Section E. NONLINEAR DYNAMICS

DIRECT AND INVERSE SCATTERING ON THE LINE FOR ONE-DIMENSIONAL SCHRÖDINGER EQUATION

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The direct and inverse scattering problems are studied for one-dimensional Schrödinger equation with a potential, which is asymptotically close to distinct periodic functions on the different half-axes. It is supposed that the background Hill operators have two bands spectra with the coinciding half-infinite band. It is also assumed that the perturbation has the second moment finite. For such a class of potentials the characterization of the scattering data is proposed and the inverse problem is solved.

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1. INTRODUCTION

In physics, in the area of scattering theory, the inverse scattering problem is the problem of determining the characteristics of an object (its shape, internal constitution, etc.) from measurement data of radiation or particles scattered from the object.

In mathematics, inverse scattering refers to the determination of the solutions of a set of differential equations based on known asymptotic solutions, that is, on solving the S-matrix.

One can consider the inverse scattering transform as the generalized Fourier transformation that is usually applied for solving linear problems. The value of the inverse scattering transform is that it essentially allows investigating a nonlinear problem by the methods of linear theory. Today, one of the most interesting applications of the inverse scattering problem is to solve exactly nonlinear evolution equations that many of them are encountered in hydrodynamics, plasma physics, nonlinear optic, and solid state physics, as well as to establish their complete integrability.

The direct problem is to define appropriate scattering data for the potentials in a reasonable class of potentials and to study their properties; the inverse problem is twofold: to give sufficient conditions on candidate scattering data to assure that these data are indeed the scattering data of some potential and to give a method for constructing that potential from that scattering data.

The inverse scattering problem for Sturm-Liouville operator with fast decreasing potential on the whole real line was studied by Kay and Moses in [1-3], Faddeev in [4] and other authors (see ref. [5]). It was well founded for the potential that has the first moment finite by Marchenko in [6]. In 1967, Gardner, Greene, Kruskal and Miura using the direct and inverse scattering problem idea, invented a method of solution for the Korteweg-de Vries equation, Ref. [7]. This method was transformed to the algebraic form by P. Lax in Ref. [8], and was initially applied to the nonlinear evolution equations: Korteweg-de Vries equation, nonlinear Schrödinger equation and sine-Gordon equation.

The inverse scattering problem for the constant step-like background on the full line were studied in [9-10], and its application to the Korteweg-de Vries in [11].

The pioneer work in the inverse problem for the periodic Sturm-Liouville operator was studied by Stankevich in [12]. Another way of looking to the problem was considered by Marchenko in Ref. [6].

The scattering problem on the periodic background, i.e. perturbation of the Hill operator was studied in [13-14].

Inverse scattering problem for the Sturm-Liouville operator whose potential tends to 0 at $+\infty$ and asymptotically periodic at $-\infty$, was considered in [15].

The present work considers the general case when the potential of the one-dimensional Schrödinger operator asymptotically closes to distinct periodic functions on the different half-axes. The results were announced in [16].

2. SUBSTITUTION OF THE PROBLEM

Suppose given self-adjoint Hill operators:

$$H_{\pm} = -\frac{d^2}{dx^2} + p_{\pm}(x), \quad x \in \mathbb{R}, \tag{1}$$

on the real line with the periodic potentials

$p_{\pm}(x+T_{\pm}) = p_{\pm}(x), \ T_{+} \neq T_{-}.$

We also assume that the spectra of these operators have the following mutual location:

$$Spec\left(H_{\pm}\right) := S_{\pm} = \Delta_{\pm} \cup \mathbb{R}_{+}, \tag{2}$$

where

$\Delta_{\pm} = \left[\mu_{\pm}, \nu_{\pm}\right], \ \mathbb{R}_{+} = \left[0, +\infty\right), \ \mu_{-} < \nu_{-} < \mu_{+} < \nu_{+} < 0.$

Let $\lambda_{\pm}$ be the points of auxiliary spectra of the operators $H_{\pm}$, such that

$$\lambda_{\pm} \in \left(\nu_{\pm}, 0\right), \ \lambda_{-} \neq \mu_{+}, \nu_{+}, \lambda_{+}. \tag{3}$$

From now on, the signs "$-$" and "$+$" are related with the background data on the left and right half-axis, respectively.

