COVARIANT AMPLITUDE DECOMPOSITION IN RELATIVISTIC FERMION SCATTERING PROBLEMS

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A parameterization of on-mass-shell relativistic fermion scattering amplitudes by a set of 4 covariant amplitudes is proposed, which in the non-relativistic limit turn to coefficients of the matrix amplitude decomposition over the unity and Pauli matrices, and in the ultra-relativistic limit – to symmetrized helicity amplitudes. In the general relativistic case, the covariant amplitudes express as spurs of the matrix amplitude supplemented by γ -matrix factors not exceeding 3-rd degree. Algebraic computation of such spurs provides a comparatively short and fully covariant approach for calculation of fermion scattering processes, allowing account for all polarization observables. For extension of the method to problems of two-fermion scattering, when permitted are both ways of transition 1→3, 2→4 and 1→4, 2→3, relativistic on-mass-shell Fierz relations interconnecting the two possible definitions of transition amplitudes are derived, under simplifying assumptions of equal fermion masses and scattering elasticity. Eigenfunctions of the on-shell Fierz relations are constructed, and advantages of their use for automatic account for contributions from cross-diagrams are demonstrated with the example of Møller scattering.

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1. INTRODUCTION

In computations of scattering processes of relativistic fermions, when those emit or absorb several quanta, evaluation of spin matrix products consumes considerable effort, especially if the goal is pursued to bring the result to a visual form. To that objective, there is no common approach, which might afford to completely rely on symbolic calculation with the aid of computer. Since the result of symbolic computation of a γ -matrix product (basing on algebraic relations, such as those of anticommutation) depends on the number of γ -matrices in the product, virtually, in a factorial manner, it is obvious, that the computation must be carried out not on the level of cross-sections, but on the level of amplitudes. But in the letter case, there arise questions, how to preserve covariance of the analysis and the correct number of spin degrees of freedom on the mass shell. For matrix element computation, a formidable number of approaches has been proposed [1-6], which footed on special forms of a basis for bispinors or/and Lorentz-vectors, or, as in [7,8], the bispinors were augmented to the form of density matrices, at the expense of appearing polarization-dependent denominators. This contribution offers formulation of an approach, in which no information on bispinor realizations is involved.

Beginning with scattering of one fermion, the on-shell matrix element may always be parameterized as

$$M_{fi} = \bar{u}'Mu = \bar{u}'\left(SI + A^{\mu}\gamma^{\mu}\gamma^{5}\right)u\,, \qquad (1)$$

given that bispinors u , $\bar{u}'$ obey Dirac equations[1]

$$(p\gamma - m)u = 0\,, \qquad \bar{u}'(p'\gamma - m') = 0\,,$$

and so are allowed for 2 spin degrees of freedom. The component of vector A^{μ} , parallel to $p^{\mu}/m + p'^{\mu}/m'$, does not contribute to (1) on account of the identity

$$\bar{u}'\left(p^{\mu}/m + p'^{\mu}/m'\right)\gamma^{\mu}\gamma^{5}u \equiv 0\,,$$

so the set of S , A^{μ} contains the correct number of 4 linearly independent components. The choice of I , $\gamma\gamma^5$ as basic matrices on-shell is not the only possible one, but it is favored because in the non-relativistic limit the spatial part of $\gamma\gamma^5$ is equivalent to Pauli matrices.

The explicit formulas for evaluating covariant amplitudes introduced in (1) read[2]

$$S = \frac{1}{pp' + mm'}\frac{1}{4}Sp\left(p'\gamma + m'\right)M\left(p\gamma + m\right); \qquad (2)$$

$$A^{\mu} = G^{\mu\nu}\,\frac{1}{pp' + mm'}\frac{1}{4}Sp\left(p'\gamma + m'\right)M\left(p\gamma + m\right)\gamma^{5}\gamma^{\nu}\,, \qquad (3)$$

where[3]

$$G^{\mu\nu} = g^{\mu\nu} - \frac{1}{4}\left(p'^{\mu}/m' - p^{\mu}/m\right)\left(p'^{\nu}/m' - p^{\nu}/m\right) + \alpha^{\mu}\left(p'^{\nu}/m' + p^{\nu}/m\right) + \left(p'^{\mu}/m' + p^{\mu}/m\right)\alpha^{\nu}\,,$$

with arbitrary vector α^{μ}. The choice of α^{μ} will further be referred to as "acceptance condition". Two choices important for practice are

$$G_p^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{m^2}\right),\quad G_{p'}^{\mu\nu} = \left(g^{\mu\nu} - \frac{p'^{\mu}p'^{\nu}}{m'^2}\right). \qquad (4)$$

Also, if in some reference frame the scattering is elastic ($m' = m$, $p'^0 = p^0 = E$), it may be convenient to make

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[1] The account for difference between initial and final fermion masses may be essential in problems of baryon excitation or change of flavour.

[2] As is straightforward to check by substituting $M = SI + A\gamma\gamma^5$ to (2), (3).

[3] The vector coefficient at $p'^{\nu}/m' + p^{\nu}/m$ in (3) has no effect on the form of the vector A^{μ} and is chosen so that tensor $G^{\mu\nu}$ is symmetric.

$G^{\mu\nu}$ have only spatial components non-zero. Then,

$$G_E^{\mu\nu} = \Pi_N^{\mu\nu} + (p' + p)^2 \left(\frac{1}{4E^2} \Pi_r^{\mu\nu} + \frac{1}{4m^2} \Pi_q^{\mu\nu} \right), \quad (5)$$

where $\Pi^{\mu\nu}$ means projector on the vector indicated at its subscript ($\Pi_x^{\mu\nu} \equiv x^\mu x^\nu / x^2$), $q^\mu = p'^\mu - p^\mu$, r^μ is the spatial part of vector $p'^\mu + p^\mu$ (i. e., $r^0 = 0$, $\boldsymbol{r} = \boldsymbol{p}' + \boldsymbol{p}$), N – a vector, orthogonal to q, r and to the time axis. According to (5), tensor $G^{\mu\nu}$ is diagonal in the orthogonal basis of q, r, N.

