vec{k}+\vec{g}_t\right\rangle\,. \quad (46)$$

The sum of the plane waves $\chi_{\vec{k}}\equiv\sum_{\vec{g}_t}A_{\vec{g}_t}|\vec{k}+\vec{g}_t\rangle$ is called *pseudo wave functions*. Substituting (46) into equation (17) one can obtain the following equation for the pseudo wave function

$$\left[-\frac{\hbar^2}{2m\gamma}(\nabla_t^2+k_t^2)+eW_p\right]\chi_{\vec{k}}=E\chi_{\vec{k}}\,, \tag{47}$$

where

$$eW_p\equiv\langle V(\vec{r})\rangle_z+\sum_{n,\vec{L}}(E-E_{n,\vec{L}})|n,\vec{L}\rangle\langle n,\vec{L}|\,, \tag{48}$$

which is called *pseudo potential.* Equation (47) is called the *"equation of the pseudo potential"*. This equation is analogous to the well-known Hartree equation which can be solved by iteration method. Pseudo potential is not local, it depends not only on coordinates, but on determining functions. But it has more advantages than deficiencies. The main convenience of method is that pseudo wave functions and pseudo potential can be multi-definited. If some combination of the channeling functions is added to the pseudo wave function, then the new pseudo wave function obtained by this way leads to the same true wave function. So we have a possibility to optimize pseudo potential. Solving equation (47) we obtain the system of equations like (33) for amplitudes $A_{\vec{g}}$ of the pseudo wave function where instead of Fourier components of the crystal potential we have Fourier components of the pseudo potential. Pseudo potential is always less and smoother than real crystal potential. Therefore solving the system approximately we can limit it by less number of equations. Using the property of multi-variant OPW-definition it is possible to add to it the linear combination of the channeling functions which makes the minimum of the functional

$$\frac{\int d^2\rho|\vec{\nabla}\chi|^2}{\int d^2\rho|\chi|^2}\,, \tag{49}$$

i.e. it makes maximum smooth pseudo potential. This method makes it possible to minimize the number of equations (like (33)) for amplitudes $\chi_{\vec{k}}$ [3].

3.6. Complete solutions of Maxwell equation (3)

3.6.1. *The derivation of the set of the dynamical equations*

At first let us assume the frequency of the electromagnetic field to be larger than the atomic and nuclear resonance frequencies $\omega\gg\omega_R$. In that case the propagation of the electromagnetic field inside the crystal can be described as a purely elastic Waller scattering process. So we will be interested in solution of the equation (20).

Inside the crystal matter vector potential $\vec{A}_\omega(\vec{r})$ we can represent in the form:

$$\vec{A}_\omega(\vec{r})=\sum_{h,\lambda}a_{h,\lambda}(z)\vec{w}_h^\lambda(\vec{r})\,, \tag{50}$$

where $a_{h,\lambda}(z)$ are slowly varying functions of coordinate z and $\vec{w}_h^\lambda(\vec{r})$ are transverse plane waves

$$\vec{w}_h^\lambda(\vec{r})=\frac{1}{\sqrt{V}}\vec{e}_h^{\,\lambda}\exp\left(i\vec{k}_h\cdot\vec{r}\right)\,. \tag{51}$$

Vectors $\vec{e}_h^{\,\lambda}$ are polarization vectors, and indexes $\lambda=1,2$ correspond to the two different transverse photon polarizations. Wave functions $\vec{w}_h^\lambda$ must satisfy Bloch's conditions. It means that the monoharmonic electromagnetic waves in a crystal matter must be represented as a package of plane waves with discrete multitude of wave vectors, and the difference between any couple of wave vectors is equal to the arbitrary reciprocal lattice vector:

$$\vec{k}_h=\vec{k}+\vec{g}_h; \qquad \left(\vec{e}_h^{\,\lambda},\vec{k}_h\right)=0\,. \tag{52}$$

Let us remember that reciprocal lattice vectors are defined as

$$\vec{g}_h=\vec{g}_{h_1h_2h_3}=\vec{b}_1h_1+\vec{b}_2h_2+\vec{b}_3h_3,$$

where integer numbers $h_j=0;\pm1;\pm2;...$. Wave functions (51) must satisfy the condition of normalization as well:

$$\int_V\vec{w}_h^{*\lambda}(\vec{r})\vec{w}_{h'}^{\lambda'}(\vec{r})d^3r=\delta_{h,h'}\delta_{\lambda,\lambda'}\,. \tag{53}$$

Substituting (50) into (20) and taking into account that amplitudes $a_{h,\lambda}(z)$ of waves $\vec{w}_h^\lambda(\vec{r})$ are slowly changing functions comparing with their phases, one can obtain the following set of the dynamical equations:

$$\begin{aligned}&\left(2i\vec{k}_h\vec{\nabla}_z+\kappa^2-\vec{k}_h^2\right)a_{h,\lambda}(z)= \qquad (54)\\ &=4\pi\alpha\sum_{h',\lambda'}F\left(\vec{k}_h-\vec{k}_{h'}\right)\left(\vec{e}_h^{\,\lambda},\vec{e}_{h'}^{\,\lambda'}\right)a_{h',\lambda'}(z)\,,\end{aligned}$$

where $\alpha\approx1/137$, $\kappa^2=\varepsilon_0\omega^2/c^2$, ε_0 is the crystal permittivity, and

$$F\left(\vec{k}_h-\vec{k}_{h'}\right)=\frac{1}{V_c}\int_{V_c}\varrho(\vec{r})\exp\left[i\left(\vec{k}_h-\vec{k}_{h'}\right)\cdot\vec{r}\right]d^3r \tag{55}$$

is the Fourier inversion of the electron density distribution inside the simple crystal cell. Expression (55) can be presented as a product of the atomic form-factor, structure-factor and thermal-factor. Equations (54) are the Bethe equations analogous to (33).

