EXACT PLASMA DISPERSION FUNCTIONS FOR COMPLEX FREQUENCIES

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On the base the theory of Cauchy type integrals is given an analytic continuation of the exact relativistic plasma dispersion functions from the real axis into the complex region and studied their analytical properties in this region.
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1. INTRODUCTION

The basic to study linear plasma waves in hot enough plasmas is an evaluation of the relativistic Maxwellian plasma dielectric tensor [1]. In order to give a recipe of it for arbitrary plasma and wave parameters the exact plasma dispersion functions (PDFs) in the form of Cauchy type integrals with purely real density, defined at the real axis and tending to zero in the infinite, were introduced [2-4]. A dielectric tensor has been presented as a finite Larmor radius expansion in terms of those PDFs, similarly to cases of non-relativistic and weakly relativistic approximations, to reduce an evaluation of the tensor to the PDFs evaluation. The exact PDFs is a generalization of the weakly relativistic PDFs [5] on the case of an arbitrary plasma temperature. Two ways evaluating the exact PDFs in the real frequency region were given and their main analytical properties were studied.

The main scope of the present work is an analytic continuation of the exact PDFs from the real axis to the complex region. On the base the theory of Cauchy type integrals we study their analytical properties in this region.

2. EXACT PDFs IN COMPLEX REGION

Exact PDFs of half-integer index q $(q \ge 3/2)$ for real frequency ω are defined by Cauchy type integrals [2-4]

$$Z_q(a,z,\mu) = \sqrt{\frac{-\beta}{2\pi\mu}} \frac{e^{-2\beta a-\mu}}{K_2(\mu)(\sqrt{a})^{q-1}} \times \int_{-\infty}^{+\infty} \frac{\left(\sqrt{a-t+t^2/(2\mu)}\right)^{q-1} K_{q-1}\left(-2\beta\sqrt{a}\sqrt{a-t+t^2/(2\mu)}\right) e^{\beta t} dt}{t-z}, \quad (1)$$

$N_{//} > 1$,

$$Z_q(a,z,\mu) = \sqrt{\frac{\pi\beta}{2\mu}} \frac{e^{-\mu\sqrt{\beta}}}{K_2(\mu)(\sqrt{a})^{q-1}} \times \int_{0}^{+\infty} \frac{\left(\sqrt{t(t/(2\mu)+1/\sqrt{\beta})}\right)^{q-1} I_{q-1}\left(2\beta\sqrt{a}\sqrt{t(t/(2\mu)+1/\sqrt{\beta})}\right) e^{-\beta t} dt}{t+z+a_r}, \quad (2)$$

$0 \le N_{//} < 1$,

where $a = \mu N_{//}^2/2$, $z = \mu(1-\omega_c/\omega)$, $a_r = \mu(1-\sqrt{1-N_{//}^2})$, $\mu = m_0c^2/T$, $\beta = 1/(1-N_{//}^2)$; $N_{//} = k_{//}c/\omega$, ω_c, m_0, T are the longitudinal refractive index, fundamental electron cyclotron frequency, rest mass of electron and plasma temperature; $I_{q-1}(x)$, $K_{q-1}(x)$ are modified Bessel function and Macdonald function of half-integer index $q-1$; square root means the positive branch of this function. If an argument z takes real values (ω is real) both integrals are divergent at the poles $t = z$ and $t = a_r - z$ (when $z \le a_r$), respectively, and must be understood as the Principal Part of these integrals in the sense of Cauchy. The contour of integration in that case is chosen to pass below the pole in the expression (1) and above the pole in the expression (2).

For real argument z those integrals can be evaluated by means of the next nonsingular integral forms [4]

$$Z_q = i\pi f_1(a,z,\mu) + \int_{-\infty}^{+\infty} \frac{f_1(a,t,\mu) - f_1(a,2z-t,\mu)dt}{t-z}, \quad (3)$$

$$Z_q = -i\pi f_2(a,a_r-z,\mu) + \int_{0}^{a_r-z} \frac{f_2(a,t,\mu) - f_2(a,2(a_r-z)-t,\mu)dt}{t+z-a_r} - \int_{-\infty}^{0} \frac{f_2(a,2(a_r-z)-t,\mu)dt}{t+z-a_r}, \quad (4)$$