In the non-relativistic limit, at any of the acceptances (4), (5), A^μ reduces to the vector coefficient of expansion of a 2×2 amplitude over Pauli matrices.

Amplitudes S and A^μ carry the full information about the scattering, and have a "minimal" structure from the viewpoint of algebraic covariant computation. These are also well-suited for expression of scattering observables, as will be shown shortly. In what follows, we shall for simplicity put $m' = m$.

With the given definitions, it is straightforward to find that the cross-section averaged over initial and final fermion spin states expresses only through squares of the covariant amplitudes:

$$\begin{aligned} \left\langle \left| \bar{u}' M u \right|^2 \right\rangle &= Sp \frac{1}{2} (p'\gamma + m) M \frac{1}{2} (p\gamma + m) \overline{M} \\ &= \left(pp' + m^2\right) \left[|S|^2 - \left|A_p\right|^2 \right] \\ &= \left(pp' + m^2\right) \left[|S|^2 - \left|A_{p'}\right|^2 \right] \\ &= \left(pp' + m^2\right) \left[|S|^2 - \left(G_E^{-1}\right)^{\mu\nu} A_E^\mu A_E^{\nu *} \right] \end{aligned} \quad (6)$$

(positive definiteness of final expressions is secured by vectors A_p, $A_{p'}$, A_E being space-like). If, further, fermion polarization effects are concerned, a correspondence of the covariant amplitudes with mean polarization vectors or spin asymmetries referred to some quantization axes has to be established, as is done below.

2. EXPRESSION OF ONE-FERMION POLARIZATION CHARACTERISTICS

2.1. The generic quantity carrying information about all polarization effects in terms of particle polarization vectors is the scattering amplitude square averaged over the density matrices with arbitrary polarization 4-vectors a^μ, a'^μ ($pa = 0$, $-a^2 \le 1$, $p'a' = 0$, $-a'^2 \le 1$):

$$\begin{aligned} &\left\langle \left| \bar{u}' M u \right|^2 \right\rangle_{a,a'} \\ &= Sp \frac{1}{2} (p'\gamma + m)\left(1 + a'\gamma\gamma^5\right) M \frac{1}{2} (p\gamma + m)\left(1 + a\gamma\gamma^5\right) \overline{M} \\ &= \left\langle \left| \bar{u}' M u \right|^2 \right\rangle \left(1 - f_{p'}^\mu a_{\to p'}^\mu - g_{p'}^\nu a'^\nu + h_{p'}^{\mu\nu} a_{\to p'}^\mu a'^\nu \right), \end{aligned} \quad (7)$$

where coefficients $f_{p'}^\mu$, $g_{p'}^\nu$, $h_{p'}^{\mu\nu}$ are assumed to be orthogonal to the momentum p', and notation

$$a_{\to p'}^\mu = a^\mu - \frac{p'a}{pp' + m^2} \left(p^\mu + p'^\mu \right), \; (a_{\to p'} p' = 0) \quad (8)$$

is utilized for the initial polarization 4-vector a, transformed by a straight Lorentz boost to position $\perp p'$. In terms of coefficients f, g and h, the final polarization 4-vector may be retrieved as

$$a'^\nu_{(fin)} = - \frac{1}{\left\langle \left| \bar{u}' M u \right|^2 \right\rangle_a} \frac{\delta \left\langle \left| \bar{u}' M u \right|^2 \right\rangle_{a,a'}}{\delta a'^\nu} = \frac{g_{p'}^\nu - h_{p'}^{\mu\nu} a_{\to p'}^\mu}{1 - f a_{\to p'}}.$$

For expression of coefficients (7) in terms of covariant amplitudes, it is natural to adopt acceptance $\perp p'$. Computation from the definition (7) then gives[4]

$$f_{p'}^\mu = \frac{2 \mathrm{Re}\left(S^* A_{p'}^\mu \right) - \frac{1}{m} i \varepsilon^{\rho\kappa\lambda\mu} p'^\rho A_{p'}^\kappa A_{p'}^{\lambda *}}{|S|^2 - \left|A_{p'}\right|^2}; \quad (9)$$

$$g_{p'}^\nu = \frac{2 \mathrm{Re}\left(S^* A_{p'}^\nu \right) + \frac{1}{m} i \varepsilon^{\rho\kappa\lambda\nu} p'^\rho A_{p'}^\kappa A_{p'}^{\lambda *}}{|S|^2 - \left|A_{p'}\right|^2}; \quad (10)$$

$$\begin{aligned} h_{p'}^{\mu\nu} = \left(|S|^2 - \left|A_{p'}\right|^2 \right)^{-1} \Big\{ &\left(|S|^2 + \left|A_{p'}\right|^2 \right) g_{p'}^{\mu\nu} \\ &- 2 \mathrm{Re}\left(A_{p'}^{*\mu} A_{p'}^\nu \right) - \frac{2}{m} \varepsilon^{\kappa\lambda\mu\nu} p'^\kappa \mathrm{Im}\left(S^* A_{p'}^\lambda \right) \Big\}. \end{aligned} \quad (11)$$

In appearance, the above expressions are straightforward generalizations of known non-relativistic ones.

