MODELLING OF MAXWELL'S EQUATIONS USING UNIFORM FINITE ELEMENTS

V.E. Moiseenko
Institute of Plasma Physics, National Science Center "Kharkov Institute of Physics and Technology", 61108 Kharkov, Ukraine

The theory of numerical stability of weighted residuals schemes for Maxwell's equations written in terms of electric field is presented. Basing on it, the numerically stable scheme using physical components of electric field and uniform trial functions is developed. The proposed scheme is tested in cylindrical geometry and compared with the numerically stable Galerkin scheme. The tests show the evidence of numerical stability of the scheme proposed. The convergence is monotonic and corresponds to the order of approximation. It is demonstrated that, unlike the Galerkin scheme, the scheme proposed is much less sensitive to the stiffness of the Maxwell's equations in plasma.
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INTRODUCTION

The Maxwell's equations in terms of electric field are degenerate. This is the origin of problems for solving them numerically. However, the numerically stable finite element Galerkin schemes are developed (see [1]) and used in practice. The differs from standard finite element schemes by the following:

for different components of electric field, the finite elements of different order should be used;

in curvilinear geometry, not physical, but covariant components of electric field should be discretized.

For example, in cylindrical geometry with discretization in the radial direction only, the test and trial functions that are conjugate each to other for Galerkin method are $\left(e_r\Lambda^{(s-1)},\ e_\varphi\Lambda^{(s)}/r,\ e_z\Lambda^{(s)}\right)$, where Λ is the finite element (hat) function, s is the finite element order. The discretized components of the electric field are $\left(E_r, rE_\varphi, E_z\right)$.

In ion cyclotron range of frequencies (ICRF) the dielectric response of plasma depends strongly on the direction of steady magnetic field. First, the dielectric response is much higher for the component of electric field parallel to the steady magnetic field. Second, under condition of the fundamental cyclotron resonance the plasma response is substantially different for left and right polarized electric field components and, only left polarized component provides the cyclotron damping.

For numerical calculations, these features introduce some kind if stiffness. To treat it correctly it is good to use left polarized, right polarized and parallel to the steady magnetic field components of electric field. This would be possible if the physical components of electric field are used and, all of them could be discretized with the same finite elements. But, this is not possible in the framework of the above-mentioned numerically stable Galerkin approach.

WEIGHED RESIDUALS SCHEME

Consider the linear eigenvalue problem for Maxwell's equations:

$$\nabla\times\nabla\times\vec{E} = \frac{\omega^2}{c^2}\hat{\varepsilon}\cdot\vec{E}, \tag{1}$$

with ω^2 as an eigenvalue and assume no dependence on ω in the dielectric tensor $\hat{\varepsilon}$. This problem has a multiple eigenvalue $\omega^2 = 0$. To provide the numerical stability this multiple eigenvalue should be reproduced in discretized equations too [1]. In other words, for $\omega^2 = 0$ the discretized system should be degenerate at least N_i times. Here $N_i = N - N_b$ is the number of internal mesh nodes, N is the total number of mesh nodes and, N_b is the number of nodes at the boundary of the domain. In the framework of the weighted residuals approach the discretization is made integrating the equations with test functions. For internal mesh nodes and $\omega^2 = 0$ this integration reads:

$$\begin{aligned}&\int f_k\vec{e}_k\cdot\nabla\times\nabla\times\vec{E}\,dV = \\ &= \int\nabla\times\left(f_k\vec{e}_k\right)\cdot\nabla\times\vec{E}\,dV = 0\end{aligned}, \tag{2}$$

where f_k are the test functions, e_k is the unit vector, i is the index enumerating the test functions. The requirement of the degeneration of the equation set (2) could be written as following:

$$\int\nabla\times\left(\sum_k C_{i,k}f_k\vec{e}_k\right)\cdot\nabla\times\vec{E}\,dV = 0, \tag{3}$$

where $C_{i,k}$ are the constants. Since equation (3) should be met for different E, the left term in scalar product must be zero:

$$\nabla\times\left(\sum_k C_{i,k}f_k\vec{e}_k\right) = 0, \tag{4}$$

or

$$\sum_k C_{i,k}f_k\vec{e}_k = \nabla\Phi_i. \tag{5}$$

Here Φ_i, the generating function is introduced. Since our consideration relates to Galerkin method too, its functions should satisfy the equation (5). In fact, this is met. For example, for one-dimensional cylindrical problem with lowest order finite elements the finite element functions are

$$\begin{aligned}&f_{3(i-1)+1}\vec{e}_{3(i-1)+1} = \Lambda^{(0)}_{i-1/2}\vec{Ge}_r,\\ &f_{3(i-1)+2}\vec{e}_{3(i-1)+2} = \Lambda^{(1)}_i Ge_\varphi/r,\\ &f_{3i}\vec{e}_{3i} = \Lambda^{(1)}_i Ge_z\end{aligned} \tag{6}$$

with $G = \exp(-im\varphi - ik_z z)$. Here index i enumerates mesh nodes, $\Lambda^{(0)}_{i-1/2}$ is the finite element of zero order (piecewise constant function that is unity at the segment (

r_{i-1}, r_i) and zero outside), $\Lambda_i^{(1)}$ is the first order finite element (hat) function. The generating function is

$$\Phi_i = \Lambda_i^{(1)} G . \qquad (7)$$

The explicit form of equation (5) for such functions is the following:

$$\frac{f_{3i+1} e_{3i+1}}{r_{i+1} - r_i} - \frac{f_{3(i-1)+1} e_{3(i-1)+1}}{r_i - r_{i-1}} - im f_{3(i-1)+2} e_{3(i-1)+2} - ik_z f_{3i} e_{3i} = \nabla \Phi_i . \qquad (8)$$

Formula (5) restricts the choice of test functions and tells nothing on trial ones. Therefore, taking an advantage from this freedom, it is possible to use physical components of electric field vector and represent them by uniform finite element functions keeping test functions the same as in Galerkin method.

NUMERICAL EXPERIMENTS

In this section we compare the numerically stable Galerkin method and the method proposed, weighted residuals method with uniform trial functions (WRMUTF). For simplicity we use first order numerical scheme in cylindrical geometry. For Galerkin method test functions are represented by formulas (6). Trial functions are conjugate. For WRMUTF, the test functions are the same as for Galerkin method. The trial functions are simply first order finite elements:

$$f^{(T)}_{3(i-1)+j} = \Lambda_i^{(1)} G^* , \qquad (9)$$

where $j = 1,2,3$ and star means conjugation. For radial component of electric field the number of test functions is less by one than the number of trial functions. Thus, one more equation is necessary to make the discretized system complete. There are a number of possibilities to do this. We choose the simplest one providing the regularity condition at the axis:

$$E_r + ipE_\varphi = 0 \qquad (10)$$

with $p = \begin{cases} m, if\ |m| = 1 \\ 0, if\ |m| \neq 1 \end{cases}$.

We study the eigenvalue problem (1) with the dielectric tensor modeling cold plasma in magnetic field directed along z-axis

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_\perp & ig & 0 \\ -ig & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_{//} \end{pmatrix} \qquad (11)$$

with the components having the parabolic radial dependence. The ideally conducting metallic wall is positioned at $r = r_w$.

The example of the calculations is shown at Fig.1. This is the eigenmode of fast magnetosonic wave with the frequency higher than ion cyclotron. The parameters of the calculation are the following: $\varepsilon_\perp = -100.0$, $g = -210.0$, $\varepsilon_{//} = -10^6$, $r_w = 10\,cm$, $m = -3$, $k_z = 0.03\,cm^{-1}$. The eigenvalue found is $\omega_{eig} = 2256186933.17224\ c^{-1}$.