respectively, where

$$f_1(a,t,\mu) = -\sqrt{\frac{-\beta}{2\pi\mu}} \frac{e^{-2\beta a-\mu}}{K_2(\mu)(\sqrt{a})^{q-1}} (\sqrt{a-t+t^2/(2\mu)})^{q-1} \times K_{q-1}\left(-2\beta\sqrt{a}\sqrt{a-t+t^2/(2\mu)}\right) e^{\beta t},$$

$$f_2(a,t,\mu) = \begin{cases} \sqrt{\frac{\pi\beta}{2\mu}} \frac{e^{-\mu\sqrt{\beta}}}{K_2(\mu)(\sqrt{a})^{q-1}} \left(\sqrt{t(t/(2\mu)+1/\sqrt{\beta})}\right)^{q-1} \times I_{q-1}\left(2\beta\sqrt{a}\sqrt{t(t/(2\mu)+1/\sqrt{\beta})}\right) e^{-\beta t}, & t > 0 \\ 0, & t \le 0. \end{cases}$$

Calculating the exact PDFs on the basis of integrals (1), (2) and using integrals forms (3), (4) allows one also to continue analytically those PDFs on total complex region on the base some facts from the theory of Cauchy type integrals.

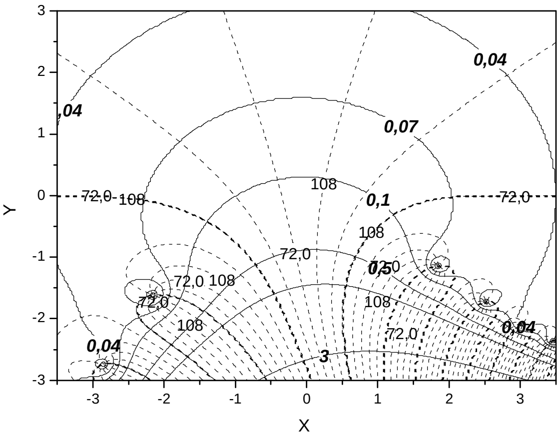

Fig.1. Module-argument diagram of function $2\sqrt{a}Z_{5/2}(a,z_1,\mu)$ *for* $N_{//}=1.1$ *and* $T_i = 100keV$

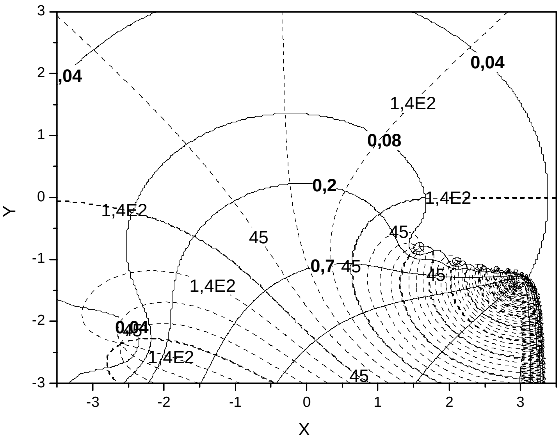

Fig.2. Module-argument diagram of function $2\sqrt{a}Z_{5/2}(a,z_1,\mu)$ *for* $N_{//}=1.1$ *and* $T_i = 1000keV$

We start from the case $N_{//} > 1$ which is rather similar to analytical continuation of the non-relativistic PDF since in this case the integrand in (1) is also the entire function at the contour of integration. Then for $z = x + iy$ in the upper semi-plane those PDFs are defined by the expressions

$$Z_q^+(a,z,\mu) = \int_{-\infty}^{+\infty} \frac{(t-x) f_1(a,t,\mu)dt}{(t-x)^2 + y^2} + i\int_{-\infty}^{+\infty} \frac{y f_1(a,t,\mu)dt}{(t-x)^2 + y^2}, \quad y>0, \qquad (5)$$

$$Z_q^+(a,x,\mu) = \int_{-\infty}^{+\infty} \frac{f_1(a,t,\mu)dt}{t-x} + i\pi f_1(a,x,\mu), \quad y = 0, \qquad (6)$$