Since the number of the observables $\left\langle \left| \bar{u}' M u \right|^2 \right\rangle$, $f_{p'}^\mu$, $g_{p'}^\nu$, $h_{p'}^{\mu\nu}$ is 16, exceeding the number of real parameters in the 4 complex amplitudes S, A^μ, measurable, moreover, up to an overall phase, the polarization observables must obey some identities. A possible way to express those is to write

$$\left(1 + h_{p'}^{\lambda\lambda}\right)\left(f_{p'}^\mu - g_{p'}^\mu\right) = \left(h_{p'}^{\mu\lambda} - h_{p'}^{\lambda\mu}\right)\left(f_{p'}^\lambda + g_{p'}^\lambda\right),$$

$$\left(g^{\mu\nu} - \frac{p'^\mu p'^\nu}{m^2} \right) + \frac{h_{p'}^{\mu\nu} + h_{p'}^{\nu\mu} - 2 h_{p'}^{\lambda\lambda} \left(g^{\mu\nu} - p'^\mu p'^\nu / m^2 \right)}{1 + h_{p'}^{\lambda\lambda}}$$

$$+ \frac{\left(f_{p'}^\mu + g_{p'}^\mu\right)\left(f_{p'}^\nu + g_{p'}^\nu\right) + \frac{1}{m^2} \left(\varepsilon^{\rho\sigma\tau\mu} p'^\rho h_{p'}^{\sigma\tau} \right)\left(\varepsilon^{\alpha\beta\gamma\nu} p'^\alpha h_{p'}^{\beta\gamma} \right)}{\left(1 + h_{p'}^{\lambda\lambda}\right)^2} = 0.$$

In turn, from the measured analyzing powers $f_{p'}^\mu$, $g_{p'}^\nu$ and spin correlation coefficients $h_{p'}^{\mu\nu}$, covariant amplitudes in acceptance $\perp p$ can be extracted up to their overall phase as follows:

$$|S|^2 = \frac{\left\langle \left| \bar{u}' M u \right|^2 \right\rangle}{pp' + m^2} \frac{1 + h_{p'}^{\lambda\lambda}}{4},$$

[4] The calculation is a simple matter when using the identity $(p'\gamma + m)(p\gamma + m)\left(1 + a\gamma\gamma^5\right)(p'\gamma + m) = 2\left(pp' + m^2\right)(p'\gamma + m)\left(1 + a_{\to p'}\gamma\gamma^5\right)$.

$$S^* A^{\mu}_{p'} = \frac{\left\langle \left|\bar{u}' M u\right|^2 \right\rangle}{pp' + m^2} \frac{f^{\mu}_{p'} + g^{\mu}_{p'} + \frac{1}{m} i \varepsilon^{\rho\sigma\tau\mu} p'^{\rho} h^{\sigma\tau}_{p'}}{4};$$

$$A^{*\mu}_{p'} A^{\nu}_{p'} = \left(S A^{*\mu}_{p'}\right)\left(S^* A^{\nu}_{p'}\right) \Big/ |S|^2 .$$

2.2. Among spin bases employed for description of relativistic fermion polarization phenomena, spiral one is among most widespread, especially in the high-energy limit of gauge theories, where helicity is virtually con-served. To express helicity amplitudes in terms of covariant ones, it is appropriate to adopt elastic acceptance (5) and the "low-energy" representation for γ-matrices and bispinors:

$$SI + A_E \gamma\gamma^5 = \begin{pmatrix} S + \mathbf{A}_E \boldsymbol{\sigma} & 0 \\ 0 & S - \mathbf{A}_E \boldsymbol{\sigma} \end{pmatrix};$$

$$u_{\lambda} = \begin{pmatrix} \sqrt{E+m}\, \chi_{\lambda} \\ \sqrt{E-m}\, \lambda \chi_{\lambda} \end{pmatrix}; \qquad (\lambda, \lambda' = \pm);$$

$$\bar{u}'_{\lambda'} = \left(\sqrt{E+m}\, \chi'^{+}_{\lambda'}, -\sqrt{E-m}\, \lambda' \chi'^{+}_{\lambda'}\right).$$

For definition of Pauli spinor components and overall phases, choose a spatial coordinate frame with x axis along vector $\mathbf{r} = \mathbf{p} + \mathbf{p}'$, and y along $\mathbf{q} = \mathbf{p}' - \mathbf{p}$. Then,

$$\mathbf{A}_E \boldsymbol{\sigma} = A^r \sigma^1 + A^q \sigma^2 + A^N \sigma^3 = \begin{pmatrix} A^N & A^r - iA^q \\ A^r + iA^q & -A^N \end{pmatrix},$$

and in the phase convention of [9],

$$\chi_{+} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta/4} \\ e^{-i\theta/4} \end{pmatrix} = \chi'^{*}_{+};$$

$$\chi_{-} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{i\theta/4} \\ e^{-i\theta/4} \end{pmatrix} = \chi'^{*}_{-}$$

(θ is the scattering angle, being, in the chosen frame, the azimuthal angle with respect to z-axis). Computing matrix elements with the given entries yields

$$\begin{aligned} (M_{++} - M_{--})/2 &= 2EA^r, \\ (M_{-+} - M_{+-})/2i &= 2mA^q; \end{aligned} \tag{12a}$$

$$\begin{aligned} (M_{++} + M_{--})/2 &= 2m\cos\frac{\theta}{2} S + 2iE\sin\frac{\theta}{2} A^N; \\ (M_{+-} + M_{-+})/2 &= -2iE\sin\frac{\theta}{2} S - 2m\cos\frac{\theta}{2} A^N. \end{aligned} \tag{12b}$$

In view of (12a), vector amplitude components A^r, A^q admit direct interpretation as P- and T-asymmetries of helicity amplitudes[5], but S and A^N are related with $(M_{++} + M_{--})/2$ and $(M_{+-} + M_{-+})/2$ via a matrix, unitary up to a common factor, which gets diagonal in the massless limit only (then, S is a helicity-flip, A^N – helicity non-flip amplitude).

