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On invariant subspaces of J-dissipative operators

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Abstract. New sufficient conditions for the existence of maximal nonnegative invariant subspaces of J-dissipative operators are given.

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1. Introduction

The existence problem of maximal semidefinite invariant subspaces of operators which act in spaces with indefinite metric is considered. It is well known that the existence problem of invariant subspaces, moreover of special types, is one of the key problems in the operator theory and its different applications. For the first time, the discussed problem for self-adjoint operators in spaces named the Pontryagin spaces later on was solved in 1944 by L. S. Pontryagin [10], but one year earlier by S. L. Sobolev [17] for $\varkappa = 1$. More details of the development of this problem till the 1990s can be found in [3, 4]. Here, we only mention that this problem was solved independently by one of the coauthors [1] and M. G. Krein and H. Langer [8] for J-dissipative operators in $\Pi_\varkappa$, for J-dissipative operators satisfying condition (**L**) (see the definition below) in [2]. Recently, this problem was considered in a few papers of A. A. Shkalikov [11–16], in particular, other weaker conditions were offered in [14] and [16] instead of condition (**L**). Let us note that [16] is the first paper in this direction, where the existence of regular points of a considered operator was not assumed.

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Before we will formulate the aim of our research, let us recall some notions of indefinite metric space theory. See more details, for example, in [4].

Let $\mathcal{H}$ be a linear space, and let $[\cdot,\cdot]$ be a sesquilinear form on $\mathcal{H}$. This form is called an indefinite metric. If $\mathcal{H}$ admits a decomposition in a direct orthogonal sum:

$$\mathcal{H} = \mathcal{H}^+[\dot{+}]\mathcal{H}^-, \tag{1.1}$$

where $\{\mathcal{H}^\pm, \pm[\cdot,\cdot]\}$ are Hilbert spaces and $[x^+, x^-] = 0$, for all $x^\pm \in H^\pm$, then $\{\mathcal{H}, [\cdot,\cdot]\}$ is called *a Krein space*, and the decomposition (1.1) is called *fundamental*. This decomposition generates the orthogonal projections from $\mathcal{H}$ onto $\mathcal{H}^\pm$, which we denote by $P^\pm$, respectively. The space $\mathcal{H}$ with the scalar product

$$(x, y) = [x_+, y_+] - [x_-, y_-], \quad x_\pm, y_\pm \in \mathcal{H}^\pm$$

is a Hilbert space; denote $\|x\| = \sqrt{(x,x)}$. In this case, $[\,\cdot,\,\cdot] = (\,J\cdot,\,\cdot)$, where J is *a fundamental symmetry*, J is self-adjoint and unitary at the same time, and $J = P^+ - P^-$, $P^\pm = \frac{1}{2}(I \pm J)$.

A subspace $\mathcal{L}$ of the Krein space $\{\mathcal{H}, [\cdot,\cdot]\}$ is called *nonnegative*, if $[x, x] \geq 0$ for all $x \in \mathcal{L}$; *positive*, if $[x, x] > 0$ for all $x \in \mathcal{L} \setminus \{0\}$; and *uniformly positive*, if the norms $[x, x]^{1/2}$ and $\|x\|$ are equivalent on $\mathcal{L}$, that is, (taking $[x, x] \leq \|x\|^2$ in account) for some $\varepsilon > 0$ and for all $x \in \mathcal{L}$, the inequality $[x, x] \geq \varepsilon\|x\|^2$ holds. Similarly, one can define nonpositive, negative, and uniformly negative subspaces.

The notion of uniformly definite subspace admits a generalization: a nonnegative (nonpositive) subspace $\mathcal{L}$ of the Krein space $\mathcal{H}$ is called *a subspace of the class* $\mathbf{h}^+$ (*of the class* $\mathbf{h}^-$), if it admits a decomposition $\mathcal{L} = \mathcal{L}_0[\dot{+}]\mathcal{L}^+$ ($\mathcal{L} = \mathcal{L}_0[\dot{+}]\mathcal{L}^-$) in a direct sum of the finite dimensional isotropic subspace $\mathcal{L}_0$ ($\dim \mathcal{L}_0 < \infty$, $\mathcal{L}_0 = \mathcal{L} \cap \mathcal{L}^{[\perp]}$) and a uniformly positive (a uniformly negative) subspace $\mathcal{L}^+$ ($\mathcal{L}^-$).

