ON SIX-DIMENTIONAL DIRAC EQUATION AND SOME RELATIONS BETWEEN POLARISATIONS OF THE SPIN ½ PARTICLE

M.V. Lyubchenko[1] and Yu.P. Stepanovsky[2]

[1]*Physics Department of Karazin National University, Kharkov, Ukraine;*
e-mail: rosomasha@yandex.ru;
[2]*National Science Center "Kharkov Institute of Physics and Technology", Kharkov, Ukraine;*
e-mail: yustep@kipt.kharkov.ua

Through the six-dimensional description of polarized spin ½ particles we obtain some formulae representing general properties of polarization phenomena. The obtained relations between polarizations are used for consideration of electron scattering in coulomb field in second Born approximation.

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1. INTRODUCTION

The physical theories with extra dimensions are very popular now [1]. Usually extra space dimensions considered to be of the size of Planck length and are compactified. But some recent theories stem from the idea that extra dimensions could after all be much larger without violating any established fact [2]. Higher dimensional space-time theories pretend on fundamental modification of the laws of nature and our view of a four dimensional universe. Our aim is much more modest. We shall show that a six-dimensional and a five-dimensional generalization of Dirac equation can be useful for consideration and calculation of probabilities of scattering processes with polarized spin ½ particles.

2. FOUR-DIMENSIONAL DESCRIPTION OF SPIN ½ POLARIZED PARTICLE

Let us consider a spin ½ polarized particle with definite 4-momentum p and wave function $\psi(x) = u(p)\exp(ipx)$. The bispinor $u(p)$ satisfies Dirac equation

$$(ip_\mu\gamma_\mu + m)u = 0. \tag{1}$$

Generally the state of polarization of such particle is described by some 4-pseudovector s, pseudoscalar σ and by following equations [3]:

$$(1 - i\gamma_5 s_\mu\gamma_\mu + \sigma\gamma_5)u = 0, \tag{2}$$

$$p^2 + m^2 = 0,\ s^2 + \sigma^2 = 1,\ (sp) - \sigma m = 0.$$

Usually in these equations σ is supposed to be zero, but there are some reasons, in particular good high energy limit, to introduce nonzero σ.

Using (1-2) we can obtain the following expression for the density matrix of the particle

$$u_\alpha \bar{u}^\beta = \left[\frac{(m - ip_\mu\gamma_\mu)(1 + i\gamma_5 s_\mu\gamma_\mu + \sigma\gamma_5)}{2}\right]_\alpha^\beta. \tag{3}$$

Expressions (2-3) are invariant under transformations

$$s \to s' = s - \alpha p,\ \sigma \to \sigma' + \alpha m. \tag{4}$$

If the particle is not massless owing (4) we can make $\sigma = 0$ and describe state of particle polarization only by 4-vector s [4]. But for a masssless particle the using of helicity σ is necessary.

3. SIX-DIMENSIONAL DESCRIPTION OF SPIN ½ POLARIZED PARTICLE

The six-dimensional structure of 4×4 Dirac matrices algebra is well known [5-8]. Let us introduce six matrices

$$(\Phi_A)^{\alpha\beta} = -(\Phi_A)^{\beta\alpha},\ A = 1,2,3,4,5,6,$$

$$\begin{aligned}(\Phi_\mu)^{\alpha\beta} &= (C^{-1} i\gamma_5\gamma_\mu)^{\alpha\beta},\ \mu = 1,2,3,4,\\ (\Phi_5)^{\alpha\beta} &= (C^{-1}\gamma_5)^{\alpha\beta},\ (\Phi_6)^{\alpha\beta} = (iC^{-1})^{\alpha\beta},\end{aligned} \tag{5}$$

where γ_μ are Dirac matrices, $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, C is the charge conjugation matrix. Matrices $(\Phi_A)^{\alpha\beta}$ are 6-vectors, that is being transformed as 6-vectors under transformations of *SO*(6,*C*)-group and as 4-spinors under *SL*(4,*C*) they are remaining invariant,

$$L_{AB}(\Phi_B)^{\gamma\delta}(\Lambda^{-1})_\gamma^\alpha(\Lambda^{-1})_\delta^\beta = (\Phi_A)^{\alpha\beta}. \tag{6}$$