3.6.2. *Solution the set of the dynamical equations, if Born approximations can be applied*

First of all let's obtain solution the set of dynamical equations in the case, when the crystal thickness is small enough. In that case $a_{0,\lambda}$ is almost constant, and the perturbation theory may be applied. We can suppose that we have one big amplitude $a_{0,\lambda}$ and small all other amplitudes $a_{h,\lambda}$. In this case we have the following solution:

$$a_{h,\lambda} = \frac{4\pi\alpha F\left(\vec{k}_h - \vec{k}_0\right)\left(\vec{e}_h^{\lambda}, \vec{e}_0^{\lambda'}\right)}{\varepsilon_0\omega^2/c^2 - \vec{k}_h^2} a_{0,\lambda'} \,. \qquad (56)$$

Substituting (56) into (50), we obtain solution of the wave equation (20) in the first Born approximation.

3.6.3. *Solution the set of the dynamical equations in the case of strong diffraction Bragg scattering*

Let us suppose the photon energy to be in the X-rays region $E_\gamma < 200keV$, i.e. the photon wave vector $\vec{k}$ to be order of the reciprocal lattice vector $\vec{g}$ but $|\vec{k}| > |\vec{g}|$. Let us suppose that a strong diffraction Bragg scattering can arise under a proper direction of incident wave. If the crystal is sufficiently thick and its perfection is high enough, then scattering wave amplitude can come up to the amplitude of the incident wave (because we have the case of small denominator $\varepsilon_0\omega^2/c^2 - \vec{k}_h^2$ in the expression (56)). Small number of vectors $\vec{g}$ satisfies this condition, so one can suppose that under a proper direction of incident wave the denominator is small for only one vector $\vec{g}$. So only two large amplitudes are possible, and the solution of (20) can be found in *two-wave approximation.* Let us designate the wave vectors of these waves as:

$$\begin{array}{lll} \vec{k}_1 = \vec{k}_0 + \eta\vec{n} & for & a\ refractive\ wave, \\ \vec{k}_2 = \vec{k}_0 + \vec{g}_h + \eta\vec{n} & for & the\ reflective\ wave, \end{array} \qquad (57)$$

where vector $\vec{n}$ is the unit vector perpendicular to the crystal plane surface, which is the boundary crystal-vacuum (i.e. $\vec{r}\vec{n} = z$). It's convenient to use such polarization vectors:

$$\vec{e}_1^{\,\sigma} = \vec{e}_2^{\,\sigma} = \frac{[\vec{k}_1, \vec{k}_2]}{|[\vec{k}_1, \vec{k}_2]|} \,, \qquad (58)$$

which describe σ-polarized quanta and π-polarized quanta:

$$\vec{e}_1^{\,\pi} = \frac{[\vec{k}_1, \vec{e}_1^{\,\sigma}]}{|\vec{k}_1|} \,, \qquad \vec{e}_2^{\,\pi'} = \frac{[\vec{k}_2, \vec{e}_2^{\,\sigma}]}{|\vec{k}_2|} \,. \qquad (59)$$

Each index λ and λ' can take significance σ or π. Solution the set of the dynamical equations one can search in the form:

$$a_{j,\lambda}(z) = b_{j,\lambda}\exp(i\eta z)$$

for both refractive and reflective waves ($j = 1, 2$). It's convenient to introduce the following designations:

$$\Delta_g = \frac{(-2\vec{k}_0 + \vec{g}_h)\cdot\vec{g}_h}{4\vec{k}_0\vec{n}} \,, \quad F_0 = \frac{4\pi\alpha F(0)}{4\vec{k}_0\vec{n}} \,,$$

$$F_g^{\lambda\lambda'} = \frac{4\pi\alpha F(\vec{g}_h)\left(\vec{e}_1^{\lambda}, \vec{e}_2^{\lambda'}\right)}{4\vec{k}_0\vec{n}} \,, \qquad (60)$$

where $F(0)$ and $F(\vec{g}_h)$ are significances of Fourier inversion of the atomic electron density distribution. Then the system of differential equations (54) leads to such system of algebraic equations:

$$\begin{aligned} (\eta + F_0)b_{1,\lambda} + F_g^{\lambda\lambda'} b_{2,\lambda'} &= 0, \\ F_g^{\lambda'\lambda} b_{1,\lambda} + (\eta + \Delta_g + F_0)b_{2,\lambda'} &= 0 \,. \end{aligned} \qquad (61)$$

Due to $F_g^{\sigma\pi'} = 0$ and $F_g^{\pi\sigma} = 0$, the set of four equations splits up into two independent subsystems of equations for photons with σ-polarization and π-polarization.[1] They have solutions if

$$\det\begin{vmatrix} \eta + F_0 & F_g \\ F_g & \eta + \Delta_g + F_0 \end{vmatrix} = 0 \,, \qquad (62)$$

where $F_g = F_g^{\sigma\sigma}$ stands for σ-polarization, and $F_g = F_g^{\pi\pi'} = F_g^{\pi'\pi}$ stands for π-polarization. The last equation has solution if parameter η satisfies the equality

$$\eta^{(\pm)} = -F_0 - \frac{\Delta_g}{2} \pm \sqrt{\frac{\Delta_g^2}{4} + F_g^2} \,. \qquad (63)$$

By the way, we can note that an important feature of result (63): *due to* $F_0 > F_g$ *both values* $\eta^{(\pm)} < 0$ *at any value of* Δ_g. Then in two-wave approximation one can write the solution of the wave equation (20) in the form of wave superposition

$$\begin{aligned} \vec{A}_\omega(\vec{r}) \;=\; & \sum_{s=1}^{4} C_s \left[\vec{e}_1^{\,(s)} \exp\left(i\vec{k}_1\vec{r}\right) + \right. \\ & \left. + \alpha_s \vec{e}_2^{\,(s)} \exp\left(i\vec{k}_2\vec{r}\right) + c.c.\right] \,, \end{aligned} \qquad (64)$$

and the amplitude of the reflected wave is

$$\alpha_s = \frac{F_g^{(s)}}{\Delta_g/2 \pm \sqrt{\Delta_g^2/4 + F_g^{(s)\,2}}}, \qquad s = (1;2;3;4) \,. \qquad (65)$$

We have four possible values of α_s because of two possible photon polarizations and two possible solutions of equation (63). The solution (64) of the wave equation (20) can be satisfied to the boundary conditions by the proper selection of the constants C_s. For the separated photon polarization the boundary conditions look like:

$$\sum_{s=1,2} C_s = 1, \qquad \sum_{s=1,2} C_s\alpha_s = 0 \,. \qquad (66)$$

Remember that $\vec{k}_1 = \vec{k}_0 + \eta\vec{n}$ where: $\vec{k}_0$ is a wave vector of X-ray wave (in an emptiness) outgoing from the crystal, $\vec{k}$ is a wave vector of X-ray wave inside the crystal and vector $\eta\vec{n}$ is a small vector, which is caused by the photon refraction on the border crystal-emptiness. Unit vector $\vec{n}$ is directed from the crystal to the emptiness and $\eta^{(\pm)} < 0$.