The bounded operator $K : K = P^-(P^+|\mathcal{L})^{-1}$, $K : P^+\mathcal{L} \to \mathcal{H}^-$, is called *the angle operator* of a nonnegative subspace $\mathcal{L}$. Every nonnegative subspace $\mathcal{L}$ has the angle operator K, and $\mathcal{L} = \{x = x^+ + Kx^+ \mid x^+ \in \mathcal{L}_+\}$, where $\mathcal{L}_+ = P^+\mathcal{L}$.

Denote, by $(\mathfrak{M}^+(\mathcal{H}) =)\mathfrak{M}^+$, $(\mathfrak{M}^-(\mathcal{H}) =)\mathfrak{M}^-$, the set of maximal nonnegative and maximal nonpositive subspaces of $\mathcal{H}$.

Let us define some classes of linear operators in spaces with an indefinite metric.

An operator $V : \mathcal{H} \to \mathcal{H}$ is called *J-noncontractive*, if $[Vx, Vx] \geq [x, x]$, $x \in \mathcal{H}$.

If both operators V and V^* are J-noncontractive, we say that V is *J-binoncontractive.*

An operator A acting in a Krein space $\mathcal{H}$ is called *J-dissipative*, if $\mathrm{Im}\,[Ax,x] \geq 0$ for all $x \in \mathrm{dom}\,A$; $\mathrm{dom}\,A$ denote the domain of A. An operator A is maximal J-dissipative, if it does not admit nontrivial J-dissipative extensions.

Analogously to the case of a Hilbert space, an operator is said to be essentially maximal J-dissipative if the closure of it is maximal J-dissipative.

Let us note the evident relation between dissipative and J-dissipative operators: A is J-dissipative if and only if JA (and then AJ) is dissipative. In this case, A is maximal J-dissipative if and only if JA (and then AJ) is maximal dissipative.

If $J = I$, our definition coincides with the definition of dissipative operators by M. S. Livschitz which describes a class of operators $\mathfrak{D}_L$ containing symmetric and self-adjoint operators. Another definition of dissipative operators belong to R.S. Phillips: $\mathrm{Re}\,(Bx,x) \leq 0$, and this class $\mathfrak{D}_{Ph}$ of operators describes problems related to the stability problem and dissipation. The following trivial proposition establishes a relation between these two definitions.

Proposition 1.1.

$$A \in \mathfrak{D}_L \qquad \Longleftrightarrow \qquad B = -iA \in \mathfrak{D}_{Ph}; \tag{1.2}$$

in this case, A is maximal dissipative by Livschitz if and only if $B = -iA$ is maximal dissipative by Phillips.

We will say that the operator A in a Krein space $\mathcal{H}$ satisfies the condition $(\mathbf{L})$ and will write $A \in (\mathbf{L})$, if $\mathcal{H}^+ \subset \mathrm{dom}\,A$. Condition $(\mathbf{L})$ was introduced by H. Langer in paper [9], where a theorem of existence of the maximal semidefinite invariant subspace of self-adjoint operators in a Krein space was proved.

As usual, $\sigma(A)$, $\rho(A)$, $\mathrm{dom}\,A$, and $\mathrm{ran}\,A$ denote the spectrum, the resolvent set, the domain of definition, and the range of values of A, respectively. The set of completely continuous (compact) operators acting in $\mathcal{H}$ is denoted by $\mathfrak{S}_\infty$.

Let us recall the *Cayley–Neumann transform* U of an operator A at $(\overline{\lambda} \neq)\lambda \notin \sigma_p(A)$: $U = I + (\lambda - \overline{\lambda})(A - \lambda I)^{-1}$. In this case, the inverse Cayley–Neumann transform is $A = \lambda I + (\lambda - \overline{\lambda})(U - I)^{-1}$.