[5] At a different choice of basis helicity states, the helicity amplitudes would enter (12a,b) with additional phases. For example, in the convention of [10],

$$\chi_{+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{-i\theta/2} \end{pmatrix} = \chi'^{*}_{+}, \quad \chi_{-} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{i\theta/2} \\ 1 \end{pmatrix} = \chi'^{*}_{-};$$

$$\left(M_{++} e^{i\theta/2} - M_{--} e^{-i\theta/2}\right)/2 = 2EA_r;$$

$$\left(M_{++} e^{i\theta/2} + M_{--} e^{-i\theta/2}\right)/2 = 2m\cos\frac{\theta}{2} S + 2iE\sin\frac{\theta}{2} A^N.$$

2.3. In the simplest physical situation, when the fermion scatters due to absorption of a single (virtual) photon with polarization e^{ν}, by virtue of parity and gauge invariance of electromagnetic interaction, the matrix amplitude admits parameterization

$$\dot{M} = 2m(\mathrm{F}_e - \mathrm{F}_m) \frac{(p+p')e}{(p+p')^2} + \mathrm{F}_m\, e\gamma, \tag{13}$$

where F_e, F_m are functions of $(p'-p)^2$, real in the domain $(p'-p)^2 < 0$ and referred to as charge and magnetic form-factors. Such not a self-evident fashion of parameterization proves advantageous because F_e and F_m appear to not interfere in the cross-section averaged over fermion spins – in contrast to form-factors introduced in a fashion suggested by perturbation theory

$$\dot{M} = \mathrm{F}_1\, e\gamma - \frac{1}{2m} \mathrm{F}_2\, \sigma^{\kappa\lambda} e^{\kappa} q^{\lambda}.$$

In this regard, it is instructive to compute the covariant amplitudes by substituting (13) into (2), (3). One finds:

$$S = m\frac{(p+p')e}{pp' + m^2} \mathrm{F}_e; \qquad A^{\mu} = i\frac{\varepsilon^{\mu\alpha\beta\nu} p^{\alpha} p'^{\beta} e^{\nu}}{pp' + m^2} \mathrm{F}_m$$

(in any of acceptances (4) or (5)). Recalling (6), the non-interference of F_e and F_m in the averaged cross-section should be regarded as natural result.

If polarization e^{ν} is linear, the direction of vector A serves as a physical quantization axis. Manifestly, if N^{μ} designates a unit real vector in the space-like direction of A^{μ} (so that $A^{\mu} = A^N N^{\mu}$), the matrix $N\gamma\gamma^5$ commutes with both $p\gamma$ and $p'\gamma$, and on account of the identity $(N\gamma\gamma^5)^2 = I$, its eigenvalues equal ±1. Thus, it is possible to pick an initial spin basis u_{σ} and a final basis $u'_{\sigma'}$ so that

$$N\gamma\gamma^5 u_{\sigma} = \sigma u_{\sigma}, \quad \bar{u}'_{\sigma'} N\gamma\gamma^5 = \sigma' \bar{u}'_{\sigma'}, \quad (\sigma, \sigma' = \pm)$$

and $\bar{u}'_{+} u_{-} = \bar{u}'_{+}\left(N\gamma\gamma^5 - N\gamma\gamma^5\right) u_{-} = 0 = \bar{u}'_{-} u_{+}$. Then,

$$\bar{u}'_{\sigma'}\left(SI + A^{\mu}\gamma^{\mu}\gamma^5\right) u_{\sigma} = \delta_{\sigma'\sigma} \bar{u}'_{\sigma} u_{\sigma}\left(S + \sigma A^N\right).$$

Owing to reality of the form-factors, thereat $|S + A^N| = |S - A^N|$, so the polarizing effect is absent and the vector amplitude acts merely as an axis of spin precession. However, for the problem of scattering in a strong, centrally-symmetric electrostatic field, the structure of invariant amplitudes remains similar (with e^{ν} substituted by the time axis direction), but these become energy-dependent and are no longer real. Under such conditions $|S + A^N| \neq |S - A^N|$, so the polarizing effect may be essential.

If, moreover, the field is not centrally-symmetric, or the process of scattering is accompanied by radiation with substantial recoil, the direction of the vector amplitude is no longer fixed by kinematics alone. In this event, the quantization axis direction can be straightforwardly computed via formula (3).

3. TWO-FERMION COVARIANT AMPLITUDES. RELATIVISTIC ON-SHELL FIERZ RELATIONS

Generalization of the covariant amplitude approach to the case of collision between two fermions can be made by introducing parameterization of the on-shell matrix element like

$$M_{fi} = \bar{u}_4\bar{u}_3 M u_2 u_1 = \bar{u}_4\bar{u}_3\left\{S_{31,42}I_{31}I_{42} + A^{\mu}_{31}\left(\gamma^{\mu}\gamma^{5}\right)_{31}I_{42} + A^{\nu}_{42}I_{31}\left(\gamma^{\nu}\gamma^{5}\right)_{42} + B^{\mu\nu}_{31,42}\left(\gamma^{\mu}\gamma^{5}\right)_{31}\left(\gamma^{\nu}\gamma^{5}\right)_{42}\right\}u_2 u_1, \quad (14)$$

where subscript 31 at matrices I, $\gamma\gamma^5$ indicates that those act from spin state of the particle with momentum p_1 into spin state of the particle with momentum p_3, and similarly for subscript 42.

Convenience of actual use of the representation (14) depends on structure of the matrix M. If $M = \int d\Xi M_{31}(\Xi)M_{42}(\Xi)$, where Ξ is a shorthand for all variables of intermediate particles exchanged between fermions, then covariant amplitudes in (14) are evaluated by formulae trivially generalizing those of one-particle theory:

$$S_{31,42} = \frac{1}{(p_3p_1 + m_3m_1)(p_4p_2 + m_4m_2)}\int d\Xi \frac{1}{4}Sp(p_3\gamma + m_3)M_{31}(\Xi)(p_1\gamma + m_1)\cdot\frac{1}{4}Sp(p_4\gamma + m_4)M_{42}(\Xi)(p_2\gamma + m_2) \quad (15)$$

and etc. for A^{μ}_{31}, A^{ν}_{42}, $B^{\mu\nu}_{31,42}$. The cross-section is correspondingly formed by a formula generalizing (6).