---

[1] The coefficients F_0 and $F_g^{\lambda\lambda'}$ in the set of equations (61) are in fact proportional to the coherent elastic scattering amplitudes of gamma quanta in the crystal cell.

In Fig.2a,b it is shown the functional connection between wave vectors of the radiated photon wave $\vec{k}_0$ and the couple of waves: refractive $\vec{k}_1 = \vec{k} = \vec{k}_0 + \eta\vec{n}$ and reflective $\vec{k}_2 = \vec{k}_h = \vec{k}_0 + \vec{g}_h + \eta\vec{n}$.

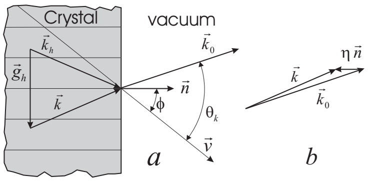

Fig.2. *Boundary conditions*

Such picture shows, that in the present model the refractive index of the crystal is $n < 1$ (see Fig.2b).

3.6.4. *The general set of dynamical equations*

As it was shown in [11, 12], current density inside the crystal matter $\vec{j}_t(\vec{r},t)$ in the general case (see Eq.3) has linear, local and synchronous dependence from the vector-potential $\vec{A}(\vec{r},t)$. *Linear dependence* stands for electromagnetic field, which is much less than internal electric field inside the crystal matter. *Locality* means the current density to be a response to the action of the field $\vec{A}(\vec{r},t)$ in the same point of the physical space. Locality exists if the wave length of the external field $\lambda \gg a_s$, where a_s is the dimension of the scattering atom (or nucleus). This condition coincides with condition of the dipole approximation applicability in the task of photon scattering. Linearity and locality can be extended on the sufficiently general case of the current-field dependence. But it's necessary to replace *synchronous* dependence by the *causative* dependence. It means that only values of $\vec{A}(\vec{r},t')$ at the times $t' < t$ determine the current density $\vec{j}_t(\vec{r},t)$ at the time t. So we can write the sufficiently general expression [11, 12]:

$$\vec{j}_t(\vec{r},t) = \int_{-\infty}^{t} \hat{G}(\vec{r},t-t') \star \vec{A}(\vec{r},t')dt' , \qquad (67)$$

where $\hat{G}(\vec{r},t-t')$ is *causative operator*. Representing $\vec{A}(\vec{r},t')$ in the form of the Fourier integral

$$\vec{A}(\vec{r},t') = \frac{1}{2\pi}\sum_{\lambda=1}^{2} \vec{e}^{\lambda} \int_0^{\infty} d\omega\ a_\lambda(\vec{r},\omega)\ e^{-i\omega t'} , \qquad (68)$$

and substituting it into (67), we obtain

$$\vec{j}_t(\vec{r},t) = \frac{1}{2\pi}\sum_{\lambda=1}^{2}\int_0^{\infty} d\omega e^{-i\omega t} a_\lambda(\vec{r},\omega) \times$$

$$\times \int_{-\infty}^{0} d\tau e^{-i\omega\tau} \hat{G}(\vec{r},\tau) \star \vec{e}^{\lambda} . \qquad (69)$$

Expression of the projection the Fourier component of the current density $\vec{j}_t(\vec{r},\omega)$ on the polarization vector $\vec{e}^{\lambda}$ is:

$$j_t^{\lambda}(\vec{r},\omega) = \sum_{\lambda'=1}^{2} G^{\lambda\lambda'}(\vec{r},\omega) \cdot a_{\lambda'}(\vec{r},\omega) , \qquad (70)$$

where

$$G^{\lambda\lambda'}(\vec{r},\omega) = \int_{-\infty}^{0} d\tau e^{-i\omega\tau} \vec{e}^{\lambda} \star \hat{G}(\vec{r},\tau) \star \vec{e}^{\lambda'} .$$

Evidently, that the condition of the translational symmetry must be added:

$$G^{\lambda\lambda'}(\vec{r},\omega) = G^{\lambda\lambda'}(\vec{r}+\vec{L},\omega) ,$$

where $\vec{L}$ is the lattice vector. Then in the two-wave approximation we can write the following system of the dynamical equations:

$$\left(2i\vec{k}_h\vec{\nabla}_z + \kappa^2 - \vec{k}_h^2\right) a_\lambda(z,\vec{k}_h,\omega) =$$

$$4\pi \sum_{h',\lambda'} \tilde{G}^{\lambda\lambda'}(\vec{k}_h - \vec{k}_{h'},\omega) a_{\lambda'}(z,\vec{k}_{h'},\omega) . \qquad (71)$$

Equation (71) coincides with equation (54) if

$$\tilde{G}^{\lambda\lambda'}(\vec{k}_h - \vec{k}_{h'},\omega) \equiv \alpha F(\vec{k}_h - \vec{k}_{h'})(\vec{e}_h^{\lambda}, \vec{e}_{h'}^{\lambda'}) ,$$

where on the right we have the matrix element of the Reyleigh elastic photon scattering by atomic electrons localized in the crystal cell. It consists of four multipliers:

$S(\vec{g}) = \sum_j e^{i\vec{g}\vec{r}_j}$	↔	*structure factor,*
$e^{-W} = e^{-\bar{u^2}g^2}$	↔	*Debye factor,*
$F_{at}(\vec{g}) = \int d^3r V_{at}(\vec{r})e^{i\vec{g}\vec{r}}$	↔	*atomic formfactor,*
$p^{\lambda\lambda'} = (\vec{e}_h^{\lambda}, \vec{e}_{h'}^{\lambda'})$	↔	*polarization factor.*

One can see that lattice vibrations were taken into account in the zero-phonon approximation. Evidently that polarization factors $p^{\sigma\pi} = p^{\pi\sigma} = p^{\sigma\pi'} = p^{\pi'\sigma} = 0$, and in this case the set of dynamical equations splits up into two independent subsets for σ- and π-polarized photons. The product $F_{at}(\vec{g})p^{\lambda\lambda'}$ is the matrix element $M^{\lambda\lambda'}(\vec{g})$ of the Thompson photon scattering on the single atom. All other processes such as: resonance elastic scattering on atoms and nuclei, photo-effect, Compton scattering, inelastic nuclear reactions can be taken into account by summation of corresponding matrix elements in the expression of the *causative matrix* $\tilde{G}^{\lambda\lambda'}$.

Now we will consider the γ-quanta propagating in the crystal matter in the case of γ-quanta energy close to that of a nuclear transition. We assume that crystal includes Mössbauer nuclei with resonance energy $\hbar\omega_0$ and we are interested in propagation photons with energies close to $\hbar\omega_0$.

We can write:

$$\tilde{G}^{\lambda\lambda'}(\vec{k}_h - \vec{k}_{h'},\omega) =$$

$$= \sum_j e^{i(\vec{k}_h - \vec{k}_{h'})\vec{r}_j} \left\{ e^{-\bar{u^2}(\vec{k}_h - \vec{k}_{h'})^2} \sum_p \left(M_{hh'}^{\lambda\lambda'}\right)_p^{(el)} + \right.$$

$$\left. + e^{-\bar{u^2}(\vec{k}_h^2 + \vec{k}_{h'}^2)} \sum_q C_j \left(M_{hh'}^{\lambda\lambda'}\right)_q^{(nucl)} \right\} ,$$

$$(72)$$

where $e^{-\bar{u^2}(\vec{k}_h^2+\vec{k}_{h'}^2)}$ is Mössbauer factor; $\left(M_{hh'}^{\lambda\lambda'}\right)_p^{(el)}$ and $\left(M_{hh'}^{\lambda\lambda'}\right)_q^{(nucl)}$ are amplitudes of atomic and nuclear photon scattering, respectively; constant $C_j = 1$ in the point $\vec{r}_j$ the Mössbauer nucleus exists and $C_j = 0$, if it is absent. In the general case both atomic electrons and nuclei will contribute to the current (see Eq.3).

But in the compound crystal we can investigate pure nuclear photon scattering by choosing such proper reciprocal lattice vector with structure factor $\sum_j e^{i(\vec{k}_h-\vec{k}_{h'})\vec{r}_j} = 0$ for photon scattering on the electron subsystem and $\sum_j C_j e^{i(\vec{k}_h-\vec{k}_{h'})\vec{r}_j} \neq 0$ for photon scattering on the Mössbauer nuclei. Introducing new designations for simplicity

$$Q_{hh'}^{\lambda\lambda'} = \frac{\pi\tilde{G}_{hh'}^{\lambda\lambda'}}{(\vec{k}_0,\vec{n})}, \qquad a_{h\lambda}(z) = e^{i\eta z} b_{h\lambda}, \qquad (73)$$

we obtain the general set of the dynamical equations in two-wave approximation

$$\begin{aligned}
-\eta b_1^{\sigma} &= Q_{11}^{\sigma\sigma} b_1^{\sigma} + Q_{12}^{\sigma\sigma} b_2^{\sigma} + Q_{11}^{\sigma\pi} b_1^{\pi} + Q_{12}^{\sigma\pi'} b_2^{\pi'}, \\
-(\eta+\Delta) b_2^{\sigma} &= Q_{21}^{\sigma\sigma} b_1^{\sigma} + Q_{22}^{\sigma\sigma} b_2^{\sigma} + Q_{21}^{\sigma\pi} b_1^{\pi} + Q_{22}^{\sigma\pi'} b_2^{\pi'}, \\
-\eta b_1^{\pi} &= Q_{11}^{\pi\sigma} b_1^{\sigma} + Q_{12}^{\pi\sigma} b_2^{\sigma} + Q_{11}^{\pi\pi} b_1^{\pi} + Q_{12}^{\pi\pi'} b_2^{\pi'}, \\
-(\eta+\Delta) b_2^{\pi'} &= Q_{21}^{\pi'\sigma} b_1^{\sigma} + Q_{22}^{\pi'\sigma} b_2^{\sigma} + Q_{21}^{\pi'\pi} b_1^{\pi} + Q_{22}^{\pi'\pi'} b_2^{\pi'},
\end{aligned}$$

$$(74)$$

which has solution if:

$$\det \begin{vmatrix} \eta + Q_{11}^{\sigma\sigma} & Q_{12}^{\sigma\sigma} & Q_{11}^{\sigma\pi} & Q_{12}^{\sigma\pi'} \\ Q_{21}^{\sigma\sigma} & \eta+\Delta+Q_{22}^{\sigma\sigma} & Q_{21}^{\sigma\pi} & Q_{22}^{\sigma\pi'} \\ Q_{11}^{\pi\sigma} & Q_{12}^{\pi\sigma} & \eta + Q_{11}^{\pi\pi} & Q_{12}^{\pi\pi'} \\ Q_{21}^{\pi'\sigma} & Q_{22}^{\pi'\sigma} & Q_{21}^{\pi'\pi} & \eta+\Delta+Q_{22}^{\pi'\pi'} \end{vmatrix} = 0. \qquad (75)$$

Equation (75) in the general case has four solutions η_m, $(m = 1,2,3,4)$. So the electromagnetic field inside the crystal matter can be presented in the form:

$$\vec{A}(\omega,\vec{r}) = \sum_{m=1}^{4}\sum_{s=1}^{2} C_s^{(m)} \left[\vec{e}_1^{\,s} \exp\left(i\vec{k}_1^{(m)}\vec{r}\right) + \right. \\ \left. +\alpha_s^{(m)}\vec{e}_2^{\,s} \exp\left(i\vec{k}_2^{(m)}\vec{r}\right) + c.c.\right], \qquad (76)$$

where index s denotes the photon polarization. Finally we must take into account the boundary condition, which looks like:

$$\sum_{m=1}^{4} C_s^{(m)} = 1, \qquad \sum_{m=1}^{4} C_s^{(m)}\alpha_s^{(m)} = 0 \qquad (77)$$

and similar to (66) (see Fig.2).

