FLUTE MODE INSTABILITY OF ROTATING PLASMA

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The influence of rotation on flute instability is studied in the frame of one-fluid magnetohydrodynamics. We consider the simplest model of gravitating cylindrical plasma in the straight nonuniform magnetic field to simulate plasma behavior in mirrors. Using linear stability analysis, we derive dispersion equation and integral expression for the increment of instability. In virtue of this expression, it is shown that rotation itself appears to be destabilizing factor due to centrifugal effect; we prove the corresponding theorem in general. Eigenmode structure and the dependence of the threshold on the frequency of rotation are calculated for linear radial profile of the angular velocity.

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1. INTRODUCTION

A popular paradigm appeared in view of phenomenon of transport barriers is that sheared plasma rotation provides a stabilizing effect and is able to reduce turbulence [1]. However, the role of rotation can be also negative, e.g., due to specific hydrodynamic instabilities. In this report, we investigate how rotation affects the flute plasma instability – one of the most well known phenomena typical for magnetic confinement systems. We solve the eigenvalue equation for small flute perturbations without using the energetic principle.

2. BASIC EQUATIONS AND DISPERSION RELATION

Consider a cylindrical plasma in a straight magnetic field with gravity ($\mathbf{g} = g\mathbf{e}_r$ simulates effectively a role of the curvature of magnetic force lines); plasma rotates in azimuthal direction. In equilibrium

$$\mathbf{B_0} = B_0\mathbf{e}_z, \tag{1}$$

$$\mathbf{V_0} = r\Omega\mathbf{e}_\varphi,$$

$$\frac{d}{dr}\left(p_0 + \frac{B_0^2}{8\pi}\right) = \rho_0(g+\Omega^2 r),$$

where pressure p_0, magnetic field B_0, density ρ_0 and angular frequency of rotation Ω – functions of radius r. MHD equation in terms of displacement of liquid element $\boldsymbol{\xi}(t,\mathbf{r})$ has well known view [2]:

$$\rho_0\ddot{\boldsymbol{\xi}} + 2\rho_0(\mathbf{V_0}\cdot\boldsymbol{\nabla})\dot{\boldsymbol{\xi}} - \mathbf{F}(\boldsymbol{\xi}) = 0, \text{ where} \tag{2}$$

$$\begin{aligned}\mathbf{F}(\boldsymbol{\xi}) &= -\delta\rho(\mathbf{V_0}\cdot\boldsymbol{\nabla})\mathbf{V_0} - \rho_0(\delta\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V_0} - \\ &- \rho_0(\mathbf{V_0}\cdot\boldsymbol{\nabla})\delta\mathbf{V} - \boldsymbol{\nabla}\delta p + ((\boldsymbol{\nabla}\times\delta\mathbf{B})\times\mathbf{B_0} + \\ &+ \frac{1}{4\pi}(\boldsymbol{\nabla}\times\mathbf{B_0})\times\delta\mathbf{B}) + \delta\rho\mathbf{g}\end{aligned}$$

– force operator consisted from perturbed values,

$$\delta\rho = -\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\xi}),$$

$$\delta\mathbf{V} = (\mathbf{V_0}\cdot\boldsymbol{\nabla})\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\mathbf{V_0},$$

$$\delta\mathbf{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B_0}).$$

For flute modes, the perturbation of magnetic field is absent,

$$\delta\mathbf{B} = 0. \tag{3}$$

Eq.(3) makes the system Eq.(2) secluded so the specification of perturbed pressure δp is unnecessary.