However, if there are two cross-subprocesses interfering in the matrix scattering amplitude:

$$M = \int d\Xi M_{31}(\Xi)M_{42}(\Xi) + \int d\Xi M_{41}(\Xi)M_{32}(\Xi), \quad (16)$$

direct calculation of coefficients in (14) with the second term of (16) yields spurs which do not factorize, unlike (15), and straightforward algebraic expansion of such formidable spurs leads to improperly extensive results. In those circumstances, it is more practical to compute contribution from this cross-subprocess in its natural representation

$$\bar{u}_4\bar{u}_3 M u_2 u_1 = \bar{u}_4\bar{u}_3\left\{S_{41,32}I_{41}I_{32} + A^{\mu}_{41}\left(\gamma^{\mu}\gamma^{5}\right)_{41}I_{32} + A^{\nu}_{32}I_{41}\left(\gamma^{\nu}\gamma^{5}\right)_{32} + B^{\mu\nu}_{41,32}\left(\gamma^{\mu}\gamma^{5}\right)_{41}\left(\gamma^{\nu}\gamma^{5}\right)_{32}\right\}u_2 u_1, \quad (14a)$$

whereas contribution from the first part of (15) – be still computed in representation (14). However, when doing so, to analyse quantitatively the sum of such amplitudes, it should be brought to some common basis. It is not obvious to propose any third basis except (14) and (14a), so there must be derived a kind of Fierz relations connecting the two possible representations.

In discussion of Fierz relations between on-mass-shell relativistic fermions, we confine ourselves to the case of elastic scattering of two equal-mass particles[6], i. e. assume identities

$$p_1^{\mu} + p_2^{\mu} = p_3^{\mu} + p_4^{\mu} \overset{def}{=} P^{\mu}, \qquad m_1 = m_2 = m_3 = m_4 \overset{def}{=} m.$$

When interrelating the amplitudes, it is expedient to adopt for them some common acceptance, a natural choice of which, by symmetry reasons, is $\perp P$ (analog of (5)). The resulting spin crossing relations are quoted below:

$$\begin{aligned}
(p_3 + p_1)^4 S_{31,42} = {} & \left(2m^2P^2 - \frac{1}{2}p_{41}^2p_{31}^2\right)S_{41,32} + 4mi\varepsilon^{\alpha\beta\gamma\kappa}p_4^{\alpha}p_3^{\beta}p_2^{\gamma}\left(A^{\kappa}_{41} + A^{\kappa}_{32}\right) \\
& -\left(2m^2P^2G_P^{\kappa\lambda} - \frac{1}{2}p_{41}^2p_{31}^2\Pi_N^{\kappa\lambda}\right)B^{\kappa\lambda}_{41,32};
\end{aligned}$$

$$\begin{aligned}
(p_3 + p_1)^4\left(A^{\mu}_{31} + A^{\mu}_{42}\right) = {} & 8mi\varepsilon^{\mu\alpha\beta\gamma}p_4^{\alpha}p_3^{\beta}p_2^{\gamma}S_{41,32} + \left(4m^2P^2G_P^{\mu\kappa} - p_{41}^2p_{31}^2\Pi_N^{\mu\kappa}\right)\left(A^{\kappa}_{41} + A^{\kappa}_{32}\right) \\
& - 4mG_P^{\mu\kappa}i\varepsilon^{\lambda\alpha\beta\gamma}p_4^{\alpha}p_3^{\beta}p_2^{\gamma}\left(B^{\kappa\lambda}_{41,32} + B^{\lambda\kappa}_{41,32}\right);
\end{aligned}$$

$$\begin{aligned}
(p_3 + p_1)^4\left(B^{\mu\nu}_{31,42} + B^{\nu\mu}_{31,42}\right) = {} & \left[2p_{41}^2p_{31}^2\Pi_N^{\mu\nu} - (p_4 + p_1)^2(p_3 + p_1)^2G_P^{\mu\nu}\right]S_{41,32} \\
& + 4m\left(G_P^{\mu\kappa}i\varepsilon^{\nu\alpha\beta\gamma}p_4^{\alpha}p_3^{\beta}p_2^{\gamma} + G_P^{\nu\kappa}i\varepsilon^{\mu\alpha\beta\gamma}p_4^{\alpha}p_3^{\beta}p_2^{\gamma}\right)\left(A^{\kappa}_{41} + A^{\kappa}_{32}\right) \\
& + \left[4m^2P^2\left(G_P^{\kappa\mu}G_P^{\lambda\nu} + G_P^{\lambda\mu}G_P^{\kappa\nu} - G_P^{\mu\nu}G_P^{\kappa\lambda}\right) - p_{41}^2p_{31}^2G_P^{\mu\nu}\Pi_N^{\kappa\lambda}\right]B^{\kappa\lambda}_{41,32};,
\end{aligned}$$

$$(p_3 + p_1)^2\left(A^{\mu}_{31} - A^{\mu}_{42}\right) = -\left(p_{41}^{\mu}p_{31}^{\kappa} - p_{31}^{\mu}p_{41}^{\kappa}\right)\left(A^{\kappa}_{41} - A^{\kappa}_{32}\right) + 2mi\varepsilon^{\mu\delta\kappa\lambda}P^{\delta}B^{\kappa\lambda}_{41,32},$$

$$\begin{aligned}
(p_3 + p_1)^2\left(B^{\mu\nu}_{31,42} - B^{\nu\mu}_{31,42}\right) = {} & \left[2mi\varepsilon^{\mu\nu\kappa\tau}P^{\tau} + \frac{1}{mP^2}\left(p_{41}^{\mu}p_{31}^{\nu} - p_{31}^{\mu}p_{41}^{\nu}\right)i\varepsilon^{\alpha\beta\gamma\kappa}p_4^{\alpha}p_3^{\beta}p_2^{\gamma}\right]\left(A^{\kappa}_{41} - A^{\kappa}_{32}\right) \\
& + \frac{1}{P^2}\varepsilon^{\delta\lambda\mu\nu}P^{\delta}\varepsilon^{\rho\sigma\tau\kappa}P^{\rho}p_{41}^{\sigma}p_{31}^{\tau}\left(B^{\kappa\lambda}_{41,32} - B^{\lambda\kappa}_{41,32}\right).
\end{aligned}$$

Here $p_{31} = p_3 - p_1$, $p_{41} = p_4 - p_1$, and $G_P^{\alpha\beta} = \Pi_N^{\alpha\beta} + (p_3 + p_1)^2\left(\frac{1}{4m^2}\Pi_{31}^{\alpha\beta} + \frac{1}{P^2}\Pi_{41}^{\alpha\beta}\right)$ (cf. (5)).