Let A be a J-dissipative operator which is the closure of the operator

$$A' := A|_{(\mathcal{H}^+ \cap \mathrm{dom}\,A) \oplus (\mathcal{H}^- \cap \mathrm{dom}\,A)} = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix}. \tag{1.3}$$

We say that a J-dissipative operator A satisfies condition $(\mathbf{S_1})$: $A \in (\mathbf{S_1})$, if

(a) $-A'_{22}$ is a maximal dissipative operator in the Hilbert space $\{\mathcal{H}^-, -[\cdot,\cdot]\}$;

(b) for $\operatorname{Im}\mu > 0$, the operator $(A'_{22}-\mu)^{-1}A'_{21}$ is bounded and densely defined in $\mathcal{H}^+$;

(c) the closure of the operator $A'_{12}(A'_{22}-\mu)^{-1}$ is compact;

(d) the transfer function $M(\mu) = A'_{11} - A'_{12}(A'_{22}-\mu)^{-1}A'_{21}$ is a bounded operator densely defined in $\mathcal{H}^+$.

If only conditions (a)–(c) are satisfied, we will say that $A \in (\mathbf{S_2})$. Conditions $(\mathbf{S_1})$ and $(\mathbf{S_2})$ are introduced by A. A. Shkalikov in [14] and [16], respectively. In [14], there is given the example of an operator which satisfies condition $(\mathbf{S_1})$, but not **(L)**. It is not difficult to construct an operator which belongs to $(\mathbf{S_2}) \setminus (\mathbf{S_1})$.

The main aim of this paper is a development of results from [14] and to obtain new theorems about the existence of an invariant subspace of a J-dissipative operator.

2. Invariant subspaces of a J-dissipative operator

Let us precede the main result of this section, Theorem 2.1, by considering the following lemma which arouses an independent interest.

Lemma 2.1. *Let*

$$A' = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix}$$

be a densely defined J-dissipative operator, and let $-A'_{22}$ be an essentially maximal dissipative operator in $\{\mathcal{H}^-, -[\cdot,\cdot]\}$. Let A be the closure of A'.

If there exists a λ with $\operatorname{Im}\lambda > 0$ such that $\lambda \in \rho(A)$ and the operator $M(\lambda) = A'_{11} - \lambda - A'_{12}(A'_{22}-\lambda)^{-1}A'_{21}$ is densely defined, then A is a maximal J-dissipative operator if and only if the operator

$$T(\lambda) := M(\lambda) + \overline{\lambda} \tag{2.1}$$

is essentially maximal in $\{\mathcal{H}^+, [\cdot,\cdot]\}$.

Proof. Below, we will use the well-known relations between operator classes in Krein spaces (see, for instance, [4, Chapter 2]). Since A is a J-dissipative operator, $\lambda \in \rho(A)$, its Cayley–Neumann transform

$U = (A - \overline{\lambda})(A - \lambda)^{-1}$ is a J-noncontractive operator. Indeed, let $f = (A - \lambda)x$ be an arbitrary vector in $\mathcal{H}$. Then it follows from

$$[Uf, Uf] - [f, f] = 4 \operatorname{Im} \lambda \operatorname{Im} [Ax, x] \tag{2.2}$$

that U is J-noncontractive if and only if A is J-dissipative. Moreover, A is a maximal J-dissipative operator if and only if U is J-binoncontractive. Let

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

be the matrix representation of a J-noncontractive operator U with respect to (1.1). One can check directly that U_{11} is the closure of the operator

$$U'_{11} = (T(\lambda) - \overline{\lambda})(T(\lambda) - \lambda)^{-1}, \tag{2.3}$$

and $\|U_{11}x^+\| \geq \|x^+\|$ for all $x^+ \in \mathcal{H}^+$. From the equality similar to (2.2), we have that $T(\lambda)$ is a dissipative operator in $\{\mathcal{H}^+, [\cdot, \cdot]\}$. Since U is J-binoncontractive if and only if $0 \in \rho(U_{11})$, and this is equivalent to the density of the range of values of the operator $T(\lambda) - \overline{\lambda}$ in $\mathcal{H}^+$, we have that $T(\lambda)$ is an essentially maximal dissipative operator if and only if A is a maximal J-dissipative operator. □

The first part (i) of the following Theorem 2.1 coincides with the main result of [14], but we give some another approach to the proof, which allows us to formulate and to prove part (ii).