Applying Fourier transformation $f(r,\varphi,t) = f(r)\cdot\exp(-i\omega t + im\varphi)$ and solving of Eq.(2), Eq.(3) we can arrive at eigenmode equation:

$$\frac{d}{dr}\left[\frac{\rho_0}{B_0}\frac{\varpi^2}{m^2}r\frac{d(r\bar{\xi}_r)}{dr}\right] + \left\{\frac{\rho_0}{B_0}\varpi\left(\frac{2r}{m}\frac{d\Omega}{dr} - \varpi\right) + \right.$$
$$\left. + \frac{d}{dr}\left(\frac{\rho_0}{B_0}\right)\left(g + r\Omega^2 + \frac{2r}{m}\varpi\Omega\right)\right\}\bar{\xi}_r = 0, \tag{4}$$

where $\bar{\xi}_r = \xi_r B_0$, ξ_r – normalized radial displacement of plasma, $\varpi = \omega - m\Omega$ – Doppler-shifted frequency. Last term in Eq.(4) is proportional to the total force acting on element of liquid volume – gravitation, centrifugal and Coriolis forces. Due to Eq.(4), in the absence of rotation ($\Omega \to 0$), the reducing with r profile of ρ_0/B_0 is unstable when $g > 0$ and stable when $g < 0$ – that corresponds to well known result for static stability [3].

3. INCREMENT OF INSTABILITY AND DESTABILIZATION BY ROTATION

Integration of Eq.4 over the full volume filled with plasma gives integral expression for the increment of instability, $\gamma = \mathrm{Im}(\omega)$:

$$\begin{aligned}\gamma^2 &= \frac{m^2}{(I-A)}\left\{\int\Omega^2\left(\frac{\rho_0}{B_0}r^2\left|\frac{d\bar{\xi}_r}{dr}\right|^2 + \right.\right. \\ &+ \left.\frac{\rho_0}{B_0}(m^2-1)|\bar{\xi}_r|^2\right)rdr - \\ &- \left.\int g\frac{d}{dr}\left(\frac{\rho_0}{B_0}\right)|\bar{\xi}_r|^2 rdr\right\} - \\ &- \frac{m^2J^2}{(I-A)^2} - \frac{m^2AI\Omega^2_{|r=1}}{(I-A)^2}.\end{aligned} \tag{5}$$

Here

$$\begin{aligned}I &= \int\left(\frac{\rho_0}{B_0}r^2\left|\frac{d\bar{\xi}_r}{dr}\right|^2 + \left(\frac{\rho_0}{B_0}(m^2-1) - \right.\right. \\ &- \left.\left. r\frac{d}{dr}\left(\frac{\rho_0}{B_0}\right)\right)|\bar{\xi}_r|^2\right)rdr,\end{aligned}$$

$$J = \int\Omega\left(\frac{\rho_0}{B_0}r^2\left|\frac{d\bar{\xi}_r}{dr}\right|^2 + \frac{\rho_0}{B_0}(m^2-1)|\bar{\xi}_r|^2\right)rdr,$$

$$A = \left(\frac{1}{2}\frac{\rho_0}{B_0}r^3\frac{d|\bar{\xi}_r|^2}{dr}\right)_{|r=1}.$$

In our case, the instability condition $\gamma^2 > 0$ results in

$$\begin{aligned} J^2 + AI\Omega^2_{|r=1} &< (I-A)\Bigg\{ \int \Omega^2 \left(\frac{\rho_0}{B_0} r^2 \left| \frac{d\bar{\xi}_r}{dr} \right|^2 + \right. \\ &+ \left. \frac{\rho_0}{B_0}(m^2-1)|\bar{\xi}_r|^2 \right) r dr - \\ &- \int g \frac{d}{dr}\left(\frac{\rho_0}{B_0}\right) |\bar{\xi}_r|^2 r dr \Bigg\}. \end{aligned} \quad (6)$$

Let's put $g = 0$ to exclude the influence of magnetic field's curvature. Put also $\int d(\rho_0/B_0)/dr|\bar{\xi}_r|^2 r^2 dr = 0,\ \ A = 0$ and enter

$$\Omega \sqrt{r \left(\frac{\rho_0}{B_0} r^2 \left| \frac{d\bar{\xi}_r}{dr} \right|^2 + \frac{\rho_0}{B_0}(m^2-1)|\bar{\xi}_r|^2 \right)} = X, \quad (7)$$

$$\sqrt{r \left(\frac{\rho_0}{B_0} r^2 \left| \frac{d\bar{\xi}_r}{dr} \right|^2 + \frac{\rho_0}{B_0}(m^2-1)|\bar{\xi}_r|^2 \right)} = Y.$$