where formula (5) follows from expression (1) by means of analytical continuation of integrand into upper semi-plane and formula (6) is obtained from (5) by limit passing $y \to 0$. Divergent expression (6) can be evaluated on the base nonsingular integral form (3). Then from (5) and formulas of Sokhotskii-Plemelj it follows analytical continuation of functions $Z_q(a,z,\mu)$ from upper semi-plane into the low one

$$Z_q^+(a,z,\mu) = Z_q^+(a,z^*,\mu) + 2\pi i f_1(a,z,\mu), \quad y<0, \qquad (7)$$

where asterisk denotes complex conjugation. This branch of $Z_q(a,z,\mu)$ corresponds to Landau rule of passing the pole. If to start from analytical continuation of (1) first into the low semi-plane and then into the upper one we will obtain the second branch $Z_q^-(a,z,\mu) = Z_q^*(a,z^*,\mu)$ which has a sense for negative values of $N_{//}$ [6].

At the Figs. 1,2 there are presented plots of module-argument diagram for function $2\sqrt{a}Z_{5/2}(a,2\sqrt{a}z_1,\mu)$ for $N_{//}=1.1$, which corresponds ICR frequency range, and $T_i = 200,\ 2000$ keV, respectively, for $|z_1| = |z/(2\sqrt{a})| \le$ $\le constant$, obtained using formulas (5)-(7) (module is presented in logarithm scale). It can be concluded from these plots that exact PDFs are loosing module symmetry which there is in non-relativistic PDF respectively of imaginary axis. This anti-symmetry becomes more and more essential with growing of ion plasma temperature.

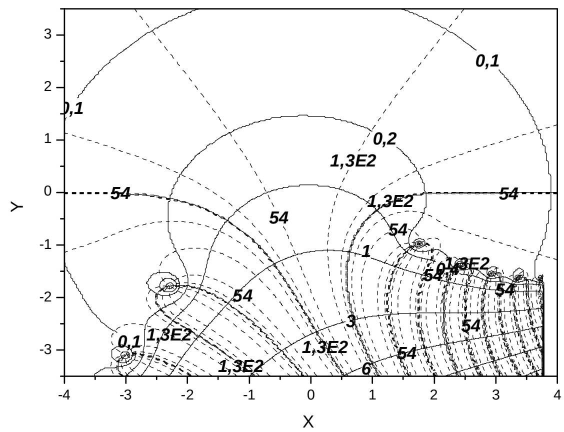

Fig.3. Module-argument diagram of function $2\sqrt{a}Z_{5/2}(a,z_1,\mu)$ *for* $T_e = 2keV$ *and* $N_{//} = 0.6$

For the case $0 \le N_{//} < 1$ on the same way we will have next formulas for analytical continuation of formula (2) for $Z_q(a,z,\mu)$ on the whole complex plane with cutting along the line $\mathrm{Re}\, z = a_r$ ($\mathrm{Im}\, z < 0$)

$$Z_q^+(a,z,\mu) = \int_0^{+\infty} \frac{(t+x-a_r) f_2(a,t,\mu)dt}{(t+x-a_r)^2 + y^2} - i\int_0^{+\infty} \frac{y f_2(a,t,\mu)dt}{(t+x-a_r)^2 + y^2}, \quad y>0, \qquad (8)$$

$$Z_q^+(a,x,\mu) = \int_0^{+\infty} \frac{f_2(a,t,\mu)dt}{t+x-a_r} - i\pi f_2(a,a_r - x,\mu), \quad y = 0, \qquad (9)$$

$$Z_q^+(a,z,\mu) = \begin{cases} Z_q^+(a,z^*,\mu) - 2\pi i f_2(a,a_r - z,\mu), & \mathrm{Re}\, z < a_r, y<0 \\ Z_q^+(a,z^*,\mu) & \mathrm{Re}\, z > a_r, y<0. \end{cases} \qquad (10)$$

The cutting line is uniquely defined by PDFs behavior for the case $N_{//} > 1$ while plasma temperature is increasing (fig. 1,2). We will call this continuation first branch of the function $Z_q(a,z,\mu)$ in this case.