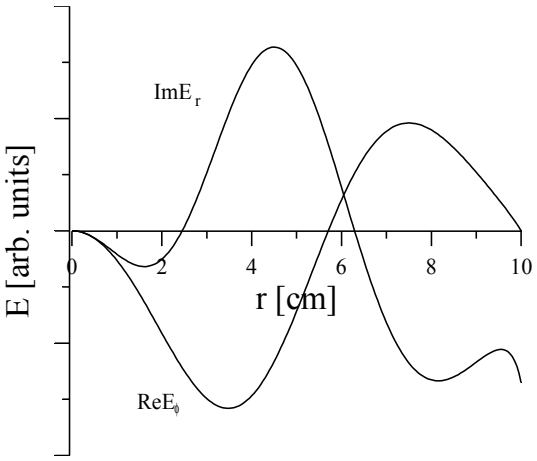

Fig.1. Distribution of $\mathrm{Im}\,E_r$ *and* $\mathrm{Re}\,E_\varphi$ *in plasma column. All other components of electric field are negligibly small*

The convergence curves, the dependence of relative error in frequency ω_{eig} on the number of mesh nodes, for both methods are shown in Fig.2. Both curves are the straight lines in logarithmic scale. This is the evidence of uniform convergence and absence of any numerical pollution. The slope of curves is almost the same meaning the same order of approximation. But WRMUTF has smaller level of the numerical error. We notice this feature in all our calculations. This could be explained by better approximation of E_r component of the electric field and by the absence of artificial singularities in equations that appear with introduction of rE_φ as a quantity.

The example of calculations shown does not exhibit the above-mentioned stiffness of Maxwell's equations in plasma. Since the axis of the dielectric tensor is z-axis the parallel component of electric field coincide with the unit vector of cylindrical geometry. Besides, the components of dielectric tensor $\varepsilon_+ = \varepsilon_\perp + g$ and $\varepsilon_- = \varepsilon_\perp - g$ are of the same order.

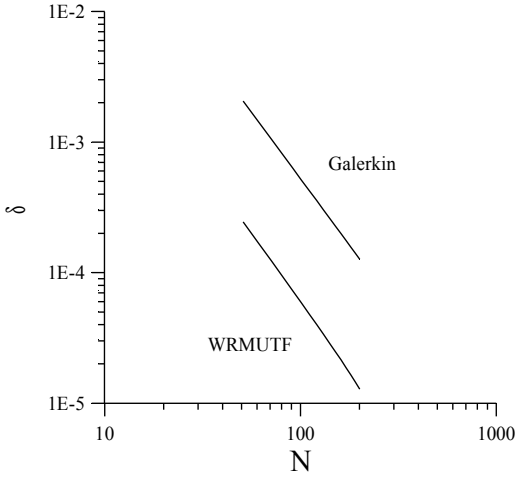

Fig.2. Relative error in ω_{eig} $\delta = \left| \frac{\omega_{eig}^{(N)} - \omega_{eig}}{\omega_{eig}} \right|$ *as a function of number of mesh points*

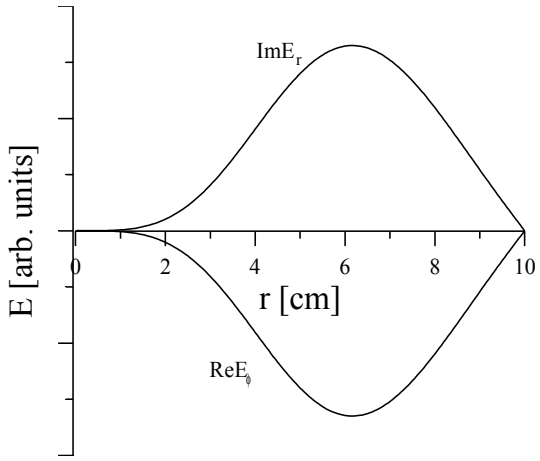

Fig.3. Distribution of $\mathrm{Im}\,E_r$ *and* $\mathrm{Re}\,E_\varphi$ *in plasma column for the eigenmode with dominantly non-hermitian dielectric tensor*