4. ALL TYPES OF COHERENT RADIATIONS

4.1. Total coherent bremsstrahlung in Born approximation

The total matrix element of the coherent bremsstrahlung of relativistic charged particles in the lowest Born approximation can be obtained in the form:

$$M_{i\to f}^{CR} = \frac{ie}{mc^2\gamma}\left[M_1^{BS} + M_2^{BS} + M^{PR}\right], \qquad (78)$$

where two of three wave functions are chosen in the zero Born approximation, and one of three in the first Born approximation:

$$M_1^{BS} = \int d^3r \phi_f^{(1)*}(\vec{r})\hat{\vec{p}}\cdot\vec{A}^{(0)*}(\vec{r})\phi_i^{(0)}(\vec{r}), \qquad (79)$$

$$M_2^{BS} = \int d^3r \phi_f^{(0)*}(\vec{r})\hat{\vec{p}}\cdot\vec{A}^{(0)*}(\vec{r})\phi_i^{(1)}(\vec{r}), \qquad (80)$$

$$M^{PR} = \int d^3r \phi_f^{(0)*}(\vec{r})\hat{\vec{p}}\cdot\vec{A}^{(1)*}(\vec{r})\phi_i^{(0)}(\vec{r}). \qquad (81)$$

Matrix elements M_1^{BS} and M_2^{BS} describe bremsstrahlung emitted by a relativistic particle, and M^{PR} bremsstrahlung emitted by a crystal. So as

$$\phi_{i,f}^{(1)}(\vec{r}) = \int d^3r' G(\vec{r},\vec{r}')V_{cr}(\vec{r}')\phi_{i,f}^{(0)}(\vec{r}'),$$

one can see: in case of M_1^{BS} relativistic electron radiates, and then crystal gets recoil momentum; in case of M_2^{BS} crystal gets recoil momentum, and then relativistic particle radiates. So as the differential cross section of radiation process (12) $d\sigma \sim \delta(E_i - E_f - \hbar\omega)$, the law of energy conservation in our case looks like $E_i = E_f + \hbar\omega$. It means, that the crystal target has the infinitely large mass $M_{cr} \to \infty$, and therefore the energy received by crystal is $E_{cr} = q^2/2M_{cr} \to 0$, where $\vec{q}$ is the recoil momentum. Taking into account the law of energy conservation we have the following identity: $K_i^2 - K_f^2 - k^2 \equiv 0$. Using this identity, we can represent the denominators in the expressions for M_1^{BS} and M_2^{BS} in the following form: $(\vec{K}_f + \vec{k})^2 - K_i^2 = -(2K_f k - 2\vec{K}_f\vec{k})$ and $(\vec{K}_i - \vec{k})^2 - K_f^2 = 2K_i k - 2\vec{K}_i\vec{k}$. Besides, the product

$\vec{e}_{k,\lambda} \cdot \vec{k} = 0$. Finally we have:

$$M_{i\to f}^{BS} = \frac{ie}{mc^2\gamma}\left[M_2^{BS} + M_1^{BS}\right] =$$
$$\frac{-ie\hbar}{mc^2\gamma}F_{cr}(\vec{q})\frac{\vec{e}_{k,\lambda}}{2k}\cdot\left\{\frac{\hat{v}_i}{1-\hat{v}_i\cdot\hat{k}} - \frac{\hat{v}_f}{1-\hat{v}_f\cdot\hat{k}}\right\}\,, \quad (82)$$

where $\vec{q} = \hbar(\vec{K}_i - \vec{K}_f - \vec{k})$ is the recoil momentum, $F_{cr}(\vec{q})$ is the Fourier transformation of the crystal potential, and $\hat{v}_i, \hat{v}_f, \hat{k}$ are unit vectors $\hat{v}_i = \vec{K}_i/K_i$, $\hat{v}_f = \vec{K}_f/K_f$, $\hat{k} = \vec{k}/k$, respectively.

$$\vec{A}_{\vec{k},\lambda}^{(1)*}(\vec{r}) = \int d^3r'\hat{G}^*_{\lambda',\lambda}(\vec{r},\vec{r}')V'_{cr}(\vec{r}')\vec{A}_{\vec{k},\lambda}^{(0)*}(\vec{r}')\,, \quad (83)$$

$$\hat{G}^*_{\lambda',\lambda}(\vec{r},\vec{r}') = \frac{1}{(2\pi)^3}\int d^3k'\frac{(\vec{e}_{k',\lambda'}\cdot\vec{e}_{k,\lambda})e^{i\vec{k}'(\vec{r}-\vec{r}')}}{\varepsilon_0\omega^2/c^2 - k'^2}\,. \quad (84)$$

$$M^{PR} = \hbar F'_{cr}(\vec{K}_i - \vec{K}_f - \vec{k}) \times$$
$$\times\sum_{\lambda'}\frac{(\vec{e}_{\vec{K}_f-\vec{K}_i,\lambda'}\cdot\vec{K}_i)(\vec{e}_{\vec{K}_f-\vec{K}_i,\lambda'}\cdot\vec{e}_{k,\lambda})}{\varepsilon_0\omega^2/c^2 - (\vec{K}_f - \vec{K}_i)^2}\,. \quad (85)$$