Theorem 2.1. *Let J-dissipative operator $A \in (\mathbf{S_1})$. Then*

(i) *the operator A has an invariant subspace $\mathcal{L} \in \mathfrak{M}^+$, $\mathcal{L} \subset \operatorname{dom} A$ with $\operatorname{Im} \sigma(A|\mathcal{L}) \geq 0$ [14];*

(ii) *if $\mathcal{L}^+$ is a nonnegative invariant subspace of A : $\mathcal{L}^+ \subset \operatorname{dom} A$, $A\mathcal{L}^+ \subset \mathcal{L}^+$, then there exists $\widetilde{\mathcal{L}^+} \in \mathfrak{M}^+$ such, that $\mathcal{L}^+ \subset \widetilde{\mathcal{L}^+}$, $\widetilde{\mathcal{L}^+} \subset \operatorname{dom} A$ and $A\widetilde{\mathcal{L}^+} \subset \widetilde{\mathcal{L}^+}$, that is, each A-invariant nonnegative subspace admits an extension to a maximal nonnegative subspace invariant with respect to A.*

Proof. First of all, we note that, according to Lemma 2.1, the operator A is maximal J-dissipative. So, its Cayley–Neumann transform $U = (A - \overline{\lambda})(A - \lambda)^{-1}$ is a J-binoncontractive operator. Condition (c) of the definition of class ($\mathbf{S_1}$) yields the compactness of $U_{12} = (U_{11} - I)\overline{A_{12}{}'(A'_{22} - \lambda)^{-1}}$. Hence, according to [5], [4, Theorem 3.2.8], the operator U has an invariant subspace $\mathcal{L} \in \mathfrak{M}^+$ such that $|\sigma(U|\mathcal{L})| \geq 1$. To prove (i), it is sufficient to check that $\mathcal{L}$ is A-invariant. Since $A|\mathcal{L}$ is the inverse Cayley–Neumann transform of $U|\mathcal{L}$, it remains to use the relation between the spectra of an operator and its Cayley–Neumann transform.

So, let us prove that $\mathcal{L}$ is A-invariant. Let K be the angle operator of the subspace $\mathcal{L}$. Then the operators $U|\mathcal{L}$ and $U_{11}+U_{12}K$ are similar:

$$U|\mathcal{L} = (P^+|\mathcal{L})^{-1}(U_{11}+U_{12}K)(P^+|\mathcal{L}).$$

Since $1 \notin \sigma_p(U)$, we have $1 \notin \sigma_p(U|\mathcal{L})$, and $1 \notin \sigma_p(U_{11}+U_{12}K)$. By condition (d) of the definition of class $(\mathbf{S_1})$, the operator $M(\lambda)$ is bounded and densely defined. It follows from (2.1) and (2.3) that $1 \in \rho(U_{11})$. Since $U_{12}K$ is compact, $1 \in \rho(U_{11}+U_{12}K)$. The latter is equivalent to $1 \in \rho(U|\mathcal{L})$, that is, $(U-I)\mathcal{L} = \mathcal{L}$. Hence, $\mathcal{L} \subset \operatorname{dom} A = \operatorname{ran}(U-I)$ and then $\mathcal{L}$ is A-invariant.

Let us prove (ii). First, we check that there exists a uniformly positive subspace $\mathcal{H}_1^+ \in \mathfrak{M}^+$ in $\operatorname{dom} A$. Let us consider the operators $A_\varepsilon = A + i\varepsilon J$. These operators are J-dissipative, $\operatorname{dom} A_\varepsilon = \operatorname{dom} A$ and, for a sufficiently small $\varepsilon > 0$, together with A belong to the class $(\mathbf{S_1})$. Moreover,

$$\operatorname{Im}[A_\varepsilon x, x] \geq \varepsilon \|x\|^2. \tag{2.4}$$

According to (i), the operator A_ε has an invariant subspace $\mathcal{L}_\varepsilon \in \mathfrak{M}^+$ such that $\mathcal{L}_\varepsilon \in \operatorname{dom} A_\varepsilon = \operatorname{dom} A$ and $\operatorname{Im}\sigma(A_\varepsilon|\mathcal{L}_\varepsilon) \geq 0$. Let G_ε be the Gram operator of the subspace $\mathcal{L}_\varepsilon$. Let us prove that $0 \in \rho(G_\varepsilon)$. Indeed, if we suppose the contrary, then there exists a normalized sequence of vectors $x_n \in \mathcal{L}_\varepsilon$, $\|x_n\| = 1$ such that $G_\varepsilon x_n \to 0$ as $n \to \infty$. But then also $\operatorname{Im}[A_\varepsilon x_n, x_n] \to 0$, which is impossible, according to (2.4). So, $0 \in \rho(G_\varepsilon)$, and this is equivalent to the fact that $\mathcal{L}_\varepsilon$ is uniformly positive. It remains to put $\mathcal{H}_1^+ = \mathcal{L}_\varepsilon$. Without loss of generality, we can assume that $\mathcal{H}_1^+ = \mathcal{H}^+$, that is, $\mathcal{H}^+ \subset \operatorname{dom} A$ and then $A \in (\mathbf{L})$ by definition.