We see that condition of instability (6) is equivalent to the famous inequality by Cauchy-Bunyakovsky

$$\left(\int XY dr \right)^2 < \int X^2 dr \int Y^2 dr. \quad (8)$$

It is identically valid for every $X, Y \neq 0$ ($\int d(\rho_0/B_0)/dr|\bar{\xi}_r|^2 r^2 dr \neq 0,\ \ A \neq 0$ just strengthen the inequality). So in our problem, rotation is destabilizing factor and could be compensated only by confining gravitationl field, $\mathbf{g} = -|g|\mathbf{e}_r$.

4. NUMERICAL CALCULATIONS

Second-order equation on eigenvalues, Eq.(4), with zero boundary conditions has been solved numerically by the algorithm similar to the algorithm used in [4] for calculation of magnetorotational instability. The firm of calculations is provided by double verification: by shooting method and by control of eigenvalues from integral expressions.

In calculations we have chosen a parabolic profile for the density and a linear profile for angular velocity of rotation:

$$\frac{\rho_0}{B_0} = \left(\frac{\rho_0}{B_0}\right)_{|r=0} (1 - r^2), \quad (9)$$

$$\Omega = \Omega_{|r=0} + \frac{d\Omega}{dr} r,\ \frac{d\Omega}{dr} = \text{const}.$$

Typical spectrum of unstable modes is presented in Fig. 1; the structure of the modes in the case $g = 0$ is given in Fig. 2.

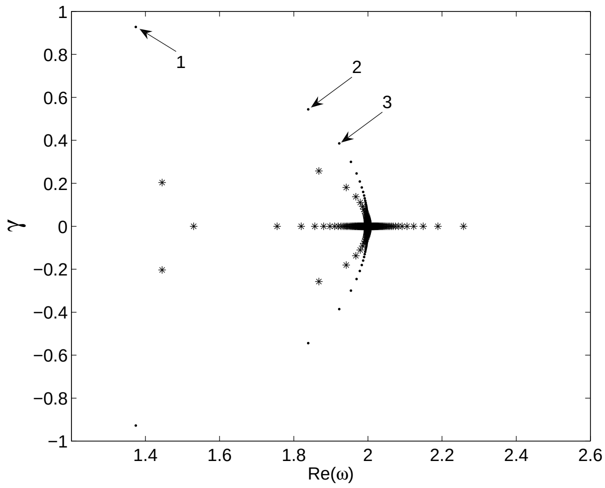

Fig.1. Typical spectrum in case $m = 2$, $\Omega_{|r=0} = 1$, $d\Omega/dr = 0$; Points – $g = 0$, stars – $g = -0.5$

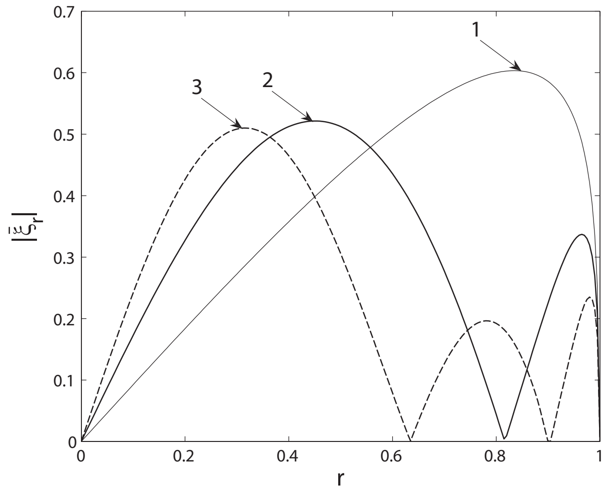

Fig.2. Eigenmode structure in case $g = 0$

The mode with minimal number of nodes (line 1) has maximal increment. In case of differential rotation, the nodes on the radial mode's profile can be smoothed. Values of increments decrease a bit in comparison with solid rotation due to reduction of averaged Ω^2.