We obtain the cross section of total radiation, which arises in the collision of relativistic electron and crystal, using expressions (82) and (85) in the expression for total matrix element $M_{i\to f}^{CR}$ (78) and substituting $M_{i\to f}^{CR}$ into expression for cross section (12). This expression differs from the cross section for a single atom on the multiplier, which is well known as a diffraction factor:

$$\left|\sum_{\vec{L}} e^{-i\vec{q}\vec{R}_{\vec{L}}}\right|^2\,, \quad (86)$$

where vectors $\vec{R}_{\vec{L}}$ denote the relative coordinates of the crystal atom, and $\vec{L}$ are lattice vectors. Taking into account expression (30) (and the following crystal description), one can see, that expression (86) is proportional to the multiplier $D(\vec{q})$, which includes δ-function $\delta(\vec{q}-\vec{g})_t$. After averaging the cross section over the thermal displacements of the crystal atoms from their equilibrium positions, one can select the coherent part of the cross section and integrate it over d^3p_f using 3 δ-functions. In this way we obtain spectral-angular distribution of radiated photons, which has extremely sharp maximum (see (30)). In the final result we transit to the limit $N_1 \to \infty$ (N_1 is the number of crystal periods in the direction of the fast electron motion). Thus the final expression for spectral-angular distribution of radiated photons in the system of units $\hbar = c = 1$ looks like:

$$\left(\frac{d^3\sigma}{d\omega d\Omega}\right)_{coh} = \frac{8\pi\omega}{V(1-\vec{v}\vec{n}_k)}\times$$
$$\times\sum_{\vec{g},\lambda}\left|M_{CR}\right|^2 S^2(\vec{g})e^{-g^2\overline{u^2}_T}\delta\left(\omega - \frac{\vec{g}\vec{v}}{1-\vec{v}\vec{n}_k}\right)\,, \quad (87)$$

where $\vec{k}$ and ω are photon momentum and photon energy, Ω is the solid angle, $\vec{n}_k = \vec{k}/k$, V is the volume of the crystal elementary cell, $|M_{CR}|^2$ is the total matrix element, $S^2(\vec{g})$ is the crystal structure factor, $e^{-g^2\bar{u}^2}$ is the Debye-Waller factor, $\bar{u}^2$ is the mean square of thermal vibrational amplitude of the crystal atoms, and the summation performed over all reciprocal lattice vectors $\vec{g}$ of the crystal and two photon polarization directions $\vec{e}_{k\lambda}$. The module square of the total matrix element in (87) is:

$$\left|M_{i\to f}^{CR}\right|^2 = \frac{e^6}{\omega^2m^2}\left|\vec{e}_{k\lambda}\left\{\frac{Z-F(\vec{g})}{\gamma g^2}\frac{\vec{g}}{1-\vec{v}\vec{n}_k} + \right.\right.$$
$$\left.\left. +F(\vec{g})\frac{\vec{v}\omega - \vec{g}}{(\vec{g}+\vec{k})^2 - k^2}\right\}\right|^2\,, \quad (88)$$

where Z and $F(\vec{g})$ are atomic number and atomic form-factor of the crystal atom. The first part of the right hand side of Eq.88 corresponds to the *coherent bremsstrahlung of the charged particle (CBS)* and the second part corresponds to the *coherent polarization radiation (CPR)*. Expressions (87) and (88) coincide with the result of [19] (see formulae Eq.21, Eq.22 of [19]). In the case if angle of radiation $\theta_k = \angle(\vec{v},\vec{k}) \le 1/\gamma$, then the first part of Eq.88 becomes dominant and Eqs.87 and 88 describe exactly CBS in the low photon energy approximation. On the other hand, if $\theta_k \gg 1/\gamma$, the contribution of CBS becomes small and can be neglected, and CPR can be observed separately. It's evident, that exists such radiation angle θ_k, that both matrix elements of CBS and CPR give equal contributions (see formula 88). In this case we can observe the maximum effect of interference of these two types coherent radiations.

4.2. Effect of the dynamical diffraction

Formulae (87) and (88), obtained for CPR in the first Born approximation are called as "kinematical theory" [16]. For low electron energies, i.e. if $1/(\beta\gamma)^2 \gg |\chi|$ with χ being the electric susceptibility of the crystal, the influence of the crystal dielectric properties can be neglected (i.e. we can accept $\varepsilon_0 = 1$). It turns out that the maximum of the intensity of CPR increases proportional to $\sim\gamma^2$ (Fig.3, function number 1).

Within the framework of the kinematical theory of CPR the influence of the crystal dielectric properties can be taken into account by changing the dispersion relation for the radiated photon to $k^2 = \varepsilon_0\omega^2/c^2$, where $\varepsilon_0 = 1+\chi$. It turns out that at high electron energies the maximum of the intensity of CPR is limited by χ and strives to the constant quantity with increasing γ (see Eq.23 in [19] and Fig.3, function number 2).

For taking into account so-called dynamical effect, we must use expression of $\vec{A}_\omega(\vec{r})$ in the form of (64) in the matrix element (13) and cross section (12). It turns out that in the expressions (87) and (88) instead of vector $\vec{g}$ we must use the sum $\vec{g}+\vec{n}\eta$, where value $\vec{g}\vec{n}=0$, $|\vec{n}|=1$, and $\eta \ll g$. In Fig.3 it's shown the relative peak intensity of CPR as a function of electron Lorentz factor γ [18]. The electron multiply scattering and photon absorbtion are taking into account.

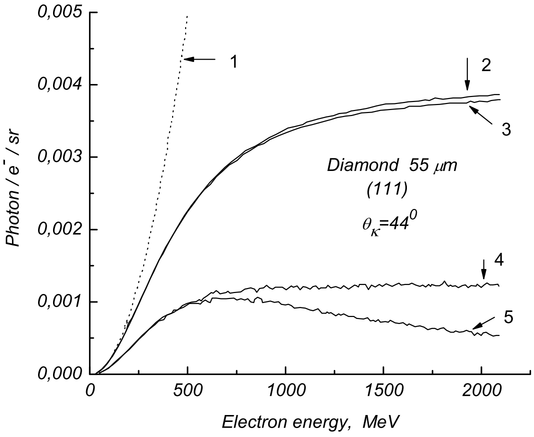

Fig.3. *Intensity of CPR (or PXR(B)) as a function of electron energy calculated: 1-using kinematical theory for the case of low electron energy and the influence of the crystal dielectric properties is neglected; 2-for the maximum intensity, i.e for angle between electron velocity direction and crystal plane* $\phi = \theta_k/2 - 1/\gamma$, *using kinematical theory; 3-the same conditions as for 2, and using dynamical theory; 4-for the minimum intensity, i.e for angle* $\phi = \theta_k/2 - 1/2\theta\gamma^2$, *using kinematical theory formula; 5-the same conditions as for 4,and using dynamical theory*

It is shown, that at low electron energy dynamical effect is negligibly small. At high energy, even in the limit $\gamma \to \infty$ dynamical effect in the maximum of CPR intensity is small too, and it is of order of several per cents (see difference between functions number 2 and number 3 in Fig.3).