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[6] It is worth reminding, that understood in approximate sense, equality of masses does not necessarily imply identity of particles. The typical example is proton and neutron, and, to a lesser accuracy, Λ-hyperon.

Before addressing specific models for particle interaction, the Fierz relations allow one to establish consequences from the suggested symmetry properties. First of all, it is apparent, that amplitudes

$$S_{31,42}\,,\; A_{31}^{\mu}+A_{42}^{\mu}\,,\; B_{31,42}^{\mu\nu}+B_{31,42}^{\nu\mu} \tag{17}$$

and

$$A_{31}^{\mu}-A_{42}^{\mu}\,,\quad B_{31,42}^{\mu\nu}-B_{31,42}^{\nu\mu} \tag{18}$$

constitute Fierz subgroups. The physical meaning of combinations (17, 18) is that amplitudes (17) are symmetric under the simultaneous permutation (1↔2, 3↔4), and conserving the initial wave function permutational symmetry, whereas amplitudes (18) are antisymmetric and permutational symmetry changing. For collision of identical fermions, the permutational symmetry of initial and final states is obliged to be the same (negative), then, only amplitudes (17) remain.

Consequences from discrete space-time symmetries are also easy to deduce. P-invariance of the matrix amplitude permits covariant amplitudes to have only components

$$S_{31,42}\,,\; A_{31}^{N}\,,\; A_{42}^{N}\,,\; B_{31,42}^{NN}\,,\; B_{31,42}^{\perp N,\perp N}\,, \tag{19}$$

which upon the cross-transformation turn to a set of similar structure: $S_{41,32}\,, A_{41}^{N}\,, A_{32}^{N}\,, B_{41,32}^{NN}\,, B_{41,32}^{\perp N,\perp N}$.

As for T-invariance, in analogy with the non-relativistic case [11], one can infer that a scattering subprocess, during which continuously 1→3, 2→4 (the first term of (15), contributes only to amplitudes

$$S_{31,42}\,,\; A_{31}^{\perp 31}\,,\; A_{42}^{\perp 31}\,,\; B_{31,42}^{31,31}\,,\; B_{31,42}^{\perp 31,\perp 31}\,. \tag{20}$$

A cross-subprocess (the second term of (15)), accordingly, yields amplitudes $S_{41,32}\,,\; A_{41}^{\perp 41}\,, A_{32}^{\perp 41}\,, B_{41,32}^{41,41}\,,$ $B_{41,32}^{\perp 41,\perp 41}$, which upon crossing transformation turn to

$$S_{31,42}\,, A_{31}^{\perp 41}+A_{42}^{\perp 41}\,, B_{31,42}^{41,41}\,, \left(B_{31,42}^{\mu\nu}+B_{31,42}^{\nu\mu}\right)^{\perp 41,\perp 41}\,,$$
$$A_{31}^{41}-A_{42}^{41}\,,\; B_{31,42}^{41,\perp 41}-B_{31,42}^{\perp 41,41}\,. \tag{21}$$

In the sum of (20) and (21), forbidden are only 3 amplitudes, namely, $A_{31}^{31}-A_{42}^{31}$, $B_{31,42}^{31,41}+B_{31,42}^{41,31}$ and $B_{31,42}^{31,N}-B_{31,42}^{N,31}$.

In the maximally restricted case, when both P- and T-invariance apply and the colliding fermions are identical, there remain only 5 non-zero amplitudes: $S_{31,42}$, $A_{31}^{N}+A_{42}^{N}$, $B_{31,42}^{N,N}$, $B_{31,42}^{31,31}$, $B_{31,42}^{41,41}$ (direct analogs of Wolfenstein's parameters known in the non-relativistic case [11]). In their terms, the averaged over all fermion spins matrix element square is obtained to be

$$\left\langle \left|\bar{u}_4\bar{u}_3 M u_2 u_1\right|^2 \right\rangle$$
$$=\left(p_3 p_1+m^2\right)^2\left(\left|S_{31,42}\right|^2+\frac{1}{2}\left|A_{31}^{N}+A_{42}^{N}\right|^2+\left|B_{31,42}^{NN}\right|^2\right)$$
$$+4m^4\left|B_{31,42}^{31,31}\right|^2+\frac{P^4}{4}\left|B_{31,42}^{41,41}\right|^2. \tag{22}$$

4. FIERZ-INVARIANT AMPLITUDES AND ACCOUNT FOR CONTRIBUTIONS FROM CROSS-DIAGRAMS

Besides amplitudes (17-18), symmetric and antisymmetric under the simultaneous permutation (1↔2, 3↔4), i. e. physical permutation of particles, of further use can be amplitudes of definite symmetry with respect to separate permutations 1↔2 and 3↔4. At determining such a symmetry, it should be minded that permutation 1↔2 or 3↔4 brings covariant amplitudes to a cross-representation, so to compare the result with the initial expression, it needs to be turned back to the initial representation, with the aid of Fierz identities. Note, that in the non-relativistic case, amplitudes of definite permutational symmetry both in the initial and in the final state admit simple physical interpretation – those are transition amplitudes between states with definite values of the two-fermion total spin. In terms of covariant amplitudes, those classes, accordingly, come as a scalar, two vector and a tensor combinations[7] – with constant coefficients, granting that non-relativistic spin does not depend on momentum.