Let us now consider the one-dimensional Schrödinger operator

$$L=-\frac{d^2}{dx^2}+q(x),\quad x\in\mathbb{R}, \tag{4}$$

whose potential is asymptotically close to the distinct functions $p_\pm(x)$, i.e.

$$\lim_{x\to\pm\infty}\left[q(x)-p_\pm(x)\right]=0,$$

with the following conditions:

$$\left|\int_0^{\pm\infty}\left|q(x)-p_\pm(x)\right|\ (1+x^2)\ dx\right|<\infty. \tag{5}$$

Our main goal is:

a. to give the characterization of the scattering data, that is the necessary and sufficient conditions for reconstruction of the potential by scattering data,

b. to solve the inverse scattering problem for the operator L by means of Marchenko's approach in the special case, when the second moment of perturbation is finite.

The next steps for investigating are: to solve the Cauchy problem for corresponding Korteweg-de Vries equation, to consider the analogous problem and study the analytic behavior of scattering data for more complicated case when the spectra S_+ and S_- have a non-empty intersection exception in the ray $\mathbb{R}_+$, i.e. $\Delta_+\bigcup\Delta_-\neq\varnothing$.

3. DIRECT SCATTERING PROBLEM

We will introduce necessary spectral characterizations of the background operators (1). Let $s_\pm(x,\lambda)$, $c_\pm(x,\lambda)$ be the sin and cos-type solutions for the equations

$$\left[-\frac{d^2}{dx^2}+p_\pm(x)\right]y(x)=\lambda y(x),\quad \lambda\in\mathbb{C}, \tag{6}$$

with the following initial conditions:

$$s_\pm(0,\lambda)=c'_\pm(0,\lambda)=0,\ c_\pm(0,\lambda)=s'_\pm(0,\lambda)=1,$$

where $y'(x,\lambda)=\frac{d}{dx}y(x,\lambda)$. For simplicity of notation, we write $s_\pm(\lambda):=s_\pm(T_\pm,\lambda)$, $c_\pm(\lambda):=c_\pm(T_\pm,\lambda)$. By definition $s_\pm(\lambda_\pm)=0$.

Let $u_\pm(\lambda)=\frac{1}{2}\left[s'_\pm(\lambda)+c_\pm(\lambda)\right]$ be the Hill discriminants and

$$m_\pm(\lambda)=\frac{s'_\pm(\lambda)-c_\pm(\lambda)\mp2\sqrt{u_\pm^2(\lambda)-1}}{2s_\pm(\lambda)}$$

the Weyl functions of the operators $H_\pm$, where

$$sign\ \sqrt{u_\pm^2(\lambda)-1}>0,\quad \lambda\to-\infty.$$

The functions $m_\pm(\lambda)$ correspond to the Floquet-Weyl solutions

$$\psi_\pm(x,\lambda)=c_\pm(x,\lambda)+m_\pm(\lambda)s_\pm(x,\lambda),$$

for the equations Eq. (1) such that

$$\psi_\pm(.,\lambda)\in L^2(\mathbb{R}_\pm),\ \lambda\in\mathbb{C}\setminus S_\pm.$$

We shall consider

$$\sigma_\pm(\lambda)=\begin{cases}\lambda-\lambda_\pm,\ \text{if } \lambda_\pm \text{ is a pole of } m_\pm(\lambda),\\ 1,\qquad \text{if } \lambda_\pm \text{ is not a pole of } m_\pm(\lambda).\end{cases}$$

By (3), the functions $\tilde{\psi}_\pm(x,\lambda):=\psi_\pm(x,\lambda)\sigma_\pm(\lambda)$ are holomorphic of λ in the domains $\mathbb{C}\setminus S_\pm$, continuous right up to the boundaries in which connection $\tilde{\psi}_\pm(x,\lambda^{\mathrm{u}})=\overline{\tilde{\psi}_\pm(x,\lambda^{\mathrm{l}})}$, where λ^{u} and λ^{l} lie symmetrically on the upper and lower sides of cuts across the spectra $S_\pm^{\mathrm{u}}$ and $S_\pm^{\mathrm{l}}$, respectively.