It follows from [4, Theorem 3.1.13] that, for $A \in (\mathbf{L})$, there exists a point $\lambda \in \mathbb{C}^+$ such that the nonnegative invariant subspaces of A and its Cayley–Neumann transform U coincide. Moreover, one can choose λ with a sufficiently large imaginary part. Let λ be such that $\lambda, \overline{\lambda} \in \rho(A|\mathcal{L}^+)$, which is possible since $\mathcal{L}^+ \subset \operatorname{dom} A$ and, hence, $A|\mathcal{L}^+$ is bounded. Consequently, $\mathcal{L}^+$ is invariant with respect to the Cayley–Neumann transform U of the operator A and, moreover, $U\mathcal{L}^+ = \mathcal{L}^+$. It follows from [4, Theorem 3.3.9] that there exists a subspace $\widetilde{\mathcal{L}^+} \in \mathfrak{M}^+$ such that $U\widetilde{\mathcal{L}^+} = \widetilde{\mathcal{L}^+}$ and $\mathcal{L}^+ \subset \widetilde{\mathcal{L}^+}$. From the choice of λ and [4, Theorem 3.1.13], we have that $\widetilde{\mathcal{L}^+}$ is a desired maximal nonnegative A-invariant subspace which is an extension of the given nonnegative A-invariant subspace $\mathcal{L}^+$. $\square$

3. Invariant subspaces of a C_0-semigroup

Let $U(t)$, $t \in [0;\infty)$ be a one-parameter C_0-semigroup of J-binoncontractive operators. Let $B = -iA$ be the generator of this semigroup:

$$Bx = \lim_{t\to 0} \frac{U(t)x - x}{t}. \tag{3.1}$$

Let us recall that B is a closed operator defined on the set of vectors $x \in \mathcal{H}$, for which limit (3.1) exists. Since the functions $[U(t)x, U(t)x]$ and $[U(t)^*x, U(t)^*x]$ are nondecreasing for every $x \in \mathcal{H}$, A is a maximal J-dissipative operator. Indeed, one can prove this in the standard way by differentiating the functions $[U(t)x, U(t)x]$ and $[U(t)^*x, U(t)^*x]$ with respect to t and taking in account that the derivative of a nondecreasing function is nonnegative:

$$\begin{aligned}[U(t)x, U(t)x]' &= 2\mathrm{Re}\,[U'(t)x, U(t)x] \\ &= 2\mathrm{Re}\,(-i[AU(t)x, U(t)x]) = 2\mathrm{Im}\,[AU(t)x, U(t)] \geq 0.\end{aligned}$$

In particular, we have $\mathrm{Im}\,[Ax, x] \geq 0$ as $t \to 0$. Similarly, $\mathrm{Im}\,[-A^*x, x] \geq 0$. For a completion of the proof, let us use the relation between J-dissipative and dissipative operators, Proposition 1.2, and the fact that (see, for example, [7, Theorem I.4.4]) a closed dissipative operator is maximal by Phillips if and only if its adjoint is dissipative.

Let us note that, in contrast to the Hilbert space case, there exist not maximal J-dissipative operators which generate C_0-semigroups, and there are maximal J-dissipative operators which do not generate C_0-semigroups.

Example 3.1. Let $A_\pm : \mathcal{H}^\pm \to \mathcal{H}^\pm$ be maximal symmetric operators with $\rho(A_\pm) = \mathbb{C}^-$. Then the operator

$$A = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix} \tag{3.2}$$

is J-dissipative but not maximal, since $-A_-$ is not maximal dissipative in $\mathcal{H}^-$. Nevertheless, the operator A generates a C_0-semigroup with the generator $B = -iA$. On the other hand, let us consider the maximal J-dissipative operator

$$\widetilde{A} = \begin{bmatrix} A_+ & 0 \\ 0 & \widetilde{A}_- \end{bmatrix},$$

where $-\widetilde{A}_-$ is a maximal dissipative extension of the dissipative operator $-A_-$. The operator $\widetilde{A}$ does not generate a C_0-semigroup, since, for instance, it has no regular points.