In the relativistic case, inception of dependence on momenta deprives total spin of its absolute sense as a vector, but concepts of permutational symmetry and correspondent state multiplicity remain valid. Based on on-shell Fierz identities, one can pick a full set of "minimal" cross-symmetric and antisymmetric covariant amplitude combinations, securing them to be non-interfering in the spin-averaged cross-section. Below quoted are expressions for singlet-singlet and triplet-triplet amplitudes, those which stay non-zero under P-, T-invariance and identity of colliding particles:

$$\begin{aligned}U&=\left(p_3+p_1\right)^2\left(S_{31,42}+B_{31,42}^{NN}\right)+4m^2B_{31,42}^{31,31}+P^2B_{31,42}^{41,41}\\&\equiv-\left(p_4+p_1\right)^2\left(S_{41,32}+B_{41,32}^{NN}\right)-P^2B_{41,32}^{31,31}-4m^2B_{41,32}^{41,41}\,;\end{aligned} \tag{23a}$$

$$\begin{aligned}W_N&=\left(p_3+p_1\right)^2\Big\{S_{31,42}-B_{31,42}^{NN}\\&\quad-\frac{1}{mP^2}\varepsilon^{\alpha\beta\delta\mu}p_4^{\alpha}p_3^{\beta}p_2^{\delta}i\left(A_{31}^{\mu}+A_{42}^{\mu}\right)\Big\}\\&\equiv\left(p_4+p_1\right)^2\Big\{S_{41,32}-B_{41,32}^{NN}\\&\quad+\frac{1}{mP^2}\varepsilon^{\alpha\beta\delta\kappa}p_4^{\alpha}p_3^{\beta}p_2^{\delta}i\left(A_{41}^{\kappa}+A_{32}^{\kappa}\right)\Big\};\end{aligned} \tag{23b}$$

$$\begin{aligned}W_A&=\left(p_3+p_1\right)^2\Big\{S_{31,42}-B_{31,42}^{NN}\\&\quad+\frac{4m}{p_{41}^2p_{31}^2}\varepsilon^{\alpha\beta\delta\mu}p_4^{\alpha}p_3^{\beta}p_2^{\delta}i\left(A_{31}^{\mu}+A_{42}^{\mu}\right)\Big\}\\&\equiv-\left(p_4+p_1\right)^2\Big\{S_{41,32}-B_{41,32}^{NN}\\&\quad-\frac{4m}{p_{41}^2p_{31}^2}\varepsilon^{\alpha\beta\delta\kappa}p_4^{\alpha}p_3^{\beta}p_2^{\delta}i\left(A_{41}^{\kappa}+A_{32}^{\kappa}\right)\Big\},\end{aligned} \tag{23c}$$

[7] The latter tensor part virtually appears in terms of a scalar, a vector (equivalent to antisymmetric tensor), and a symmetric tensor with zero trace.

$$W_{NQ} = (p_3 + p_1)^2 \left(S_{31,42} + B_{31,42}^{NN}\right) - 4m^2 B_{31,42}^{31,31} - P^2 B_{31,42}^{41,41}$$
$$\equiv (p_4 + p_1)^2 \left(S_{41,32} + B_{41,32}^{NN}\right) - P^2 B_{41,32}^{31,31} - 4m^2 B_{41,32}^{41,41}; \quad \text{(23d)}$$

$$W_Q = 4m^2 B_{31,42}^{31,31} - P^2 B_{31,42}^{41,41} \equiv P^2 B_{41,32}^{31,31} - 4m^2 B_{41,32}^{41,41}. \quad \text{(23e)}$$

The spin-averaged cross-section (22) rewrites through the above amplitudes as

$$\left\langle \left| \bar{u}_4 \bar{u}_3 M u_2 u_1 \right|^2 \right\rangle = \frac{1}{16}|U|^2 + \frac{1}{16}|W_{NQ}|^2 + \frac{1}{8}|W_Q|^2$$
$$+ \frac{1}{2(p_4+p_1)^2 (p_3+p_1)^2} \left(m^2 P^2 |W_N|^2 + \frac{p_{41}^2 p_{31}^2}{4} |W_A|^2 \right). \quad \text{(24)}$$

It appears, that Fierz-invariant amplitudes can supply an efficient tool for performing calculations, because contribution to them from any Feynman diagram with continuous fermionic lines may be calculated via that representation of covariant amplitudes, which corresponds to the order of fermionic transitions in this diagram. Moreover, diagrams differing only by crossing of fermion ends need not actually be calculated both – with contributions from one diagram evaluated, it suffices to make momentum and charge substitutions to recover that from the other.

To illustrate efficiency of this way of conducting calculations, it may be applied to scattering of two electrons via one-photon exchange (Møller scattering). In Feynman's gauge, the corresponding matrix amplitude is [9,10]:

$$M = \frac{\left(\gamma^\mu\right)_{31}\left(\gamma^\mu\right)_{42}}{p_{31}^2} - \frac{\left(\gamma^\mu\right)_{41}\left(\gamma^\mu\right)_{32}}{p_{41}^2}. \quad \text{(25)}$$

Computing Fierz-invariant amplitudes for the first term of (25) and deducing those from the other, one finds

$$U = 4p_2 p_1 \left(\frac{1}{p_{31}^2} + \frac{1}{p_{41}^2} \right);$$

$$W_{NQ} = \frac{4p_4 p_1}{p_{31}^2} - \frac{4p_3 p_1}{p_{41}^2} = 4p_2 p_1 \left(\frac{1}{p_{31}^2} - \frac{1}{p_{41}^2} \right);$$

$$W_Q = 1 + 1 = 2;$$

$$W_N = \frac{2(p_4+p_1)^2 - (p_3+p_1)^2}{p_{31}^2} - \frac{2(p_3+p_1)^2 - (p_4+p_1)^2}{p_{41}^2}$$
$$= 4p_2 p_1 \left(\frac{1}{p_{31}^2} - \frac{1}{p_{41}^2} \right) = W_{NQ};$$

$$W_A = -\frac{2(p_4+p_1)^2 + (p_3+p_1)^2}{p_{31}^2} - \frac{(p_4+p_1)^2 + 2(p_3+p_1)^2}{p_{41}^2}$$
$$= -2\left[1 + \left(2m^2 + P^2\right)\left(\frac{1}{p_{31}^2} + \frac{1}{p_{41}^2} \right) \right].$$

Note that all those quantities stay finite in the limit $m \to 0$, whereas contribution to (24) from W_N in this limit is vanishing. In the opposite, non-relativistic limit, achieved through $p_{31}, p_{41} \to 0$, the main contribution to (24) comes from U (singlet Coulomb scattering) and from $W_{NQ} = W_N$ (triplet Coulomb scattering).