For infinitesimal

$$L_{AB} = \delta_{AB} + \Delta\omega_{AB},\quad \Delta\omega_{AB} = -\Delta\omega_{BA}, \tag{7}$$

$$\Lambda = I + \frac{i}{2}\Sigma_{AB}\Delta\omega_{AB},\quad \Sigma_{AB} = -\Sigma_{BA}. \tag{8}$$

We may express 15 infinitesimal operators in terms of the Dirac γ-matrices,

$$\begin{aligned}&\Sigma_{5\mu} = -\Sigma_{\mu 5} = \frac{1}{2}\gamma_\mu,\ \Sigma_{6\mu} = -\Sigma_{\mu 6} = \frac{1}{2i}\gamma_5\gamma_k,\\ &\Sigma_{56} = -\Sigma_{65} = \frac{1}{2}\gamma_5,\quad \Sigma_{\mu\nu} = \frac{\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu}{4i}.\end{aligned} \tag{9}$$

The Σ_{AB} satisfy the relations

$$\begin{aligned}\Sigma_{AB}\Sigma_{CD} &= \frac{1}{4}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})I - \frac{1}{4}\varepsilon_{ABCDEF}\Sigma_{EF}\\ &+ \frac{i}{2}(\delta_{AC}\Sigma_{BD} - \delta_{AD}\Sigma_{BC} + \delta_{BD}\Sigma_{AC} - \delta_{AC}\Sigma_{BD}),\end{aligned} \tag{10}$$

where ε_{ABCDEF} is unit completely antisymmetric symbol, $\varepsilon_{123456} = 1$. The relations (10) represent the Dirac matrices algebra in six-dimensional form.

Now the equations (2) can be written in the six-dimensional form

$$P_A(\Phi_A)^{\alpha\beta} u_\beta = 0; \quad S_A(\Phi_A)^{\alpha\beta} u_\beta = 0; \quad P^2 = 0, S^2 = 0, (SP) = 0. \tag{11}$$

Expressions (11) are invariant under transformations

$$S \to S' = S + \alpha P. \tag{12}$$

6-vectors P_A and S_A have the following components: $(p_1, p_2, p_3, ip_0, m, 0)$, $(s_1, s_2, s_3, is_0, -\sigma, i)$.

The next step we may make it is to consider the real pseudo-Euclidean six-dimensional space. It is necessary in order to introduce the self-adjoint bilinear forms to construct the matrix elements for calculations of the quantum-mechanical transition probabilities.

The local isomorphism between groups *SO(6), SO(5,1), SO(4,2), SO(3,3)* and the groups *SU(4), SU*(4), SU(2,2), SL(4,R)* were discovered by É. Cartan in 1914 [9] and are known to mathematicians as the constituent of *Cartan List* [10]. The groups *SO(4,2), SO(5,1)* are of interest of our purpose.

Let us consider at first real pseudo-Euclidean six-dimensional space with metrics signature (+ + + – – +) and introduce analogue of Dirac conjugate bispinor $(\tilde{u})^\alpha = (\bar{u} i\gamma_5)^\alpha = (u^* i\gamma_4\gamma_5)^\alpha$. Then we can obtain from (11) the equations

$$\tilde{u}^\beta P_A(\Phi_A)_{\beta\alpha} = 0; \quad \tilde{u}^\beta S_A(\Phi_A)_{\beta\alpha} = 0 \tag{13}$$

and write the expression (3) for the density matrix in such form

$$u_\alpha \tilde{u}^\beta = \frac{-iP_A(\Phi_A)_{\alpha\gamma} S_B(\Phi_B)^{\gamma\beta}}{2} = P_A S_B \Sigma_{AB}. \tag{14}$$

The matrices $(\Phi_A)_{\alpha\beta}$ in (13) and (14) are the following matrices

$$(\Phi_\mu)_{\alpha\beta} = (i\gamma_5\gamma_\mu C)_{\alpha\beta}, \ \mu = 1,2,3,4; \quad (\Phi_5)_{\alpha\beta} = (\gamma_5 C)_{\alpha\beta}, (\Phi_6)_{\alpha\beta} = (-iC)_{\alpha\beta}. \tag{15}$$

The relation (14) permits substantially simplify the calculation of quadric forms of bilinear combinations of bispinors presented in [11].

