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About the extension of a linear functional

Alexander V. Revenko

(*Presented by M. M. Malamud*)

Abstract. We give a criterion for the extension of a linear functional subordinate to an arbitrary function. This makes it possible to obtain new necessary and sufficient conditions for the extension of functionals with the given properties, as well as analogs of the Hahn–Banach theorem for convex continuous functions.

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The versions of the Hahn–Banach theorem and its applications for various tasks which are known to the author are based either on the subordination of a linear functional to a sublinear (calibrating) function p [2, 5, 6, 8, 9], or these tasks do not concern the extension, but deal with the existence of a linear functional with given properties [7, 10]. There is a question, if we can deliver from the sublinear function p. In light of this, we present a criterion of the extension of a linear functional.

Theorem 1. *Let ϕ be a real function on the real vector space X, and let f_0 be a linear functional which is defined on the subspace X_0 and satisfies the condition*

$$f_0(x) \le \phi(x) \ (x \in X_0). \tag{1}$$

Then f_0 admits a linear extension on the whole space X with preserving inequality (1) *on it if and only if the following condition holds:*

$$\inf\Bigg\{ \sum_{k=1}^{n} \frac{1}{\lambda_k}\phi(\lambda_k x_k) - f_0\Bigg(\sum_{k=1}^{n} x_k\Bigg) \Bigg|$$

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$$\sum_{k=1}^{n} x_k \in X_0,\ x_k \in X,\ \lambda_k > 0,\ n \in N \Bigg\} = 0. \quad (2)$$

Proof. Necessity. Let inequality (1) be satisfied, and let f be a linear extension of the functional f_0 such that $f(x) \leq \phi(x)$ $(x \in X)$. Then, for any sum $\sum_{k=1}^{n} x_k \in X_0$ and for any combination of positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, the following inequality holds:

$$f_0\left(\sum_{k=1}^{n} x_k\right) = f\left(\sum_{k=1}^{n} x_k\right) = \sum_{k=1}^{n} \frac{1}{\lambda_k} f(\lambda_k x_k) \leq \sum_{k=1}^{n} \frac{1}{\lambda_k} \phi(\lambda_k x_k).$$

Condition (2) of the theorem obviously follows from the last inequality.

Sufficiency. Let conditions (1) and (2) be satisfied. Then, for any $x \in X$, any sum $\sum_{k=1}^{n} x_k = x$, and any combination of positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, we get

$$\phi(-x) + \sum_{k=1}^{n} \frac{1}{\lambda_k} \phi(\lambda_k x_k) - f_0\left(\sum_{k=1}^{n} x_k - x\right) \geq 0.$$

The last statement yields

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} \phi(\lambda_k x_k) \geq -\phi(-x),$$

and, hence, the function

$$p(x) = \inf\left\{ \sum_{k=1}^{n} \frac{1}{\lambda_k} \phi(\lambda_k x_k) \;\middle|\; \sum_{k=1}^{n} x_k = x,\ x_k \in X,\ \lambda_k > 0,\ n \in N \right\}$$

is finite on X. From (1), we get $\phi(0) \geq 0$, and, therefore, $p(0) = 0$. For any $\alpha > 0$, we obtain

$$\begin{aligned} &p(\alpha x) \\ &\quad = \inf\left\{ \sum_{k=1}^{n} \frac{1}{\lambda_k} \phi(\lambda_k x_k) \;\middle|\; \sum_{k=1}^{n} x_k = \alpha x,\ x_k \in X,\ \lambda_k > 0,\ n \in N \right\} \\ &= \inf\left\{ \alpha \sum_{k=1}^{n} \frac{1}{\alpha\lambda_k} \phi(\alpha\lambda_k x_k) \;\middle|\; \sum_{k=1}^{n} \frac{1}{\alpha} x_k = x,\ \frac{x_k}{\alpha} \in X,\ \lambda_k > 0,\ n \in N \right\} \\ &\qquad = \alpha p(x). \end{aligned}$$

For any $x, y \in X$ and any $\varepsilon > 0$, let us choose sums $\sum_{k=1}^{n} x_k = x$, $\sum_{k=1}^{m} y_k = y$ and combinations of positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$; $\mu_1, \mu_2, \ldots, \mu_m$ in such a way that

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} \phi(\lambda_k x_k) < p(x) + \varepsilon$$

and

$$\sum_{k=1}^{n} \frac{1}{\mu_k} f(\mu_k y_k) < p(y) + \varepsilon.$$

Then we get

$$p(x+y) \leq \sum_{k=1}^{n} \frac{1}{\lambda_k} \phi(\lambda_k x_k) + \sum_{k=1}^{m} \frac{1}{\mu_k} \phi(\mu_k y_k) < p(x) + p(y) + 2\varepsilon.$$

By virtue of the arbitrariness of $\varepsilon > 0$, the last inequality implies that p is a gauge function on the space X.

