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Optimal stopping problem for processes with independent increments

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Abstract. We consider the optimal stopping problem for processes with independent increments with the exponential $g(x) = (1 - e^{-x})^+$ or logarithmic $g(x) = (\ln x)^+$ payoff function. For the exponential payoff function, it is shown that the optimal stopping time is the first time of hitting a certain level. For the logarithmic payoff function, it is proved that a moment of the first hitting of a level cannot be optimal.

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1. Introduction

Consider a model of financial market with the single risky asset. The price process of this asset can be modeled by a process with independent increments $\{X_t,\ t \in T\}$, $X_0 = x \in \mathbb{R} = (-\infty, \infty)$. This process is defined on the probability space $(\Omega, \mathcal{F}, P)$ with natural filtration $\mathcal{F}_t = \sigma\{X_s,\ s \leq t\}$, $\mathcal{F}_0 = \{\varnothing, \Omega\}$. A market model can be discrete (in this case, the parametric set $T \subset \mathbb{Z}^+ = \{0, 1, 2, \dots\}$) or continuous ($T \subset \mathbb{R}^+ = [0, \infty)$). The risk-free interest rate is assumed to be constant and equal to $q \geq 0$.

The problem of the optimal exercise of a perpetual contingent claim of the American type with payoff function g can be formulated as follows: to maximize the expected discounted payoff

$$\mathbf{E}(g(X_\tau) e^{-q\tau} \mathbf{I}\{\tau < \infty\})$$

in the class M of all $(\mathcal{F}_t)$-Markov moments τ taking values in $[0, \infty]$. In

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other words, the problem is to find a "value" function

$$V(x) = \sup_{\tau \in M} \mathbf{E}(g(X_\tau)e^{-q\tau}\mathbf{I}\{\tau < \infty\}). \tag{1.1}$$

We call τ^* the optimal stopping moment, if

$$V(x) = \mathbf{E}(g(X_{\tau^*})e^{-q\tau^*}\mathbf{I}\{\tau^* < \infty\}), \quad x \in \mathbb{R}. \tag{1.2}$$

The optimal stopping problem for the payoff function $g(x) = (x^+)^\upsilon = (\max\{x, 0\})^\upsilon$ with $\upsilon = 1, 2, \ldots$ in discrete time was solved in [1, 4], and these results were extended for arbitrary $\upsilon > 0$ in [3]. In [4], the problem with the payoff function $g(x) = (1 - e^{-x})^+$ for random walks was solved. In our work, we extend the results obtained in [4] to processes with independent increments and consider the optimal stopping problem for processes with independent increments and the payoff function $g(x) = (\ln x)^+$.

Similarly to [3], we will look for an optimal stopping moment in the form

$$\tau^* = \tau_a = \inf\{t \geq 0 : X_t \geq a\}, \tag{1.3}$$

where the optimal value of the parameter a depends on the function $g(x)$.

2. Appell functions

To solve the optimal stopping problem, we need the concept of Appell functions. Appell functions are some generalization of Appell polynomials (see, e.g., [5]).

Appell polynomials generated by a random variable η such that $\mathbf{E}\,|\eta|^n < \infty$ for all $n \geq 1$ are the polynomials

$$Q_k(y; \eta) = (-1)^k \frac{d^k}{du^k}\Big(\frac{e^{-uy}}{\mathbf{E}e^{-u\eta}}\Big)\Big|_{u=0}, \quad k = 1, 2, \ldots, n. \tag{2.1}$$