Now, we consider the equation

$$[-\frac{d^2}{dx^2}+q(x)]\ y(x)=\lambda y(x),\quad \lambda\in\mathbb{C}, \tag{7}$$

with the potential $q(x)$, satisfying the conditions (5). The Jost functions $\varphi_\pm(x,\lambda)$, are the solutions of Eq. (7) such that

$$\lim_{x\to\pm\infty}\left[\varphi_\pm(x,\lambda)-\psi_\pm(x,\lambda)\right]=0.$$

It is shown in [11] that for all λ belongs to the spectra $S_\pm$, there exist the transformation operators as the Jost solutions can be represented as:

$$\varphi_\pm(x,\lambda)$$
$$=\psi_\pm(x,\lambda)\pm\int_x^{\pm\infty}K_\pm(x,y)\psi_\pm(y,\lambda)dy,\ \pm x\le\pm y. \tag{8}$$

The real-valued and continuous functions $K_\pm(x,y)$ are the kernel of the transformation operators that satisfy the following inequalities:

$$\left|K_\pm(x,y)\right|\le\pm C_\pm(x)\int_{\frac{x+y}{2}}^{\pm\infty}\left|q(t)-p_\pm(t)\right|\ dt, \tag{9}$$

where $C_\pm(x)$ are positive continuous functions and bounded as $x\to\pm\infty$. Moreover,

$$q(t)-p_\pm(t)=\mp\frac{d}{dx}K_\pm(x,x). \tag{10}$$

It is evident that the functions

$$\tilde{\varphi}_\pm(x,\lambda):=\varphi_\pm(x,\lambda)\sigma_\pm(\lambda)$$

are also holomorphic in $\mathbb{C}\setminus S_\pm$, continuous right up to the boundaries and $\tilde{\varphi}_\pm(x,\lambda^{\mathrm{u}})=\overline{\tilde{\varphi}_\pm(x,\lambda^{\mathrm{l}})}$, for $\lambda^{\mathrm{u,l}}\in S_\pm^{\mathrm{u,l}}$, respectively.

We will denote by

$$\omega(\lambda)=\left\langle\varphi_+(x,\lambda),\varphi_-(x,\lambda)\right\rangle,\ \lambda\in\mathbb{C}\setminus(S_+\bigcup S_-)$$

the Wronskian of the Jost solutions and consider the function $W(\lambda)=\omega(\lambda)\sigma(\lambda)$, where

$$\sigma(\lambda)=\sigma_+(\lambda)\sigma_-(\lambda).$$

One can show that the continuous spectrum of the operator L defined by (1)-(5) is

$$Spec_{\mathrm{c}}(L)=S:=\Delta_+\bigcup\Delta_-\bigcup\mathbb{R}_+.$$

The discrete spectrum of L is finite, simple and coincides with the zeros of $W(\lambda)$, that is:

$$Spec_{\mathrm{d}}(L)=\{\lambda_k\}\subset\mathbb{R}\setminus S,\ \ W(\lambda_k)=0,\ k=1,...,p.$$

Put

$$\left(m_k^\pm\right)^2=\int_{\mathbb{R}}\tilde{\varphi}_\pm^2(x,\lambda_k)dx,\quad k=1,...,p. \tag{11}$$

The functions $\varphi_+(x,\lambda)$, $\overline{\varphi_+(x,\lambda)}$ and $\varphi_-(x,\lambda)$, $\overline{\varphi_-(x,\lambda)}$ form fundamental systems of solutions for the Eq. (7) on the sets $S_+^{u,l}$ and $S_-^{u,l}$, respectively. In particular, we have

$$T_\pm(\lambda)\varphi_\mp(x,\lambda)=\overline{\varphi_\pm(x,\lambda)}+R_\pm(\lambda)\varphi_\pm(x,\lambda),\ \lambda\in S_\pm^{u,l}. \quad (12)$$

The transmission and reflection coefficients $T_\pm(\lambda)$ and $R_\pm(\lambda)$ make up the scattering matrix and possess the following properties:

Theorem1.

I. *The functions $T_\pm(\lambda)$ and $R_\pm(\lambda)$ are continuous on the sets $S_\pm^{u,l}$, but at the points $\mu_\pm,\nu_\pm,0,$ and satisfy the following equalities:*

$$T_\pm(\lambda^u)=\overline{T_\pm(\lambda^l)},\quad R_\pm(\lambda^u)=\overline{R_\pm(\lambda^l)},\quad \lambda^{u,l}\in S_\pm^{u,l}.$$

$$|R_\pm(\lambda)|=1,\quad \lambda\in\Delta_\pm^{u,l}.$$

$$1-|R_\pm(\lambda)|^2=\frac{\sqrt{u_\mp^2(\lambda)-1}}{\sqrt{u_\pm^2(\lambda)-1}}\ \frac{s_\pm(\lambda)}{s_\mp(\lambda)}\ |T_\pm(\lambda)|^2,\ \lambda\in\mathbb{R}_+^{u,l}.$$