One can verify the validity of the following simple relations (if we put the restrictions on matrix C – $\det C = 1, \varepsilon^{\alpha\beta\gamma\delta}(C)_{\alpha\beta}(C)_{\gamma\delta} = -8$):

$$\varepsilon_{\alpha\beta\gamma\delta}(\Phi_A)^{\alpha\beta}(\Phi_B)^{\gamma\delta} = 8\delta_{AB};$$

$$\varepsilon^{\alpha\beta\gamma\delta}(\Phi_A)_{\alpha\beta}(\Phi_B)_{\gamma\delta} = 8\delta_{AB};$$

$$\begin{aligned}
&\varepsilon_{\alpha\beta\gamma\delta}(\Phi_A)^{\gamma\delta} = -2(\Phi_A)_{\alpha\beta};\\
&\varepsilon^{\alpha\beta\gamma\delta}(\Phi_A)_{\gamma\delta} = -2(\Phi_A)^{\alpha\beta};\\
&(\Phi_A)_{\alpha\beta}(\Phi_A)_{\gamma\delta} = 2\varepsilon_{\alpha\beta\gamma\delta};\\
&(\Phi_A)^{\alpha\beta}(\Phi_A)^{\gamma\delta} = 2\varepsilon^{\alpha\beta\gamma\delta};\\
&(\Phi_A)_{\alpha\beta}(\Phi_A)^{\gamma\delta} = -2(\delta_\alpha^\gamma\delta_\beta^\delta - \delta_\alpha^\delta\delta_\beta^\gamma).
\end{aligned} \tag{16}$$

The considering of real pseudo-Euclidean six-dimensional space with metrics signature (+ + + – + +) and using last relation from (16) permits easily to obtain the useful formulae [11] which allow expressing an arbitrary bilinear combination of bispinors describing particles with definite 4-momenta in the next form:

$$M = \overline{u_2}(i\gamma_\mu A_\mu - iB\gamma_5 + C)u_1, \tag{17}$$

where A_μ, B and C are components of some 6-vector $(A_1, A_2, A_3, iA_0, C, B)$.

4. FIVE-DIMENSIONAL DESCRIPTION OF SPIN ½ POLARIZED PARTICLE

Five-dimensional theories ascending to T. Kaluza and O. Klein after many years of disregarding are popular now again. Five-dimensional description of spin ½ polarized particle is similar to six-dimensional one of previous section and is simple and useful.

Let us introduce five matrices $(\Gamma_a)_\alpha^\beta$, a = 1,2,3,4,5,

$$(\Gamma_\mu)_\alpha^\beta = (i\gamma_5\gamma_\mu)_\alpha^\beta, (\Gamma_5)_\alpha^\beta = (\gamma_5)_\alpha^\beta. \tag{18}$$

Now we can write the relations (2-4) in the form

$$\begin{aligned}
&\langle P\rangle u = 0, (1 - \langle S\rangle)u = 0;\\
&P^2 = 0, S^2 = 1, (SP) = 0;\\
&u_\alpha \tilde{u}^\beta = \left(\frac{i\langle P\rangle(1-\langle S\rangle)}{2}\right)_\alpha^\beta;\\
&S \to S' = S - \alpha P,
\end{aligned} \tag{19}$$

where P and S are the following 5-vectors with components (p_1, p_2, p_3, ip_0, m), $(s_1, s_2, s_3, is_0, -\sigma)$; $\langle P\rangle = P_a\Gamma_a, \langle S\rangle = S_a\Gamma_a$.

From relations (13) we have

$$\tilde{u}\langle P\rangle = 0, \tilde{u}(1 + \langle S\rangle) = 0. \tag{20}$$

Using the relation following from (16),

$$\begin{aligned}
&(\Gamma_a)_\alpha^\beta(\Gamma_a)_\gamma^\delta - (I)_\alpha^\beta(I)_\gamma^\delta\\
&= -[(\Gamma_a)_\alpha^\delta(\Gamma_a)_\gamma^\beta - (I)_\alpha^\delta(I)_\gamma^\beta],
\end{aligned} \tag{21}$$

we can express an arbitrary bilinear combination of bispinors describing particles with definite 5-momenta in the next form:

$$\begin{aligned}
(\tilde{u}_2 Q u_1) = &\frac{1}{(P_2P_1)}\Big[\frac{1}{4}Sp(\langle P_2\rangle Q\langle P_1\rangle)(\tilde{u}_2 u_1)\\
&-\frac{1}{4}Sp(\langle P_2\rangle Q\langle P_1\rangle\Gamma_a)(\tilde{u}_2\Gamma_a u_1)\Big],
\end{aligned} \tag{22}$$

where Q is arbitrary 4×4 matrix. Thus we can consider the following form of the matrix element of arbitrary form (in according to (18))

$$M = \tilde{u}_2 \big(i \langle A \rangle + B \big) u_1 = -\bar{u}_2 \big(i\gamma_\mu A_\mu + A_5 - iB\gamma_5 \big) u_1 . \quad (23)$$

The addition of the 5-vector P_2 or P_1 to 5-vector A does not change the matrix element (23). Without loss of generality we can put

$$(AP_2) = (AP_1) = 0. \quad (24)$$

The calculation of the square of matrix element absolute value (22) gives

$$|M|^2 = -((A^*A) + B^*B)(P_2P_1) + (RS_1) + (TS_2) + L_{ab}S_{2a}S_{1b}, \quad (25)$$

where

$$R_a = -i(B^*A_a - BA_a^*)(P_2P_1) + \{aP_2P_1A^*A\}, \quad (26)$$

$$T_a = i(B^*A_a - BA_a^*)(P_2P_1) + \{aP_2P_1A^*A\}, \quad (27)$$

$$L_{ab} = ((A^*A) - B^*B) \times [P_{2a}P_{1b} + P_{2b}P_{1a} - (P_2P_1)\delta_{ab}] + (A_a^*A_b + A_b^*A_a)(P_2P_1) - iB\{abP_2P_1A^*\} - iB^*\{abP_2P_1A\}, \quad (28)$$

$$\{abcde\} = \varepsilon_{abcde}. \quad (29)$$

5. FIVE-DIMENSIONAL RELATIONS BETWEEN POLARIZATIONS OF SPIN ½ POLARIZED PARTICLES

From (25) we easily obtain the following relations between polarizations of spin ½ polarized particles:

$$S_{2a}(S_1) = \frac{e_a + l_{ab}S_{1b}}{1 + (fS_1)}, \quad (30)$$

where

$$e_a = \frac{T_a}{((A^*A) + B^*B)(P_2P_1)}, \quad (31)$$

$$f_a = \frac{R_a}{((A^*A) + B^*B)(P_2P_1)}, \quad (32)$$

$$l_{ab} = \frac{L_{ab}}{((A^*A) + B^*B)(P_2P_1)}. \quad (33)$$

We can obtain from (30-33) the following simple relation

$$1 - S_2^2(S_1) = \frac{1 - S_1^2}{[1 + (fS_1)]^2} \frac{|(A^2 + B^2)|^2}{[(A^*A) + B^*B]^2}. \quad (34)$$

We see from (34) that, if $S_1^2 = 1$, then $S_2^2 = 1$ also (completely polarized particle remains completely polarized as it must be). If $S_1^2 = 0$, then

$$S_2^2(0) = 1 - \frac{|(A^2 + B^2)|^2}{[(A^*A) + B^*B]^2}. \quad (35)$$

It follows from (35) that, if initial particle is not polarized, appearance of final particle polarization take place only if A and B take complex values.

Using (35) we can rewrite (34) in the form

$$1 - S_2^2(S_1) = \frac{1 - S_1^2}{[1 + (fS_1)]^2}(1 - S_2^2(0)). \quad (36)$$

6. APPLICATION TO THE ELECTRON SCATTERING IN COULOMB FIELD

For the electron scattering in coulomb field the pseudoscalar B=0 and formulae we obtained are simplifying. In the first Born approximation all quantities A_a are real and we have

$$S_{2a}(S_1) = l_{ab}S_{1b}; \quad (37)$$

$$l_{ab} = \frac{P_{2a}P_{1b} + P_{2b}P_{1a}}{(P_2P_1)} - \delta_{ab} + 2\frac{A_aA_b}{A^2}. \quad (38)$$

We can easily verify that $l_{ac}l_{bc}=\delta_{ab}$, so (37) gives 5-rotation (5-Lorentz transformation) of 5-vector of polarization.