Assume now that η is a nonnegative random variable, and

$$P(\eta < \varepsilon) > 0 \text{ for all } \varepsilon > 0. \tag{2.2}$$

Define the Appell function of order υ for $\upsilon < 0$ by the equation

$$Q_\upsilon(y; \eta) = \int_0^\infty u^{-\upsilon-1} \frac{e^{-uy}}{\mathbf{E}e^{-u\eta}} \frac{du}{\Gamma(-\upsilon)}, \quad y > 0,\ \upsilon < 0, \tag{2.3}$$

where $\Gamma(z)$ is the Euler gamma function. According to this definition, the function $Q_\upsilon(y;\eta)$ is continuous in υ and y. Note that

$$\lim_{\upsilon\uparrow 0} Q_\upsilon(y;\eta) = 1 \tag{2.4}$$

and extend $Q_\upsilon(y;\eta)$ to $\upsilon = 0$ continuously, by setting

$$Q_0(y;\eta) = 1 \text{ for all } y > 0. \tag{2.5}$$

Now we define $Q_\upsilon(y;\eta)$ for real $\upsilon > 0$ using the equation

$$Q_\upsilon(y;\eta) = Q_\upsilon(0;\eta) + \upsilon \int_0^\infty Q_{\upsilon-1}(z;\eta)\,dz, \quad y > 0,\ \upsilon > 0, \tag{2.6}$$

and put

$$Q_\upsilon(0;\eta) = -\upsilon \mathbf{E}\left(\int_0^\eta Q_{\upsilon-1}(z;\eta)\,dz \right). \tag{2.7}$$

It is not difficult to show that, for Appell functions defined in this such way, the following properties hold (see [3]):

$$\frac{d}{dy} Q_\upsilon(y;\eta) = \upsilon Q_{\upsilon-1}(y;\eta), \tag{2.8}$$

$$\mathbf{E} Q_\upsilon(y+\eta;\eta) = y^\upsilon. \tag{2.9}$$

We have also the following lemma.

Lemma 2.1. *Let* (2.2) *and* $\mathbf{E}(\eta^n) < \infty$ *hold for all* $n \geq 1$. *Then, for all* $\upsilon > 0$, *there exists such* a_υ *that*

- $Q_\upsilon(y;\eta) \leq 0$ *for* $0 < y < a_\upsilon$, $Q_\upsilon(a_\upsilon;\eta) = 0$,
- $Q_\upsilon(y;\eta)$ *increases for* $y \geq a_\upsilon$.

Proof of this Lemma is given in [3]. □

3. Some facts about the distribution of maximum

Consider an exponential distributed random variable θ, which is independent of X_t, with parameter q, i.e.

$$P(\theta > t) = e^{-qt}. \tag{3.1}$$

Denote

$$M_\theta = \sup_{0 \leq t < \theta} (X_t - x), \tag{3.2}$$

and, for $q = 0$,

$$M_\infty = \sup_{0 \le t < \infty} (X_t - x). \tag{3.3}$$

In this case, $\mathbf{E}(X_1^+) < \infty$ and $\mathbf{E}(X_1 - x) < 0$.

Lemma 3.1. *If $q \ge 0$, then, for all $\varepsilon > 0$,*

$$P(M_\theta < \varepsilon) > 0. \tag{3.4}$$

Lemma 3.2. *Let $\upsilon > 0$, and let the following conditions hold:*

1. *if $q = 0$, then $\mathbf{E}(X_1) < 0$, $\mathbf{E}((X_1^+)^{\upsilon+1}) < \infty$;*
2. *if $q > 0$, then $\mathbf{E}((X_1^+)^{\upsilon}) < \infty$.*

Then $\mathbf{E}(M_\theta^\upsilon) < \infty$.

Proofs of Lemmas 3.1 and 3.2 are given in [3]. □

Lemma 3.3. 1. *Let $\tau_a = \inf\{t \ge 0 : X_t \ge a\}$, $a \ge x$. Then, for all $u \le 0$,*

$$\mathbf{E}(\mathbf{I}\{\tau_a < \infty\} e^{uX_{\tau_a}} e^{-q\tau_a}) = \frac{\mathbf{E}(\mathbf{I}\{M_\theta + x \ge a\} e^{u(M_\theta + x)})}{\mathbf{E}(e^{uM_\theta})}. \tag{3.5}$$

2. *Under the assumptions of Lemma 3.2 for all $a \ge x$ and all υ, the following equality holds:*

$$\mathbf{E}(\mathbf{I}\{\tau_a < \infty\} X_{\tau_a}^\upsilon e^{-q\tau_a}) = \mathbf{E}(\mathbf{I}\{M_\theta + x \ge a\} Q_\upsilon(M_\theta + x; M_\theta)). \tag{3.6}$$