$$\overline{R_\pm(\lambda)}\ T_\pm(\lambda)=-R_\mp(\lambda)\ \overline{T_\pm(\lambda)},\ \lambda\in\mathbb{R}_+^{u,l}.$$

$$T_\pm(\lambda)=1+O\left(\frac{1}{\sqrt{|\lambda|}}\right),\quad R_\pm(\lambda)=O\left(\frac{1}{\sqrt{|\lambda|}}\right),\quad |\lambda|\to\infty.$$

II. *(Analyticity) $T_\pm(\lambda)$ are extended as meromorphic functions in the domain $\mathbb{C}\setminus S$ with the simple poles at $\lambda_1,\ldots,\lambda_p$ For all $\lambda\in\mathbb{C}\setminus S$ the following relation holds*

$$\frac{T_+(\lambda)s_+(\lambda)}{\sqrt{u_+^2(\lambda)-1}}=\frac{T_-(\lambda)s_-(\lambda)}{\sqrt{u_-^2(\lambda)-1}}:=\frac{1}{\omega(\lambda)},$$

in which connection the function $W(\lambda)=\omega(\lambda)\sigma(\lambda)$ is an analytic function in $\mathbb{C}\setminus S$, continuous right up to the boundaries and

$$\left[\frac{dW(\lambda)}{d\lambda}|\lambda=\lambda_k\right]^2=\left(m_k^+m_k^-\right)^{-2},$$

where $m_k^\pm$ are defined by (11).

III. *At the edges of the spectrum S, the elements of scattering matrix satisfy the following conditions:*

$$\lim_{\lambda\to\mu_\pm,\nu_\pm,0}\sqrt{u_\pm^2(\lambda)-1}\ \frac{R_\pm(\lambda)+1}{T_\pm(\lambda)}=0,\qquad \omega(0)\neq 0,$$

$$\lim_{\lambda\to 0}\frac{[R_+(\lambda)+1]\ [R_-(\lambda)+1]}{T_+(\lambda)\ T_-(\lambda)}=1,\qquad \omega(0)=0.$$

Using these properties, we derive the fundamental integral equations that are called the Gel'fand-Levitan-Marchenko equations.

4. INVERSE SCATTERING PROBLEM

Theorem 2. *The kernels $K_\pm(x,y)$ of the transformation operators satisfy the following integral equations:*

$$K_\pm(x,y)+F_\pm(x,y)\pm\int_x^{\pm\infty}K_\pm(x,t)+F_\pm(t,y)\ dt,\ \pm x\le\pm y, \quad (13)$$

where

$$F_\pm(x,y)=\int_{S_\pm^u\cup S_\pm^l}R_\pm(\lambda)\ \psi_\pm(x,\lambda)\ \psi_\pm(y,\lambda)\ \rho_\pm(\lambda)\ d\lambda$$
$$+\int_{\Delta_\mp^u}\frac{|T_\mp(\lambda)|^2}{\sigma_\pm(\lambda)}\ \tilde\psi_\pm(x,\lambda)\ \tilde\psi_\pm(y,\lambda)\ \rho_\mp(\lambda)\ d\lambda$$
$$+\sum_{\lambda_k}\tilde\psi_\pm(x,\lambda_k)\ \tilde\psi_\pm(y,\lambda_k)\ (m_k^\pm)^{-2},$$

$$\rho_\pm(\lambda)=\frac{i\ s_\pm(\lambda)}{4\pi\ \sqrt{u_\pm^2(\lambda)-1}}. \quad (14)$$

The conditions (5) and theorem 1 imply the estimates for the functions $F_\pm(x,y)$ and their derivatives at infinity, analogous the estimates obtained in [14, (7.3)-(7.9), (7.16)-(7.17)] in which

$$\left|\int_a^{\pm\infty}\frac{d}{dx}\ F_\pm(x,x)\ \left(1+|x|^2\right)\ dx\right|\le C(a),\quad a\in\mathbb{R}, \quad (15)$$