If to start from analytical continuation of expression (2) into low semi-plane and then by similar way into upper semi-plane we will obtain the function

$Z_q^-(a,z,\mu) = Z_q^*(a,z^*,\mu)$ with cutting along the line $\mathrm{Re}z = a_r$ ($\mathrm{Im}\, z > 0$). We will call this continuation which has a sense for negative values of $N_{//}$ by second branch of the function $Z_q(a,z,\mu)$ in the case $0 \le N_{//} < 1$. Obviously, both those branches are identical for $\mathrm{Re}\, z > a_r$.

Thus, we have obtained a double-valued analytical function defined on the whole complex plane excepting the points $z = a_r$ and $z = \infty$, since it is known [5] that function $Z_q(a,z,\mu)$ and its derivatives till $(q-1/2)$th are continues at the point $z = a_r$, and the $(q-1/2)$th derivative has a single pole at that point. These branches are separating when $\mu \to \infty$ since $a_r \to +\infty$ in this case.

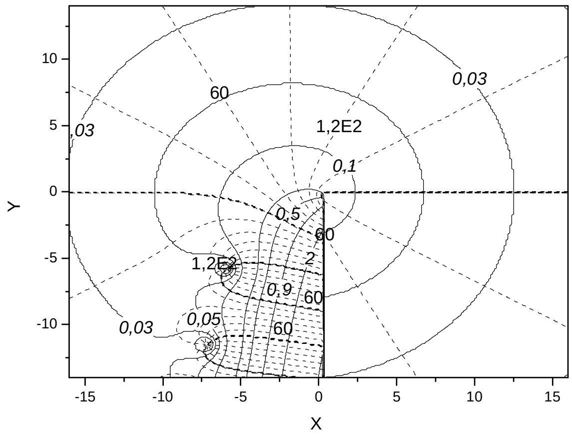

Fig.4. Module-argument diagram of function $2\sqrt{a}Z_{5/2}(a,z_1,\mu)$ *for* $T_e = 2keV$ *and* $N_{//} = 0.06$

At the Figs. 3,4 there are presented the same plots as in Figs. 1,2 for $T_e = 2$ keV and $N_{//} = 0.6$, 0.06, respectively, which are relevant to ECR frequency range, obtained using formulas (8)-(10). It can be conclude from these plots that exact PDFs are loosing module symmetry which there is in non-relativistic PDF respectively of imaginary axis with decreasing of $N_{//}$. This anti-symmetry becomes more and more essential (similar to the case $N_{//} > 1$ with increasing of T) with decreasing of $N_{//}$. It worth to note that the number of zeroes in region $\mathrm{Re}\, z_1 > o$ in this case is finite and defined by the cutting line, in difference with the case $N_{//} > 1$ where the number of such zeroes is infinite.

CONCLUSIONS

The next conclusions can be drawn from this study.

1. On the base the theory of Cauchy type integrals it was studied analytical properties of exact PDFs in complex frequency region.
2. For the case $|N_{//}| < 1$, relating to the ECR frequency range, it was shown that every exact PDF is two branched analytic function for $\mathrm{Re}\, z_n < a_r$ (one branch has a sense for $N_{//} > 0$ and second branch for $N_{//} < 0$) with cutting line $\mathrm{Re}\, z_n = a_r$ (these branches coincides for $\mathrm{Re}\, z_n > a_r$).
3. In the alternative case, $|N_{//}| > 1$, relating to ICR frequency range, these branches are separating, as in non-relativistic approximation.

These results can be useful to study the properties of plasma wave instabilities and collisionless dumping in the frame of the initial value problem in relativistic regimes.

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ТОЧНЫЕ ПЛАЗМЕННЫЕ ДИСПЕРСИОННЫЕ ФУНКЦИИ ДЛЯ КОМПЛЕКСНЫХ ЧАСТОТ

С.С. Павлов, Ф. Кастехон, Н.Б. Древаль

На основе теории интегралов типа Коши дается аналитическое продолжение точных релятивистских плазменных дисперсионных функций с реальной оси в комплексную область и изучаются их аналитические свойства в этой области.

ТОЧНІ ПЛАЗМОВІ ДІСПЕРСІЙНІ ФУНКЦІЇ ДЛЯ КОМПЛЕКСНИХ ЧАСТОТ

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На основі теорії інтегралів типу Коші дається аналітичне продовження точних релятивістських плазмових дисперсійних функцій з реальної осі на комплексну область та вивчаються їх аналітичні властивості в цій області.