Presence of stabilizing gravitational field ($g < 0$) reduces increments of instability and the number of unstable modes. System stabilized when $g \leq g_c = -|g_c|$ – threshold of instability. At the threshold, the real part of all unstable modes tend to Doppler resonanse, which takes place on the border for chosen profile of density Eq.(9). The value of the threshold is determined by centrifugal acceleration on the border and doesn't depend on the azimuthal mode's number:

$$|g_c| = r\Omega^2_{|r=1}. \quad (10)$$

The values of increments grow with growing frequency of rotation, but aren't determined by it's value at the border. They depend on both $\Omega_{|r=0}$ and $d\Omega/dr$, and also grow with m.

While coming to the threshold, the eigenfunctions lose their individual features. All modes tend to δ-functions independently on azimuthal number and on the profile of rotation. Also we studied piecewise linear profile of angular velocity. Increment has a minimum as a function of the peak of trigonal Ω profile for all modes, except $m = 1$. Increment of first mode goes down to constant while peak of rotation grows. By the way it's not correct to associate this reducing of increment with stabilizing action of rotation with trigonal profile, because the value of threshold reduces too.

Reduction of increment could be explained by changes in the mode's structure. In Fig. 3, it is shown that perturbation aspires to escape from the area of fastest rotation.

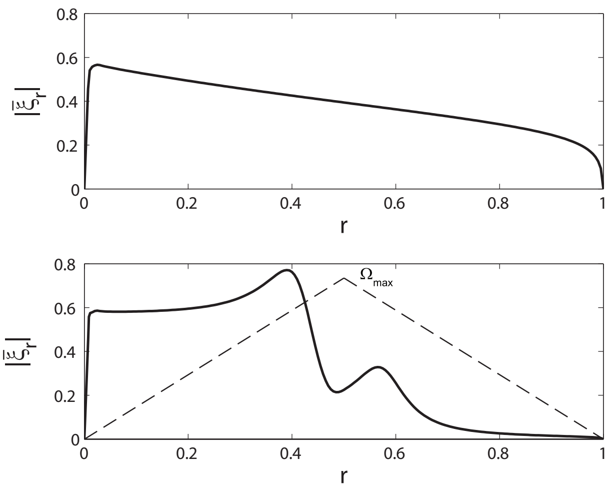

Fig. 3. Deformation of eigenfunction $m = 1$, $g = 2$ by trigonal profiled rotation with peak $\Omega_{max} = 11$

CONCLUSION

It is shown that rotation of cylindrical gravitating plasma in a straight magnetic field appears to be destabilizing factor due to centrifugal effect – it reduces the threshold of flute instability. For the instability, eigenvalues and eigenfunctions are calculated. At the instability threshold, the eigenmodes are shown to have singularities in hydrodynamic resonance points. Also we demonstrate that the special profiling of the rotation (which forms a sheared layer in the area of the mode localization) could affect the mode structure and slightly reduce the increment of instability. Nevertheless, since the rotation reduces the threshold, a possible favorable effect of rotation associates with nonlinear self-organization of the resulting turbulence rather then with a suppression of linear instability.

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ЖЕЛОБКОВАЯ НЕУСТОЙЧИВОСТЬ ВРАЩАЮЩЕЙСЯ ПЛАЗМЫ

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В рамках одножидкостной магнитной гидродинамики исследуется влияние вращения на желобковую неустойчивость. Предполагается простейшая модель цилиндрической гравитирующей плазмы в прямом неоднородном магнитном поле для моделирования поведения плазмы в зеркальных ловушках. С помощью линейного анализа получено дисперсионное уравнение и интегральное выражение для инкремента неустойчивости. Показано, что в данной постановке задачи вращение из-за центробежного эффекта является сугубо дестабилизирующим фактором; в общем случае доказывается соответствующая теорема. Для линейного радиального профиля угловой скорости вращения рассчитана структура собственных мод и зависимость порога неустойчивости от частоты вращения.