If $x \in X_0$, then it follows from condition (2) that

$$f_0(x) \leq \sum_{k=1}^{n} \frac{1}{\lambda_k} \phi(\lambda_k x_k)$$

for any sum $\sum_{k=1}^{n} x_k = x$ and any numbers $\lambda_k > 0$. Therefore, $f_0(x) \leq p(x)$ $(x \in X_0)$. Now we use the Hahn–Banach theorem and find a linear extension f of the functional f_0 such that $f(x) \leq p(x)$ $(x \in X)$.

It is obvious that $p(x) \leq \phi(x)$ $(x \in X)$. So, Theorem 1 is proved. □

From Theorem 1, we will get the following theorem.

Theorem 2. *Let ρ be a positive homogeneous real function on a real vector space X, and let f_0 be a linear functional which is defined on a subspace X_0 and satisfies the following condition on it:*

$$f_0(x) \leq \rho(x) \quad (x \in X_0). \tag{3}$$

Then f_0 admits a linear extension on the whole space X with preserving inequality (3) *on it if and only if the following condition holds:*

$$\inf\left\{ \sum_{k=1}^{n} \rho(x_k) - f_0\left(\sum_{k=1}^{n} x_k \right) \,\middle|\, \sum_{k=1}^{n} x_k \in X_0,\ x_k \in X,\ n \in N \right\} = 0. \tag{4}$$

Theorems 3–5 will be proved below with the help of Theorem 2.

Theorem 3. *Let Γ be a family of calibrating functions on a real vector space X, and let f_0 be a linear functional which is defined on a subspace X_0 and satisfies the following condition on it:*

$$f_0(x) \leq p(x) \quad (x \in X_0,\ p \in \Gamma). \tag{5}$$

Then f_0 admits a linear extension on the whole space X with preserving inequality (5) on it if and only if the following condition holds for any finite combination $p_1, p_2, \dots, p_n$ of gauge functions of a family Γ:

$$\inf\left\{ \sum_{k=1}^{n} p_k(x_k) - f_0\left(\sum_{k=1}^{n} x_k\right) \,\middle|\, \sum_{k=1}^{n} x_k \in X_0,\ x_k \in X \right\} = 0. \tag{6}$$

Proof. Necessity. If f is a linear extension of the functional f_0 which satisfies inequality (5) on X_0, then, for any finite combination $p_1, p_2, \dots, p_n$ of gauge functions of the family Γ and any sum $\sum_{k=1}^{n} x_k \in X_0$, the following condition holds:

$$f_0\left(\sum_{k=1}^{n} x_k\right) = f\left(\sum_{k=1}^{n} x_k\right) = \sum_{k=1}^{n} f(x_k) \leq \sum_{k=1}^{n} p(x_k).$$

Condition (6) follows from the last statement.

Sufficiency. Let condition (6) hold. From (6) for $n = 2$, $x_1 = x$, $x_2 = -x$ and for any $p_1, p_2 \in \Gamma$, we get $p_1(x) + p_2(-x) - f_0(x - x) \geq 0$. That is, $p_1(x) \geq p_2(-x)$ $(x \in X)$. The last inequality provides the finiteness of the function $\rho(x) = \inf\{p(x) \mid p \in \Gamma\}$ on the space X. The function ρ is positive homogeneous. For any fixed $n \in N$, relation (6) yields obviously the inequality

$$\sum_{k=1}^{n} \rho(x_k) - f_0\left(\sum_{k=1}^{n} x_k\right) \geq 0 \quad \left(\sum_{k=1}^{n} x_k \in X_0,\ x_k \in X\right).$$

So, condition (4) of Theorem 2 is true. By virtue of Theorem 2, there is a linear extension f of the functional f_0 which satisfies the condition $f(x) \leq \rho(x) \leq p(x)$ $(x \in X,\ p \in \Gamma)$. The theorem is proved. □

For $X_0 = \{0\}$, we get the next implication.