3. *Let conditions of 3.2 and the initial condition $x \ge 1$ hold. Then, for all $a \ge x$, the equality*

$$\mathbf{E}(\mathbf{I}\{\tau_a < \infty\} \ln X_{\tau_a} e^{-q\tau_a}) = \mathbf{E}\left(\mathbf{I}\{M_\theta + x \ge a\} \int_0^\infty \frac{e^{-u(M_\theta + x)}}{u} \left(1 - \frac{1}{\mathbf{E}e^{-uM_\theta}}\right) du \right) \tag{3.7}$$

holds.

Proof. The proof of items 1 and 2 is given in [3]. We will prove item 3.

We compute the left derivative $\frac{\partial_-}{\partial v} Q_v(y;\eta)|_{v=0}$. By definition, we have

$$
\begin{aligned}
\frac{\partial_-}{\partial v} Q_v(y;\eta)\Big|_{v=0} &= \frac{\partial}{\partial v} \int_0^\infty u^{-v-1} \frac{e^{-uy}}{\mathbf{E}e^{-u\eta}} \frac{du}{\Gamma(-v)}\Bigg|_{v=0} \\
&= \lim_{v\to 0-} \frac{1}{v} \int_0^\infty u^{-v-1} \frac{e^{-uy}}{\mathbf{E}e^{-u\eta}} \frac{du}{\Gamma(-v)} - Q_0(y;\eta) \\
&= \lim_{v\to 0-} \frac{1}{v} \int_0^\infty u^{-v-1} \frac{e^{-uy}}{\Gamma(-v)} \Big(\frac{1}{\mathbf{E}e^{-u\eta}} - y^{-v}\Big) du \\
&= \lim_{v\to 0-} \frac{1}{v} \int_0^\infty u^{-v-1} \frac{e^{-uy}}{\Gamma(-v)} \Big(\frac{1}{\mathbf{E}e^{-u\eta}} - 1\Big) du \\
&\quad + \lim_{v\to 0-} \frac{1}{v} \int_0^\infty u^{-v-1} \frac{e^{-uy}}{\Gamma(-v)} (1 - y^{-v})\, du \\
&= \int_0^\infty u^{-1} e^{-uy} \Big(1 - \frac{1}{\mathbf{E}e^{-u\eta}}\Big) du + 0 \\
&= \int_0^\infty u^{-1} e^{-uy} \Big(1 - \frac{1}{\mathbf{E}e^{-u\eta}}\Big) du. \qquad (3.8)
\end{aligned}
$$

For both terms, it is possible to pass to the limit under the integral sign due to the Lebesgue monotone convergence theorem. We also used the fact that $Q_0(y;\eta) = 1 = \int_0^\infty u^{-v-1} y^{-v} \frac{e^{-uy}}{\Gamma(-v)} du$ by the definition of the gamma function and $Q_0(y;\eta)$.

Setting $y = M_\theta + x$, $\eta = M_\theta$, we obtain

$$
\frac{\partial_-}{\partial v} Q_v(M_\theta + x, M_\theta)|_{v=0} = \int_0^\infty u^{-1} e^{-u(M_\theta + x)} \Big(1 - \frac{1}{\mathbf{E}e^{-uM_\theta}}\Big) du.
$$

Let us take the left derivative with respect to the parameter v at $v = 0$:

$$
\mathbf{E}(\mathbf{I}\{\tau_a < \infty\} \ln X_{\tau_a} e^{-q\tau_a}) = \mathbf{E}\Big(\mathbf{I}\{M_\theta + x \geq a\} \frac{\partial_-}{\partial v} Q_v(M_\theta + x, M_\theta)\Big|_{v=0}\Big)
$$

The left-hand side is differentiable due to the monotone convergence theorem. On the right-hand side, the expression under the mathematical expectation can be divided into two parts as it was done above, and the monotone convergence holds for each of them. This concludes the proof of the theorem. $\square$

Lemma 3.4. *Let $t \in \mathbf{Z}^+$, $q \geq 0$, and let $f(x)$ $g(x)$ be nonnegative functions such that, for all x, $f(x) \geq g(x)$, and*

$$f(x) \geq e^{-q}\mathbf{E}f(X_1). \tag{3.9}$$

Then, for all x,

$$f(x) \geq \sup_{\tau \in M} \mathbf{E}(g(X_\tau)e^{-q\tau}\mathbf{I}\{\tau < \infty\}). \tag{3.10}$$

Proof of Lemma 3.4 see in [3].