Analogous result was obtained by H. Nitta in [21] too. Thus the kinematical theory of CPR with taking into account the influence of the crystal dielectric properties (see [16]-[21]) is a good approach to the reality.

Dynamical effect becomes considerable only in minimum of CPR intensity, i.e for angle $\phi \approx \theta_k/2 = \theta_B$, where θ_B is Bragg angle (see difference between functions number 4 and number 5 in Fig.3) [18]. Such considerable difference between functions 4 and 5 is due to effective redistribution of the CPR intensity between reflective and refractive waves at the radiation angles, which satisfy the exact Bragg conditions. However such case is not of interest for creation the sources of coherent radiations.

4.3. Channeling radiation

Planar channeling electron wave function has the form:

$$\psi_n(\vec{r}) = \sqrt{\frac{E+mc^2}{2EL^3}} \begin{pmatrix} \vec{u}_n \\ \frac{c\hat{\vec{\sigma}}\hat{\vec{p}}}{E+mc^2}\vec{u}_n \end{pmatrix} \phi_n(r_t)e^{i\vec{K}_l\vec{r}_l}\,, \tag{89}$$

where $\vec{u}_n$ is two-component spinors, $\phi_n(r_t)$ is the wave function of the transverse motion, $\sqrt{E+mc^2/2E}e^{i\vec{K}_l\vec{r}_l}$ is wave function of longitudinal motion, r_t and $\vec{r}_l$ are transverse and longitudinal components of vector $\vec{r}$, and the designation $\vec{K}=\vec{p}/\hbar$ is used. The energy of the channeling radiation $\hbar\omega \ll E_i$, therefore we can assume $E_i \approx E_f = E$, and formulae (15), (12) can be applied for obtaining cross section of channeling radiation.

Therefore matrix element is:

$$\begin{aligned} M^{Ch}_{i\to f} &= \frac{ie}{mc^2\gamma}\int e^{-i\vec{k}\cdot\vec{r}}\phi^*_f(r_t)e^{-i\vec{K}^{(f)}_l\vec{r}_l} \times \\ &\times \vec{e}_{\vec{k},\lambda}\cdot\hat{\vec{p}}\phi_i(r_t)e^{i\vec{K}^{(i)}_l\vec{r}_l}d^3r = \\ &= \frac{ie}{mc^2\gamma}M^{\lambda}_{i\to f}\,(2\pi)^2\delta(\vec{k}_l+\vec{K}^{(f)}_l-\vec{K}^{(i)}_l)\,, \end{aligned} \tag{90}$$

where:

$$M^{\lambda}_{i\to f} = \int e^{-ik_tr_t}\phi^*_f(r_t)\vec{e}_{\vec{k},\lambda}\cdot(\hat{p}_t+\vec{p}_l)\phi_i(r_t)dr_t\,. \tag{91}$$

The square of the first δ-function is

$$\begin{aligned} &\left|(2\pi)^2\delta\left(\vec{k}_l+\vec{K}^{(f)}_l-\vec{K}^{(i)}_l\right)\right|^2 = \\ &= (2\pi L)^2\delta\left(\vec{k}_l+\vec{K}^{(f)}_l-\vec{K}^{(i)}_l\right)\,, \end{aligned} \tag{92}$$

where L is the length of the box $K_j = (2\pi/L)n_j$, which was used for *normalization in the box* wave function (6).

The Dirac δ-functions in the expressions (12) and (90) reflect the energy and the longitudinal momentum conservation in the radiation process:

$$\begin{aligned} &\hbar\omega + E_f - E_i = 0\,, \\ &\vec{k}_l + \vec{K}^{(f)}_l - \vec{K}^{(i)}_l = 0\,. \end{aligned} \tag{93}$$

Designating the unit vector in the direction of photon emission by $\vec{n}_k$ and the unit vector perpendicular to the channeling plane by $\vec{e}_x$, we can write the wave vector of the charged particle in the final state in the following form:

$$\vec{K}^{(f)}_l = \vec{K}^{(i)}_l - \frac{\omega}{c}\left[\vec{n}_k - (\vec{n}_k\cdot\vec{e}_x)\,\vec{e}_x\right]\,,$$

After integration (12) over the space of longitudinal momentum two first δ-functions vanish. Then taking into account, that the total electron energy E has the following correlation with its longitudinal momentum $\hbar\vec{K}$ and transverse energy $E_\perp$:

$$E^2 = \hbar^2c^2K^2_l + m^2c^4 + 2EE_\perp(E)\,, \tag{94}$$

let's write the explicit expression of the argument of δ-function, which reflects the law of energy conservation. From (94) we have:

$$E = E_{\perp} + \sqrt{\hbar^2 c^2 K_l^2 + m^2 c^4}\,, \tag{95}$$

where the square root can be interpreted as an energy of longitudinal motion. Evidently that for the emission process the following inequality $E_i \gg \hbar\omega$ is satisfied. So the second square root in the expression of E_f, can be written in the form:

$$E_f = E^{(i)}{}_l - \frac{1}{2E^{(i)}{}_l}\left(2\hbar c\omega \vec{p}_i \cdot \vec{n}_k - \right.$$
$$\left. -\hbar^2\omega^2\left[1 - (\vec{n}_k \cdot \vec{e}_x)^2\right]\right) + E^{(f)}{}_{\perp}\,. \tag{96}$$