Corollary 1. *A linear functional on the real vector space X submitting to all gauge functions of a family Γ on it exists if and only if, for any finite combination $p_1, p_2, \dots, p_n$ from Γ, the following condition holds:*

$$\inf\left\{ \sum_{k=1}^{n} p_k(x_k) \,\middle|\, \sum_{k=1}^{n} x_k = 0,\ x_k \in X \right\} = 0. \tag{7}$$

Remark 1. If Γ consists of a finite number of gauge functions $p_1, p_2, \ldots,$ p_n, then (from the proof of Theorem 3) relation (6) is the necessary and sufficient condition of the extension. In particular, for two functions p_1 and p_2, this condition looks like

$$\inf\{p_1(x) + p_2(y) - f_0(x+y) \mid x+y \in X_0\} = 0. \tag{8}$$

Here, the respective condition (7) looks like $p_1(x)+p_2(-x) \geq 0$ $(x \in X)$. If p and $-q$ are the gauge functions which satisfy the inequality $q(x) \leq f_0(x) \leq p(x)$ on X_0, then (designating $p_1(x) = p(x)$ and $p_2(x) = -q(-x)$) condition (8) yields the necessary and sufficient condition for a linear extension of the last double inequality on the whole space X:

$$\inf\{p(x) - q(y) - f_0(x+y) \mid x+y \in X_0\} = 0. \tag{9}$$

Theorem 4. *Let p be a gauge function on the ordered real vector space X, and let X_0 be such a subspace that*

$$\forall\, x \in X \ \exists\, y \in X_0 \ (x \leq y). \tag{10}$$

Then the positive (concerning the induced structure of order) linear functional f_0 on X_0, which satisfies the condition

$$f_0(x) \leq p(x) \quad (x \in X_0), \tag{11}$$

admits a positive linear extension f on the whole space X with saving inequality (11) *on it if and only if the following condition holds:*

$$\forall\, x \in X, \ y \in X_0 \ (y \leq x \Rightarrow f_0(y) \leq p(x)). \tag{12}$$

Proof. Necessity. Let f be a positive linear extension of the functional f_0 such that, for any $x \in X, \ y \in X_0$. Then, with regard for the inequality $y \leq x$, we get $f_0(y) = f(y) \leq f(x) \leq p(x)$. The necessity is proved.

Sufficiency. Let conditions (10), (11), and (12) be satisfied. Then, according to the proof of the theorem about positive extension [1], the gauge function

$$\rho(x) = \inf\{f_0(s) \mid s \in X_0, \ s \geq x\} \quad (x \in X)$$

is finite on X, $f_0(x) = \rho(x)$ $(x \in X_0)$, and any extension f of the functional f_0 that satisfies the condition $f(x) \leq \rho(x)$ $(x \in X)$ is positive on X.

Now, for any sum $x + z = t \in X_0$ and functions p, ρ, using condition (12), we get

$$
\begin{aligned}
p(x) + \rho(z) - f_0(x+z) &= p(x) + \rho(t-x) - f_0(t) \\
&= p(x) + \inf\{f_0(s) \mid s \in X_0,\ s \geq t-x\} - f_0(t) \\
&= p(x) + \inf\{f_0(t) - f_0(y) \mid y \in X_0,\ x \geq y\} - f_0(t) \\
&= p(x) + \inf\{-f_0(y) \mid y \in X_0,\ x \geq y\} \\
&= p(x) - \sup\{f_0(y) \mid y \in X_0,\ x \geq y\} \geq 0.
\end{aligned}
$$

So the functions p and ρ satisfy (8), according to which there is a linear extension f of the functional f_0 submitting to both gauge functions on X. The theorem is proved. □

Remark 2. If X is an ordered locally convex space and p is a prenorm on it, then the last theorem is a criterion of the extension of a continuous positive linear functional. The existence of a continuous positive extension of a linear functional on an ordered topological vector space with positive cone P such like $\operatorname{int}(X_0 \cap P) \neq \emptyset$ was proved by M. G. Krein [1].