The theorems below establish the relations between invariant subspaces of a semigroup, its generator, and its co-generator.

Theorem 3.1. *Let $U(t)$ be a C_0-semigroup, $\|U(t)\| \le M \exp(\omega t)$, $M > 0$, $\omega \ge 0$, let $-iA$ be the generator of $U(t)$, and let $V = (A+\omega+I)(A-\omega-I)^{-1}$ be the co-generator of $U(t)$. Then the following assumptions are equivalent:*

(i) *the subspace $\mathcal{L}$ is $U(t)$-invariant for every t;*

(ii) *the subspace $\mathcal{L}$ is A-invariant and $\omega+1 \in \rho(A|_{\mathcal{L}})$;*

(iii) *the subspace $\mathcal{L}$ is V-invariant.*

Proof. (i) $\Rightarrow$ (ii) immediately follows from the assumption that $U(t)|_{\mathcal{L}}$ is a C_0-semigroup, $\|U(t)|_{\mathcal{L}}\| \le \|U(t)\| \le M \exp(\omega t)$, and $A|_{\mathcal{L}}$ is the generator of $U(t)$.

(ii) $\Rightarrow$ (iii) is true since $\omega+1 \in \rho(A|_{\mathcal{L}})$.

(iii) $\Rightarrow$ (ii). The subspace $\mathcal{L}$ is $(A-\omega-1)^{-1}$-invariant. Assume

$$\rho_\omega(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\}.$$

Since all $\lambda \in \rho_\omega(A)$ are regular points of A and $\omega+1 \in \rho_\omega(A)$, it follows that $\mathcal{L}$ is $(A-\lambda)^{-1}$-invariant for all $\lambda \in \rho_\omega(A)$. Let us note that A is the limit of the operators $A_n := -\lambda_n - \lambda_n^2(A-\lambda_n)^{-1}$ in the the strong operator topology for real $\lambda_n \to +\infty$, $A_n\mathcal{L} \subset \mathcal{L}$. Therefore, $\operatorname{dom} A|_{\mathcal{L}} = \operatorname{dom} A \cap \mathcal{L}$ and $\mathcal{L}$ is A-invariant. Since $\mathcal{L}$ is V-invariant, we have $\omega+1 \in \rho(A|_{\mathcal{L}})$.

(ii) $\Rightarrow$ (i). The set $\rho_\omega(A)$ consists of points of the regular type of $A|_{\mathcal{L}}$ and $\omega+1 \in \rho(A|_{\mathcal{L}})$. Hence, $\rho_\omega(A) \subset \rho(A|_{\mathcal{L}})$. So, $\mathcal{L}$ is $(A-\lambda)^{-1}$-invariant as $\lambda > \omega$. Since, for $x \in \operatorname{dom} A$ and $t > 0$, the equality

$$U(t)x = -\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \exp(\lambda t)(A-\lambda)^{-1}x\, d\lambda$$

holds, $x \in \operatorname{dom} A|_{\mathcal{L}}$ implies $U(t)x \in \mathcal{L}$. It follows from the continuity of $U(t)$ that $U(t)\mathcal{L} \subset \mathcal{L}$. □

Remark 3.1. Let the conditions of Theorem 3.1 hold, and let the subspace $\mathcal{L}$ be V-invariant. Then the operators $-iA|_{\mathcal{L}}$ and $V|_{\mathcal{L}}$ are the generator and the co-generator of the C_0-semigroup $U(t)|_{\mathcal{L}}$, respectively.

We say that an operator A belongs to *the class* $\mathbf{H}$, if there exists a pair $\mathcal{L}_\pm \in \mathfrak{M}^\pm$ of A-invariant subspaces and each such subspaces belong to the classes $\mathbf{h}^\pm$, respectively. By definition, an operator A belongs to *the class* $\mathbf{K}(\mathbf{H})$, if there is a J-binoncontractive operator $B \in \mathbf{H}$ such that the resolvents of A and B commute: $(BA \subseteq AB)$.

Similarly, one can define C_0-semigroups $U(t)$ of the classes $\mathbf{H}$ and $\mathbf{K}(\mathbf{H})$.