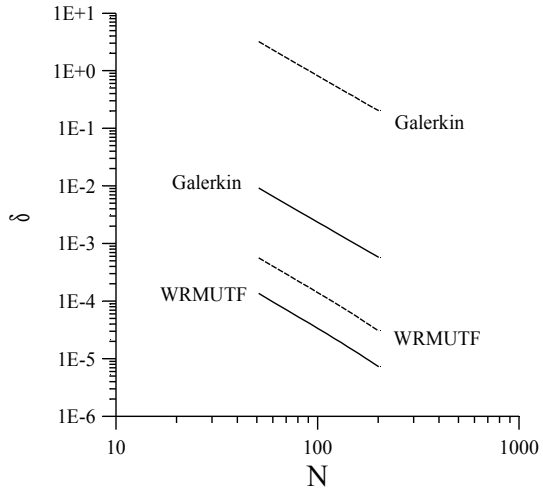

Fig.4. Relative error in $\mathrm{Re}\omega_{eig}$ *and* $\mathrm{Im}\omega_{eig}$ *as a function of number of mesh points*

To introduce the stiffness we add a big imaginary part to the ε_+ component of the dielectric tensor $\delta\varepsilon_+ = i10^4$. This corresponds to the case of fundamental cyclotron resonance. Under this condition the eigenmode of fast magnetosonic wave has almost right–polarized electric field (see Fig.3) and, its cyclotron damping is small. Indeed, the eigenvalue found $\omega_{eig} = (2005382553.78989 - i5736613.76168439)\ \mathrm{c}^{-1}$ has the imaginary part small compared with the real part regardless that non-hermitian part in dielectric tensor is dominant.

Fig.4 displays the convergence curves for real and imaginary part of eigenvalue. The curves for real part of frequency are similar to those ones of Fig.2 except that the difference between Galerkin method and WRMUTF becomes larger. The convergence in imaginary part of frequency is also uniform but figures of relative error for Galerkin method are inadmissibly high. WRMUTF demonstrates excellent convergence. The accuracy in this calculation is even better than in previous one. So, the introduction of stiffness has slight influence on WRMUTF.

CONCLUSIONS

We introduced and tested weighted residuals method with uniform trial functions (WRMUTF). As well as the Galerkin method that is frequently used for discretization of Maxwell's equations in terms of electric field, it is numerically stable. It is more efficient than Galerkin method when the stiffness pertinent to Maxwell's equations in plasma is important. It is more comfortable because all the components of electric field are represented uniformly. Technically it is similar to the Galerkin method and could be used in all cases in which the Galerkin method could.

REFERENCE

[1] R.Gruber and J.Rappaz *Finite Element Methods in Linear Ideal Magnetohydrodynamic* Springer-Verlag, Berlin, 1985.

МОДЕЛЮВАННЯ РІВНЯНЬ МАКСВЕЛА З ВИКОРИСТАННЯМ ОДНОРІДНИХ СКІНЧЕНИХ ЕЛЕМЕНТІВ

В.Є. Моісеєнко

В роботі подана теорія числової стійкості схем зважених нев'язок, що застосовані до рівнянь Максвела з виключеним магнітним полем. На її основі розроблена чисельно стійка схема, яка використовує фізичні компоненти електричного поля та однорідні пробні функції. Для цієї схеми проведено тестування у порівнянні зі схемою Гальоркіна. Воно підтвердило числову стійкість запропонованої схеми. Аналіз збігання показав, що воно є монотонне і відповідне до порядку апроксимації. Тестові розрахунки продемонстрували, що в порівнянні зі схемою Гальоркіна запропонована схема є суттєво менш чуйною до жорсткості рівнянь Максвела в плазмовому середовищі.

МОДЕЛИРОВАНИЕ УРАВНЕНИЙ МАКСВЕЛЛА С ИСПОЛЬЗОВАНИЕМ ОДНОРОДНЫХ КОНЕЧНЫХ ЭЛЕМЕНТОВ

В.Е. Моисеенко

В работе представлена теория численной устойчивости схем взвешенных невязок применительно к уравнениям Максвелла с исключенным магнитным полем. На ее основе разработана численно устойчивая схема, использующая физические компоненты электрического поля и однородные пробные функции. Для этой схемы проведено тестирование в сравнении со схемой Галеркина. Оно подтвердило численную устойчивость предложенной схемы. Анализ сходимости показал, что она является монотонной и соответствует порядку аппроксимации. Тестовые расчеты продемонстрировали, что по сравнению со схемой Галеркина предложенная схема значительно менее чувствительна к жесткости уравнений Максвелла в плазменной среде.