Remark 3. From the proof of the last theorem and Theorem 3, it is easy to see that the necessary and sufficient condition of the extension of the positive functional f_0 which satisfies the inequality $f_0(x) \leq p(x)$ $(x \in X_0,\ p \in \Gamma)$ is the condition

$$
\forall n \in N;\ p_1, p_2, \ldots, p_n \in \Gamma;\ x_1, x_2, \ldots, x_n \in X;\ y \in X_0
$$
$$
\left(y \leq \sum_{k=1}^{n} x_k \Rightarrow f_0(y) \leq \sum_{k=1}^{n} p_k(x_k) \right).
$$

Now let G be an arbitrary family of endomorphisms of the real vector space X, and let p be a gauge function on X which satisfies the condition

$$
p(u(x)) \leq p(x) \quad (x \in X,\ u \in G). \tag{13}
$$

If G is a commutative semigroup of endomorphisms or a solvable group of endomorphisms of the space X, then there is a linear invariant extension of a similar functional f_0 defined on the subspace X_0 relative to G (the Agnew–Morse theorem) [1].

In connection with the theorem of Agnew–Morse, let us define a gauge function p_u on X for any $u \in G$ with the help of the inequality

$$
p_u(x) = \varlimsup_{n} \frac{1}{n} p\left(\sum_{k=1}^{n} u^k(x) \right). \tag{14}
$$

Theorem 5. *Let X_0 be a subspace of the real vector space X invariant relative to the family G $(u(X_0) \subset X_0,\ u \in G)$, and let p be a gauge function which satisfies inequalities* (13) *on X. Let a linear functional f_0 be defined on X_0 and satisfy the conditions*

$$f_0(u(x)) = f_0(x) \leq p(x) \quad (x \in X_0,\ u \in G). \tag{15}$$

Then f_0 admits a linear extension f on the whole space X which satisfies condition (15) on it if and only if, for any finite combination $p_{u_1}, p_{u_2}, \ldots, p_{u_n}$ of functions (14), *the next equality holds:*

$$\inf\left\{\sum_{k=1}^{n} p_{u_k}(x_k) - f_0\left(\sum_{k=1}^{n} x_k\right) \,\Bigg|\, \sum_{k=1}^{n} x_k \in X_0,\ x_k \in X\right\} = 0. \tag{16}$$

Proof. Necessity. Let f be a linear extension of the functional f_0 on the whole space X and $f(x) = f(u(x)) \leq p(x)$ $(x \in X,\ u \in G)$. Then

$$f(x) = f\left(\frac{1}{n}\sum_{k=1}^{n} u^k(x)\right) \leq p\left(\frac{1}{n}\sum_{k=1}^{n} u^k(x)\right) \leq \frac{1}{n}\sum_{k=1}^{n} p(u^k(x)) \leq p(x)$$

for any $n \in N,\ x \in X$. From the last statement, we get

$$f(x) \leq p_u(x) = \overline{\lim_n}\, \frac{1}{n} p\left(\sum_{k=1}^{n} u^k(x)\right) \leq p(x).$$

Now, for any finite combination $p_{u_1}, p_{u_2}, \ldots, p_{u_n}$ $(u_k \in G)$ (with the help of Theorem 3), we get

$$\inf\left\{\sum_{k=1}^{n} p_{u_k}(x_k) - f_0\left(\sum_{k=1}^{n} x_k\right) \,\Bigg|\, \sum_{k=1}^{n} x_k \in X_0,\ x_k \in X\right\} = 0.$$

The necessity is proved.

Sufficiency. Let f_0 satisfy condition (15). Then (as in the case of proving the necessity) we get $f_0(x) \leq p_u(x)$ $(x \in X_0,\ u \in G)$. Using (16) and Theorem 3, we will find a linear extension f of the functional f_0, for which $f(x) \leq p_u \leq p(x)$ $(x \in X,\ u \in G)$. Because of

$$\begin{aligned} p_u(\pm(u(x) - x)) &= \overline{\lim_n}\, \frac{1}{n} p\left(\pm\sum_{k=1}^{n}(u^{k+1}(x) - u^k(x))\right) \\ &= \overline{\lim_n}\, \frac{1}{n} p(\pm(u^{n+1}(x) - u(x))) \leq \overline{\lim_n}\, \frac{1}{n}(p(x) + p(\pm x)) = 0, \end{aligned}$$

the last inequalities yield the invariance of the functional f relative to the family G on the space X. The theorem is proved. $\square$