is the most important ones. The properties of scattering data described in the theorem 1 and mentioned in [14], are their characterization which are sufficient conditions for solving the inverse scattering problem in the class (1)-(5). For each scattering matrix with such properties, the one-dimensional Schrödinger operator is reconstructed uniquely with the potential belongs to the class (5). Namely, let $H_\pm=-\frac{d^2}{dx^2}+p_\pm(x)$ be two Hill operators with the spectra $S_\pm$ and auxiliary spectra $\lambda_\pm$, satisfying the conditions (2)-(3). Let $u_\pm(\lambda),s_\pm(x,\lambda),\psi_\pm(x,\lambda)$ be their Hill discriminants, sin-type solutions and Weyl solutions of the equations Eq. (6). Consider any arbitrary set of points $\lambda_1,\ldots,\lambda_p\in\mathbb{C}\setminus(S_+\cup S_-)$ and positive real numbers $m_1^\pm,\ldots,\ m_p^\pm$. We define the functions $R_\pm(\lambda),T_\pm(\lambda)$ on the domains $S_\pm^{u,l}$, and suppose that all collection

$$\{R_+(\lambda),R_-(\lambda),\ T_+(\lambda),T_-(\lambda),\ \lambda_1,\ldots,\lambda_p,\ m_1^\pm,\ldots,m_p^\pm\} \quad (16)$$

possesses the properties, which are described in the conditions I and II of the theorem 1. Then the functions $F_\pm(x,y)$ constructed by formulas (14), are well defined and have real-valued for all x and y. We assume the data (16) are such as the functions $F_\pm(x,y)$ satisfy the inequalities mentioned in [14].

Lemma 1. *If the scattering data satisfies the conditions I and II of the theorem 1 and the estimates (15) then the integral equations (13) have the unique solutions $K_\pm(x,y)$ which are real-valued and continuously differentiable functions with respect to both variables. Moreover, they satisfy the same type of estimates as for $F_\pm(x,y)$.*

We solve the Gel'fand-Levitan-Marchenko equations with regard to $K_\pm(x,y)$ and put

$$q_\pm(x) = \mp\frac{d}{dx}K_\pm(x,x) + p_\pm(x). \qquad (17)$$

Applying lemma 1 and the inequalities (15) described for $K_\pm(x,y)$ we conclude that

$$\int_a^{\pm\infty} |q_\pm(x) - p_\pm(x)| \left(1+|x|^2\right)dx < \infty.$$

Furthermore, the functions $\varphi_\pm(x,y)$ constructed by formulas (8) become the Jost solutions for the equations

$$\left[-\frac{d^2}{dx^2} + q_\pm(x)\right] y(x) = \lambda y(x)$$

with the potentials (17).

Lemma 2. *If the condition III of the theorem 1 holds too, then the potentials* $q_+(x)$ *and* $q_-(x)$ *coincide.*

Therefore, it would hold the following theorem that is the main result.

Main Theorem. *The data (16) possessing the properties mentioned in the theorem 1 and satisfying the estimates as (15) becomes the scattering data for (1)-(5) with the potential* $q(x) := q_+(x) = q_-(x)$, *defined by one of the formulas (10).*

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ПРЯМАЯ И ОБРАТНАЯ ЗАДАЧИ РАССЕЯНИЯ НА ВСЕЙ ОСИ ДЛЯ ОДНОМЕРНОГО УРАВНЕНИЯ ШРЁДИНГЕРА

Дж. Базарган

Изучаются прямая и обратная задачи рассеяния для одномерного уравнения Шрёдингера с потенциалом, который асимптотически близок к различным периодическим функциям на разных полуосях. Предполагается, что фоновые операторы Хилла имеют двухзонные спектры с общей полубесконечной зоной. Также предполагается, что возмущение имеет второй суммируемый момент. Для такого класса потенциалов предлагаются полные характеристики данных рассеяния и решается обратная задача рассеяния.

ПРЯМА ТА ЗВОРОТНЯ ЗАДАЧА РОЗСІЮВАННЯ НА УСІЙ ОСІ ДЛЯ ОДНОВИМІРНОГО РІВНЯННЯ ШРЬОДІНГЕРА

Дж. Базарган

Вивчається пряма та зворотна задача розсіювання для одновимірного рівняння Шрьодінгера з потенціалом, який асимптотично наближений до різноманітних періодичних функцій на різних напівосях. Припускається, що фонові оператори Хілла мають двухзонні спектри зі спільною напівнескінченною зоною. Також припускається, що збудження має другий момент, що підсумовується. Для такого класу потенціалів пропонуються повні характеристики даних розсіювання і розв`язується зворотна задача розсіювання.