Introducing designations: $\epsilon_{if} = E^{(i)}{}_{\perp} - E^{(f)}{}_{\perp}$ and $\vec{\beta}_l = \vec{p}_l^{(i)} c / E^{(i)}{}_l$ and neglecting terms less than $\hbar\omega/E$, the latest δ-function in (12) we can write in the form:

$$\delta\left(\hbar\omega(1 - \vec{\beta}_l \cdot \vec{n}_k) - \epsilon_{if}\right) =$$
$$= \frac{1}{1 - \vec{\beta}_l \cdot \vec{n}_k}\,\delta\left(\hbar\omega - \frac{\epsilon_{if}}{1 - \vec{\beta}_l \cdot \vec{n}_k}\right). \tag{97}$$

The probability of transitions during the unit time interval of the emission process $e \to e' + \gamma$ is

$$\frac{dP^{\lambda}_{i\to f}}{dz d(\hbar\omega) d\Omega} = 2\pi e^2 \frac{1}{E^2} \frac{\hbar^2\omega}{1 - \vec{\beta}_l \cdot \vec{n}_k} \left|M^{\lambda}_{fi}\right|^2 \times$$
$$\times \delta\left(\hbar\omega - \frac{\epsilon_{if}}{1 - \vec{\beta}_l \cdot \vec{n}_k}\right), \tag{98}$$

where $M^{\lambda}_{i\to f}$ is defined by Eq.91.

5. CONCLUSIONS

It is shown, that within the framework of the quantum electrodynamics in Furry's notation, all known types of coherent radiations, which arise in the collision of relativistic electron with crystal, can be obtained using the generalized coordinates of fields in the form of the particular solutions of the wave equations (2) and (3).

The origin of different almost monochromatic radiations are caused by the discrete spectra of the transverse recoil momenta or the transverse recoil energies. The transverse recoil momenta are equal to the discrete multitude of reciprocal lattice vectors because the transverse components of relativistic electron wave functions and transverse components of wave functions of radiation field satisfy the Bloch theorem. Heterogeneity of the crystal matter in the longitudinal direction obviously does not play a vital part. Such details were discussed in [16]. The discrete spectrum of relativistic electron energies is caused by the localization of electron function near the separate crystal plane.

Let us pay attention to the conditions of coherent maxima arising. In the case of the ordinary coherent bremsstrahlung and in the case of coherent polarization radiation the condition of coherent maximum is (see formula (87)):

$$\delta\left(\omega - \frac{\vec{g}\vec{v}}{1 - \vec{v}\vec{n}_k}\right). \tag{99}$$

In the case of the channeling radiation the condition of coherent maximum is (see formula (97)):

$$\delta\left(\hbar\omega - \frac{\epsilon_{if}}{1 - \vec{\beta}_l \cdot \vec{n}}\right). \tag{100}$$

These formulae differ from each other only by the discrete values $\epsilon_{if} \leftrightarrow \vec{g}\vec{v}$. In [22] considered the case of CPR generated by the channeling particles, where the condition of coherent maxima arising keeps both these discrete values.

All these types of coherent radiations have the bremsstrahlung nature, so as all matrix elements, which correspond to them, are inversely proportional to the mass of radiating particle (see Eq.(88) and Eq.(90)), and during the radiation process relativistic electron loses the discrete part of transverse momentum or the discrete part of transverse energy.

The condition of the conservation the coherency of radiation, in the main, is the constancy of the relativistic particle velocity $\vec{v}_l$ (see formulae (99) and (100)). Particle velocity changes due to the close collisions with crystal atoms and electrons. The middle length, where particle velocity does not change vital, is the *length of coherency.* The length of coherency can be limited by the: electron multiple scattering, photon absorption, crystal defects. All these factors can be taken into account using Monte-Carlo simulations [18].

Let us emphasize, that matrix element of polarization radiation, which includes a refractive photon wave with wave vector $\vec{k}_1 = \vec{k}_0 + \eta\vec{n}$ (see formulae (57) and (64)), is equal zero because for ordinary crystals parameter η is always $\eta < 0$, and therefore such process is forbidden by the laws of momentum-energy conservation. Only matrix element of polarization radiation, which includes a reflective photon wave with wave vector $\vec{k}_1 = \vec{k}_0 + \vec{g}_h + \eta\vec{n}$ is not equal zero. In Fig.2 it's shown, that on the border of crystal vector $\eta\vec{n}$ is directed from emptiness into the crystal. In this case the refraction factor $n < 1$, and thus the radiation is not a kind of Čerenkov radiation.

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ВСЕ ВИДЫ КОГЕРЕНТНЫХ ИЗЛУЧЕНИЙ, КОТОРЫЕ ВОЗНИКАЮТ ПРИ СТОЛКНОВЕНИИ РЕЛЯТИВИСТСКИХ ЭЛЕКТРОНОВ С КРИСТАЛЛОМ

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Кратко изложена общая теория, которая объясняет все типы когерентных рентгеновского и γ-излучений, возникающих при столкновении релятивистских электронов с кристаллической мишенью. Такие типы излучений, как: обычное когерентное тормозное излучение и излучение при каналировании, испускаемые релятивистским электроном; разные виды когерентных поляризационных излучений, испускаемых атомами и ядрами кристалла, – могут быть описаны в рамках данной теоретической модели.

ВСІ ВИДИ КОГЕРЕНТНИХ ВИПРОМІНЮВАНЬ, ЯКІ ВИНИКАЮТЬ ПРИ ЗІТКНЕННІ РЕЛЯТИВІСТСЬКИХ ЕЛЕКТРОНІВ З КРИСТАЛОМ

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Стисло викладена загальна теорія, яка пояснює всі типи когерентних рентгенівського та γ-випромінювань, які виникають при взаємодії релятивістських електронів з кристалічною мішенню. Такі типи випромінювань, як: звичайне когерентне гальмове випромінення та випромінення при каналюванні, випромінювані релятивістським електроном; різні види когерентних поляризаційних випромінень, джерелами яких є атоми та ядра кристала, – можуть бути описані в рамках даної теоретичної моделі.