Now let us have a look on the topological vector space X and a finite convex function φ on it. If $\varphi(0) \geq 0$, then the envelope of the function φ will be a function p defined on X with the help of the inequality

$$p(x) = \inf_{\alpha > 0} \frac{\varphi(\alpha x)}{\alpha} \quad (x \in X). \tag{17}$$

Lemma 1. *The envelope of a function φ is a continuous gauge function on the topological vector space X.*

Proof. First of all, we will prove that the function p is finite on X. Actually, if, for any $x \in X$,

$$\inf_{\alpha > 0} \frac{\varphi(\alpha x)}{\alpha} = -\infty,$$

then there is a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers α_n such that

$$\frac{\varphi(\alpha_n x)}{\alpha_n} < -n \quad (n \in N). \tag{18}$$

From the continuity of a function φ and (18), we now get that the mentioned sequence has no finite positive limit points, and 0 is not its limit point, when $\varphi(0) > 0$. If $\varphi(0) = 0$ and 0 is a finite point of the sequence $\{\alpha_n\}_{n=1}^{\infty}$, then inequality (18) contradicts the existence of a finite right derivative of a continuous convex real function $M_x(\alpha) = \varphi(\alpha x)$ $(\alpha \in R)$ at 0 [3]. So $\alpha_n \to +\infty$ for $n \to \infty$. Now, for sufficiently great n, using the convexity of a function φ and inequality (18), we get

$$-1 > \frac{1}{n\alpha_n}\varphi(\alpha_n x) \geq \varphi\Big(\frac{1}{n}x\Big) - \Big(1 - \frac{1}{n\alpha_n}\Big)\varphi(0).$$

The last statement contradicts the condition of a lemma that proves the finiteness of a function p. It is obvious that $p(0) = 0$ and, for any $\lambda > 0$, it is true that

$$p(\lambda x) = \inf_{\alpha > 0} \frac{\varphi(\lambda \alpha x)}{\alpha} = \lambda p(x).$$

Now let x and y be arbitrary points of the space X. Then, for any $\alpha, \beta > 0$, it is true that

$$\begin{aligned} p(x+y) &\leq \frac{\alpha+\beta}{\alpha\beta}\varphi\Big(\frac{\alpha\beta}{\alpha+\beta}(x+y)\Big) \\ &\leq \frac{\alpha+\beta}{\alpha\beta}\Big(\frac{\beta}{\alpha+\beta}\varphi(\alpha x) + \frac{\alpha}{\alpha+\beta}\varphi(\beta y)\Big) \\ &= \frac{\varphi(\alpha x)}{\alpha} + \frac{\varphi(\beta y)}{\beta}. \end{aligned}$$

For any $\varepsilon > 0$, let us choose α and β in such a way that $\frac{\varphi(\alpha x)}{\alpha} < p(x) + \varepsilon$ and $\frac{\varphi(\beta y)}{\beta} < p(y) + \varepsilon$. We get $p(x + y) < p(x) + p(y) + 2\varepsilon$. The last statement yields the semiadditivity of a function p. It is obvious that $p(x) \le \varphi(x)$ $(x \in X)$. Because the continuity of a continuous convex function φ and its boundedness from above in some neighborhood of a point [3] are equivalent, the function p is continuous on X. The lemma is proved. □

Lemma 2. *If φ is a continuous convex function on the topological vector space X such that $\varphi(0) \ge 0$, then its envelope function is the largest among all gauge functions r such that $r(x) \le \varphi(x)$ $(x \in X)$.*

Proof. Let the conditions of a theorem be true. We suppose an opposite statement that there is $x_0 \in X$ such that $p(x_0) < r(x_0) \le \varphi(x_0)$. Then, by the definition of a function p, there is $\alpha > 0$ such that $\frac{\varphi(\alpha x_0)}{\alpha} < r(x_0)$. From the last, we get $\varphi(\alpha x_0) < r(\alpha x_0)$. The last statement contradicts the condition of a lemma. The derived contradiction proves the lemma. □

Using two last lemmas, we get the next analogs of Hahn–Banach theorems for real and complex topological vector spaces.

Theorem 6. *Let φ be a continuous convex function on the real topological vector space X, and let f_0 be a continuous linear functional which is defined on the space X_0 and satisfies the condition*

$$f_0(x) \le \varphi(x) \quad (x \in X_0) \tag{19}$$

on it. Then we can extend f_0 linearly on the whole space X with saving the last inequality on it.