4. The main results

The following theorem was proved in [3].

Theorem 4.1. *Let $g(x) = (x^+)^\upsilon$, $\upsilon > 0$, the conditions of Lemma 3.2 hold, and let a_υ be a positive root of the equation*

$$Q_\upsilon(a_\upsilon; M_\theta) = 0. \tag{4.1}$$

Then the stopping time

$$\tau_{a_\upsilon} = \inf\{t \geq 0 : X_t \geq a_\upsilon\} \tag{4.2}$$

is optimal and

$$V(x) = \mathbf{E}(Q_\upsilon(M_\theta + x; M_\theta)\mathbf{I}\{M_\theta + x \geq a_\upsilon\}). \tag{4.3}$$

With the use of the same methods, we now prove a similar statement for the exponential payoff function.

Theorem 4.2. *Let $g(x) = (1 - e^{-x})^+$, the assumptions of Lemma 3.2 hold, and*

$$a^* = -\ln \mathbf{E}e^{-M_\theta}. \tag{4.4}$$

Then the stopping time

$$\tau_{a^*} = \inf\{t \geq 0 : X_t \geq a^*\} \tag{4.5}$$

is optimal and

$$V(x) = \mathbf{E}(1 - e^{-M_\theta - x}(\mathbf{E}e^{-M_\theta})^{-1})^+. \tag{4.6}$$

Proof. Consider the function

$$\hat{V}(x) = \sup_{\tau_a \in \hat{M}} \mathbf{E}(g(X_{\tau_a})e^{-q\tau_a}\mathbf{I}\{\tau < \infty\}), \tag{4.7}$$

where $\hat{M}$ is the class of stopping times $\tau_a = \inf\{t \geq 0 : X_t \geq a\}, a \geq x$. Obviously, $\hat{V}(x) \leq V(x)$, as the supremum is taken over a narrower class of stopping times. Note item 1 of Lemma 3.3 with $u = -1$ yields

$$\mathbf{E}(\mathbf{I}\{\tau_a < \infty\}e^{-X_{\tau_a}}e^{-q\tau_a}) = \frac{\mathbf{E}(\mathbf{I}\{M_\theta + x \geq a\}e^{-(M_\theta+x)})}{\mathbf{E}(e^{-M_\theta})}, \tag{4.8}$$

and so

$$\mathbf{E}(\mathbf{I}\{\tau_a < \infty\}g(X_{\tau_a})e^{-q\tau_a}) = \mathbf{E}\mathbf{I}\{M_\theta + x \geq a\}\Big(1 - \frac{e^{-(M_\theta+x)}}{\mathbf{E}(e^{-M_\theta})}\Big). \tag{4.9}$$

Since the function $1 - \frac{e^{-a}}{\mathbf{E}(e^{-M_\theta})}$ is monotone in a and has a single root $a^* = -\ln \mathbf{E}e^{-M_\theta}$, the left-hand side of (4.9) achieves its maximum at the point $a = a^*$, and

$$\hat{V}(x) = \mathbf{E}\mathbf{I}\{M_\theta + x \geq a^*\}\Big(1 - \frac{e^{-(M_\theta+x)}}{\mathbf{E}(e^{-M_\theta})}\Big) = \mathbf{E}\Big(1 - \frac{e^{-(M_\theta+x)}}{\mathbf{E}(e^{-M_\theta})}\Big)^+. \tag{4.10}$$