The following result was proved in [6]:

Lemma 3.1. *Let A be a maximal J-dissipative operator and $C_+^a = \{\lambda \,|\, \operatorname{Im}\lambda > a\} \subset \rho(A)$. Then, for $\lambda \in C_+^a$,*

$$A \in \mathbf{H} \quad \Longleftrightarrow \quad U = (A - \overline{\lambda})(A - \lambda)^{-1} \in \mathbf{H}.$$

Moreover, the invariant subspaces of A and U are the same.

Theorem 3.2. *Let $U(t)$ be a C_0-semigroup of J-binoncontractive operators, and let $-iA$ be the generator of this semigroup. Then the following implications hold:*

(i) $U(t) \in \mathbf{H} \Longleftrightarrow A \in \mathbf{H}$;

(ii) $U(t) \in \mathbf{K}(\mathbf{H}) \Longleftrightarrow A \in \mathbf{K}(\mathbf{H})$.

Proof. (i) Assume $U(t) \in \mathbf{H}$. It follows from the definition that there exist subspaces $\mathcal{L}_\pm$ such, tat $\mathcal{L}_\pm \in \mathfrak{M}^\pm \cap \mathbf{h}^\pm$, and $U(t)\mathcal{L}_\pm \subset \mathcal{L}_\pm$ for every $t \in (0;\infty)$. Then, for each $x \in \mathcal{L}_\pm$, we have

$$(A - \lambda I)^{-1}x = \int_0^\infty e^{-\lambda t}U(t)x\,dt \in \mathcal{L}_\pm.$$

This is equivalent to $U\mathcal{L}_\pm \subset \mathcal{L}_\pm$, that is, $U \in \mathbf{H}$. Hence, according to Lemma 3.1, we have also $A \in \mathbf{H}$.

If $A \in \mathbf{H}$, that is, there are subspaces $\mathcal{M}_\pm \in \mathfrak{M}^\pm \cap \mathbf{h}^\pm$, for which $A(\mathcal{M}_\pm \cap \operatorname{dom} A) \subset \mathcal{M}_\pm$, then, for every $x \in \mathcal{M}_\pm$ and $U(t)$, the following equality holds:

$$U(t)x = \frac{1}{2\pi i}\int_0^\infty e^{\lambda t}(A - \lambda I)^{-1}x\,d\lambda \in \mathcal{M}_\pm.$$

So, $U(t)\mathcal{M}_\pm \subset \mathcal{M}_\pm$ and again, by Lemma 3.1, $U(t) \in \mathbf{H}$.

(ii) Let $U(t) \in \mathbf{K}(\mathbf{H})$. This means that there is a $V \in \mathbf{H}$ which commutes with $U(t)$ for every $t \in (0;\infty)$. Since $Ax = \lim_{t\to t_0} \frac{U(t)-I}{it}x$ $(x \in \operatorname{dom} A)$, the operator A commutes with V as well:

$$VAx = \lim_{t\to 0} V\frac{U(t)-I}{it}x = \lim_{t\to 0}\frac{U(t)-I}{it}Vx = AVx,$$

and then $A \in \mathbf{K}(\mathbf{H})$.

Suppose $A \in \mathbf{K}(\mathbf{H})$. Then there exists the operator $V \in \mathbf{H}$ such that $VA \subseteq AV$. The latter is equivalent to $(A-\lambda)^{-1}V = V(A-\lambda)^{-1}$ for $\lambda \in \rho(A)$.

Hence, for every $U(t)$,

$$U(t)Vx = \frac{1}{2\pi i}\int_0^\infty e^{\lambda t}(A-\lambda I)^{-1}x\,d\lambda V$$

$$= V\frac{1}{2\pi i}\int_0^\infty e^{\lambda t}(A-\lambda I)^{-1}x\,d\lambda = VU(t)x.$$

The theorem is completely proved. □

We give some *examples* of operators of the class $\mathbf{H}$. Consider a Cauchy problem

$$\begin{cases} \ddot{x} + iB\dot{x} - Cx = 0, \\ x(0) = x_0,\ \dot{x}(0) = \dot{x}_0, \end{cases} \tag{3.3}$$

for the differential equation in a Hilbert space $\mathcal{G}$. Here, C is a positive bounded operator, and $-B$ is a dissipative operator.