Proof. Let f_0 satisfy inequality (19). Then $\varphi(0) \ge 0$.. Because a functional f_0 is a gauge function on X_0, Lemma 2 implies that the envelope p of a function φ satisfies the inequality $f_0(x) \le p(x)$ $(x \in X_0)$ on X_0. Now, from the definition of a function p, Lemma 1, and the Hahn–Banach theorem, we get the existence of a linear extension f of the functional f_0 such that $f_0(x) \le p(x) \le \varphi(x)$ $(x \in X_0)$. The theorem is proved. □

Theorem 7. *Let φ be a continuous nonnegative convex function on the complex topological vector space X such that $\varphi(xe^{i\theta}) = \varphi(x)$ $(x \in X,\ \theta \in R)$, and let f_0 be a linear functional which is defined on a subspace X_0 and satisfies the inequality $|f_0(x)| \le \varphi(x)$ $(x \in X_0)$ on it. Then we can extend f_0 linearly on the whole space X with saving the last inequality on it.*

Proof. Let the condition of the theorem be true. Then, from the definition of the envelope, we get that p is nonnegative, and, for any complex number λ, the inequality $p(\lambda x) = |\lambda| p(x)$ is true. So, p is a prenorm on X, and the statement of a theorem now follows (as in the previous theorem) from Lemmas 1 and 2 and the respective Hahn–Banach theorem for a complex topological vector space. □

Theorem 8. *Let φ and $-\psi$ be continuous convex functions on the real topological vector space X such that $\psi(x) \leq \varphi(x)$ $(x \in X)$, and let a linear functional f_0 defined on a subspace X_0 satisfy the inequality*

$$\psi(x) \leq f_0(x) \leq \varphi(x) \quad (x \in X_0).$$

Then we can extend f_0 linearly on the whole subspace X with saving the previous inequality on it if and only if

$$\inf_{\alpha,\beta>0} (\beta\varphi(\alpha x) - \alpha\psi(\beta x)) \geq 0 \tag{20}$$

and the envelopes p and q (of the functions φ and $-\psi$, respectively) satisfy condition (9)*.*

Proof. Necessity. Let f be a linear extension of the functional f_0 such that

$$\psi(x) \leq f(x) \leq \varphi(x) \quad (x \in X). \tag{21}$$

Then $\psi(0) \leq 0 \leq \varphi(0)$, and, according to Lemma 2 for the envelope p of a function φ, we get $f(x) \leq p(x)$ $(x \in X)$. From the left-hand side of inequality (21), we similarly get that, for any envelope $-q$ of a function $-\psi$, there is the inequality $f(x) \geq q(x)$ $(x \in X)$. It is obvious that

$$q(x) = \sup_{\beta>0} \frac{\psi(\beta x)}{\beta} \quad (x \in X).$$

From two last inequalities, we obtain

$$\sup_{\beta>0} \frac{\psi(\beta x)}{\beta} \leq \inf_{\alpha>0} \frac{\varphi(\alpha x)}{\alpha} \quad (x \in X). \tag{22}$$

From inequality (22), we get condition (20) of the theorem. The rest conditions of the theorem follow from Remark 1.

Sufficiency. Let the conditions of a theorem be satisfied. Then the inequality $q(x) \leq p(x)$ $(x \in X)$ obviously follows from (20), and so all the conditions of Remark 1 are true. Thus, there is a linear extension f of the functional f_0 which satisfies the inequality $q(x) \leq f(x) \leq p(x)$ $(x \in X)$. From the definition of envelopes, we now get now inequality $\psi(x) \leq f(x) \leq \varphi(x)$. The theorem is proved. □

Corollary 2. *Let φ and $-\psi$ be continuous convex functions on a topological vector space X which satisfy the conditions $\psi(x) \le \varphi(x)$ $(x \in X)$, $\psi(0) \le 0 \le \varphi(0)$. Then a linear continuous functional f such that $\psi(x) \le f(x) \le \varphi(x)$ $(x \in X)$ exists if and only if condition* (20) *holds.*

The results which are similar to the last statements ("sandwich theorems") were get by H. König [5, 6].

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Contact information

Alexander V. Revenko

Mathematical Analysis Department,
Luhansk National University,
2, Oboronnaya Str.,
Luhansk 91011,
Ukraine
E-Mail: RevenkoAV@mail.ru