Thus, we have shown that $\hat{V}(x)$ achieves its maximum at a^*. Now we have to prove the inequality $\hat{V}(x) \geq V(x)$. To this end, we consider the function $f(x) = \mathbf{E}(1 - \frac{e^{-(M_\theta+x)}}{\mathbf{E}(e^{-M_\theta})})^+$. By the Jensen inequality,

$$f(x) \geq \Big(1 - \frac{\mathbf{E}e^{-(M_\theta+x)}}{\mathbf{E}(e^{-M_\theta})}\Big)^+ = (1 - e^{-x})^+ = g(x). \tag{4.11}$$

Let $\xi = X_1 - x$. Consider a random variable γ such that

$$P(\gamma = 1) = 1 - P(\gamma = 0) = e^{-q}. \tag{4.12}$$

Then $\hat{M}_\theta = (\gamma M_\theta + \xi)^+$ in law, and the following inequalities hold:

$$\begin{aligned} e^{-q}\mathbf{E}f(X_1) &= e^{-q}\mathbf{E}\Big(1 - \frac{e^{-(M_\theta+X_1)}}{\mathbf{E}(e^{-M_\theta})}\Big)^+ \\ &= e^{-q}\mathbf{E}\Big(1 - \frac{e^{-(M_\theta+x+\xi)}}{\mathbf{E}(e^{-M_\theta})}\Big)^+ = \mathbf{E}\Big(e^{-q} - \frac{e^{-q}e^{-(M_\theta+x+\xi)}}{\mathbf{E}(e^{-M_\theta})}\Big)^+ \\ &\leq \mathbf{E}\Big(1 - \frac{e^{-(\gamma M_\theta+x+\xi)}}{\mathbf{E}(e^{-M_\theta})}\Big)^+ = \mathbf{E}\Big(1 - \frac{e^{-(M_\theta+x)}}{\mathbf{E}(e^{-M_\theta})}\Big)^+ = f(x). \end{aligned} \tag{4.13}$$

It follows from (4.11), (4.13), and Lemma 3.4 that $f(x) = \hat{V}(x) \geq V(x)$, and this concludes the proof of the theorem. $\square$

Now let $g(x) = (\ln x)^+$, $q \geq 2$, let $X_0 = x \geq 1$, and let the assumptions of Lemma 3.2 be true. We will try to find the optimal stopping time τ_a for this case in the form

$$\tau_a = \inf\{t \geq 0 : X_t \geq a\}, \tag{4.14}$$

where $a \geq x$. Consider the function

$$\hat{V}(x) = \sup_{\tau_a \in \hat{M}} \mathbf{E}(g(X_{\tau_a})e^{-q\tau_a}\mathbf{I}\{\tau_a < \infty\}), \tag{4.15}$$

where $\hat{M}$ is the class of stopping times $\tau_a = \inf\{t \geq 0 : X_t \geq a\}, a \geq x$. Following the lines of the previous theorem, we consider $\hat{V}(x) \leq V(x)$. Note that, by item 3 of Lemma 3.3, the equality

$$\begin{aligned} &\mathbf{E}(\mathbf{I}\{\tau_a < \infty\} \ln X_{\tau_a} e^{-q\tau_a}) \\ &\quad = \mathbf{E}\left(\mathbf{I}\{M_\theta + x \geq a\} \int_0^\infty u^{-1} e^{-u(M_\theta + x)}\Big(1 - \frac{1}{Ee^{-uM_\theta}}\Big)du\right) \end{aligned} \tag{4.16}$$

holds. Since the function e^{-ua} is nonnegative and decreasing, and the function $1 - \frac{1}{Ee^{-ua}e^{ux}}$ for $u \geq 0$ is negative and increasing, the integral on the right-hand side of (4.16) is negative and increasing, and the left-hand side achieves its maximum at infinity. So, in this case, there exists no optimal stopping time in form (1.3).

5. Conclusions

We have considered the optimal stopping problem for processes with independent increments and shown that, for the exponential payoff function, an optimal stopping moment is the first moment of crossing the level which is found in the explicit form. In the case of a logarithmic payoff function, we have proved that the optimal stopping time doesn't exist in the class of first moments of crossing the level.

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