The quantities A_a become complex-valued in the second Born approximation (complex-valuedness arising due to radiative corrections gives negligible contribution to polarization phenomena). In the second Born approximation we obtain for nonzero components e_a and f_a

$$\vec{e} = \vec{f} = \vec{\tau}\Delta(\theta); \quad \vec{\tau} = \frac{[\vec{p}_2\vec{p}_1]}{|[\vec{p}_2\vec{p}_1]|};$$
$$\Delta(\theta) = 2Z\alpha \frac{V\sqrt{1 - V^2}}{1 - V^2\sin^2\frac{\theta}{2}} \frac{\sin^3\frac{\theta}{2}}{\cos\frac{\theta}{2}} \ln\frac{1}{\sin\frac{\theta}{2}}, \quad (39)$$

V is the velocity of the electron, θ is the angle of scattering. The value $\Delta(\theta)$ is well known Mott result [4]. From (36) we obtain

$$1 - S_2^2(S_1) = \frac{1 - S_1^2}{[1 + \Delta(\theta)(\vec{\tau}\vec{S}_1)]^2}(1 - \Delta^2(\theta)). \quad (40)$$

DISCUSSION

We demonstrated that Dirac equation and polarization description of spin ½ particle have simple six-dimensional and five-dimensional generalizations and used them for obtaining the relations (30) and (34) between polarizations of the particle. It would be very hard to obtain these relations using usual four-dimensional calculations. As an application of these results we considered polarization phenomena for the electron scattering in coulomb field in second Born approximation and obtained some new relations (40).

In the summary we would like to tell some sad story about one unexpected profit once obtained from investigations of extra dimensions.

One of pioneers of higher dimension theories was Yu.B. Rumer [12]. His first paper on fifth-dimension theory he published in 1931 [13]. Fifth-dimensional generalization of general relativity was very popular at that time. A. Einstein himself published the papers on this subject. Yu.B. Rumer met with A. Einstein and talk with him [14]. But Einstein did not become interested in

Rumer's work. In 1938 Yu.B. Rumer was arrested as a member of anti-soviet organization "created" by L.D. Landau. After ten years of imprisonment Yu.B. Rumer was exiled on five years in Yeniseisk. From Yeniseisk he sent his papers on fifth-dimensional relativity (5-optics) to the editors of ZhETF. "Poor Rumochka is going crazy!" – said L.D. Landau about these papers [14]. But papers were published and this publication helped Rumer's friends to liberate Yu.B. Rumer from exile on two and half years ahead of time.

Yu.B. Rumer was the first who proposed the compactification of fifth coordinate, identified it with the action and identified its period of change with Plank constant. Due to this identification the possibility has arisen to quantize electric charge and to consider the gauge transformations as transformations of fifth coordinate. Yu.B. Rumer published ten papers on 5-optics in ZhETF and one book [15]. But at that time it was already very unpopular subject and Yu.B. Rumer said himself "wieder nichts" ("never more") and given an undertaking to publish nothing again on fifth-dimensional relativity [14].

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О ШЕСТИМЕРНОМ УРАВНЕНИИ ДИРАКА И НЕКОТОРЫХ СООТНОШЕНИЯХ МЕЖДУ ПОЛЯРИЗАЦИЯМИ ЧАСТИЦЫ СО СПИНОМ ½

М.В. Любченко, Ю.П. Степановский

С помощью шестимерного описания поляризованной частицы со спином ½ были получены формулы, представляющие основные свойства поляризационных явлений. Полученные соотношения между поляризациями использованы для рассмотрения рассеяния электрона в кулоновском поле во втором борновском приближении.

ПРО ШОСТИВИМІРНЕ РІВНЯННЯ ДІРАКА І ДЕЯКІ СПІВВІДНОШЕННЯ МІЖ ПОЛЯРИЗАЦІЯМИ ЧАСТИНКИ ЗІ СПІНОМ ½

М.В. Любченко, Ю.П. Степановський

За допомогою шестивимірного представлення поляризованої частинки зі спином ½ були одержані формули, що представляють основні властивості поляризаційних явищ. Одержані співвідношення між поляризаціями використані для розгляду розсіяння електрона в кулонівському полі в другому борновському наближенні.