As is observed, expressions for cross-invariant amplitudes (even those resulting from individual diagrams) prove to be more simple then the covariant amplitudes they add up of in (23). Computation of loop and weak corrections to such expressions can also be facilitated.

5. SUMMARY

The main objective of this note was to demonstrate, that reducing the problem of calculation of relativistic massive fermion scattering observables to evaluation of the introduced covariant amplitudes, in a proper acceptance, is advantageous from the most points of view. In the one-fermion case, it requires computation of spurs of the matrix amplitude multiplied by polynomials of γ-matrices not exceeding 3 degree. Such spurs may be calculated using conventional algebraic techniques. The relation with the scattering observables is also well suited – the scattering cross-section averaged over initial and final fermion polarizations expresses through the scalar and the vector amplitude squares. In case if the vector amplitude is proportional to a real vector, the latter serves as a physical quantization axis. In most important limiting cases, the covariant amplitudes reduce to conventional basis sets: in the non-relativistic limit the 4-vector amplitude turns to the 3-vector one, in the ultra-relativistic limit it turns to spin-non-flip amplitudes, and the scalar amplitude – to the spin-flip one.

For two fermion scattering, the generalization of the covariant approach is rather straightforward, however, a complication arises if the amplitude is an interference of two cross-running subprocesses. Then, it is preferred to compute contribution from each cross-diagram in the spin representation, corresponding to the order of continuous fermionic transitions in it. For bringing the quantities so evaluated to a common basis, relativistic on-mass-shell Fierz relations must be derived (herein presented for simplifying conditions of equal fermion masses and absence of accompanying hard radiation). At practice, performing computations may turn out to be more simple via cross-invariant amplitudes – eigenfunctions of the on-shell Fierz relations. In the considered example of Møller scattering, expressions for the cross-invariant amplitudes in arbitrary kinematics acquire very simple structure.

In applications, it should be minded, that specific composition of the matrix amplitude M, as it results from Feynman diagrams or non-perturbative dynamical solutions, may suggest basic γ-matrix structures, different from (1) (or (14)). Some of such self-suggesting bases may even occur to be mutually orthogonal, so that the cross-section averaged over the fermion spins expresses through squares of those amplitudes, as good as through (6). If that degree of knowledge about scattering is sufficient, there is no actual need for transformation to amplitudes of the form (1). However, when there is also a need to estimate fermion polarization effects, the amplitudes (1) are convenient in view of their simple and transparent connection to polarization observables, so using them to encode the information about scattering may be worth doing.

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МЕТОД КОВАРИАНТНЫХ АМПЛИТУД В ЗАДАЧАХ РАССЕЯНИЯ РЕЛЯТИВИСТСКИХ ФЕРМИОНОВ

Н.В. Бондаренко

Предложена параметризация амплитуды рассеяния релятивистского фермиона на массовой поверхности набором 4 ковариантных амплитуд, которые в нерелятивистском пределе переходят в коэффициенты разложения матричной амплитуды по единичной матрице и матрицам Паули, а в ультрарелятивистском пределе – в симметризованные спиральные амплитуды. В общем релятивистском случае ковариантные амплитуды выражаются через шпуры от матричной амплитуды, помноженной на γ-матричные факторы степени не выше 3. Алгебраическое вычисление таких шпуров дает сравнительно короткий и полностью ковариантный путь расчета процессов рассеяния фермионов, с учетом всех поляризационных наблюдаемых. Для обобщения метода на задачи рассеяния двух фермионов, в условиях когда допустимы оба пути переходов 1→3, 2→4 и 1→4, 2→3, выведены релятивистские тождества Фирца на массовой поверхности, связывающие два возможных определения амплитуд перехода, при упрощающих предположениях равенства масс всех фермионов и упругости рассеяния. Построены собственные функции тождеств Фирца, и продемонстрированы преимущества их использования для автоматического учета вкладов от кросс-диаграмм на примере Меллеровского рассеяния.

МЕТОД КОВАРІАНТНИХ АМПЛІТУД У ЗАДАЧАХ РОЗСІЯННЯ РЕЛЯТИВІСТСЬКИХ ФЕРМІОНІВ

М.В. Бондаренко

Запропоновано параметризацію амплітуди розсіяння релятивістського ферміона на масовій поверхні набором 4 коваріантних амплітуд, які в нерелятивістському наближенні переходять у коефіцієнти розкладення матричної амплітуди по одиничній матриці та матрицям Паулі, а в ультререлятивістській границі – у симетризовані спіральні амплітуди. В загальному релятивістському випадку коваріантні амплітуди виражаються через шпури від матричної амплітуди з додатковими γ-матричними факторами, ступінь яких не перевищує 3. Алгебраїчне обчислення таких шпурів дає порівняно короткий та повністю коваріантний шлях розрахунку процесів розсіяння ферміонов, з урахуванням усіх поляризацій. Для узагальнення методу до задач розсіяння двох ферміонів, в умовах коли можливі обидва шляхи переходів 1→3, 2→4 та 1→4, 2→3, виведено релятивістські тотожності Фірца на масовій поверхні, які зв'язують визначення амплітуд переходу, за спрощуючих припущеннях рівності мас усіх ферміонів та пружності розсіяння. Побудовано власні функції тотожностей Фірца, та продемонстровано переваги їх використання для автоматичного врахування вкладів від перехресних діаграм на прикладі Мелеровського розсіяння.