Assume that at least one of the following conditions holds:

a) C is *a compact operator and* $\lambda = 0$ *is a regular point of* $-B$*, or it is an isolated point of the spectrum which is an eigenvalue with a finite multiplicity, that is,* 0 *is a normal point* [4, *Definition* 2.1.4].

b) $B = D + F$, *where* D^{-1} *and* $\overline{D^{-1}F}$ *both are compact operators.*

If $x = C^{-1/2}z$, then $\dot{x} = C^{-1/2}\dot{z}$. For $y = -iC^{-1/2}\dot{z}$, let us rewrite the problem as follows:

$$\begin{pmatrix} \dot{z} \\ \dot{y} \end{pmatrix} = i\begin{pmatrix} 0 & C^{1/2} \\ -C^{1/2} & B \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix}, \quad \begin{pmatrix} z(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} C^{1/2}x_0 \\ -i\dot{x}_0 \end{pmatrix}.$$

Denote

$$W = \begin{pmatrix} z \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 0 & C^{1/2} \\ -C^{1/2} & B \end{pmatrix}, \quad W(0) = W_0,$$

where

$$W_0 = \begin{pmatrix} C^{1/2}x_0 \\ -i\dot{x}_0 \end{pmatrix}.$$

Then (3.3) can rewritten as

$$\dot{W} = iAW,\ W(0) = W_0. \tag{3.4}$$

We note that A is J-dissipative in $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, $\mathcal{H}_\pm = \mathcal{G}$, $J = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$.

Let us show that both conditions a) and b) imply $A \in \mathbf{H}$. Assume that condition a) holds. Then $\mathcal{H}_+ \subset \operatorname{dom} A$, and, therefore, there exists the A-invariant nonnegative subspace $\mathcal{L} = \{x + Kx\}_{x \in P^+\mathcal{L}}$, where K is the angle operator of $\mathcal{L}$. The assumption $A\mathcal{L} \subset \mathcal{L}$ is equivalent to the fact that, for every $x \in \mathcal{H}^+$, there exists $y \in \mathcal{H}^+$ which is a solution of the equation $A(x + Kx) = y + Ky$, that is:

$$\begin{pmatrix} 0 & C^{1/2} \\ -C^{1/2} & B \end{pmatrix} \begin{pmatrix} x \\ Kx \end{pmatrix} = \begin{pmatrix} y \\ Ky \end{pmatrix}.$$

Hence,

$$-C^{1/2} + BK - KC^{1/2}K = 0. \tag{3.5}$$

Since $\lambda = 0$ is a regular point of B, we multiply both sides by B^{-1} and get

$$-B^{-1}C^{1/2} + K - B^{-1}KC^{1/2}K = 0$$

or

$$K = B^{-1}C^{1/2} + B^{-1}KC^{1/2}K.$$

Since the operators on the right-hand side are compact, the operator on the left-hand side also is compact. That is, K is compact. This implies $\mathcal{L} \in \mathbf{h}^+$ and, therefore, $A \in \mathbf{H}$.

Let b) hold. Then we replace the operator B in (3.5) by $D + F$, multiply both sides by D^{-1}, and obtain that K is compact. Hence, the operator $-iA$ is a generator of the semigroup $\{\exp(-itA)\}_{t=0}^{\infty} \in \mathbf{H}$. Therefore, a solution of (3.4) has the form $W(t) = \exp(-itA)W_0$. So, a solution of the Cauchy problem (3.3) is $x(t) = C^{-1/2}PW(t)$, where P is the orthogonal projection onto $\mathcal{G} \oplus 0$.

Theorems 3.1 and 3.2 yield immediately the following result:

Theorem 3.3. *If $A \in \mathbf{K}(\mathbf{H})$, then the C_0-semigroup $U(t)$ of J-binon-contractive operators has a maximal nonnegative subspace of the class $\mathbf{h}^+$ and a maximal nonpositive subspace of the class $\mathbf{h}^-$.*

Moreover, if $\mathcal{L}^\pm \in \mathbf{h}^\pm$ are invariant subspaces of A, then there are subspaces $\widetilde{\mathcal{L}^\pm} \in \mathbf{h}^\pm \cap \mathfrak{M}^\pm$, $\mathcal{L}^\pm \subset \widetilde{\mathcal{L}^\pm}$ which are $U(t)$-invariant.

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