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Some problems with homogeneous boundary conditions for degenerate nonlinear equations

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Abstract. We consider a boundary-value problem for the nonlinear degenerate elliptic equation and an initial boundary-value problem for the nonlinear degenerate parabolic equation with nonstandard growth conditions. The existence theorems for the considered problems are proved.

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Introduction

Our aim is to prove the existence theorems for some nonlinear degenerate elliptic and parabolic equations. For example, we consider the initial boundary-value problem

$$\begin{aligned} |u|^{r-2}u_t - \sum_{i=1}^{n}(|u|^{\gamma_i-2}u_{x_i})_{x_i} + |u|^{q(x)-2}u = f(x,t),& \\ x \in \Omega \subset \mathbb{R}^n, \quad t \in (0,T),& \\ u|_{\partial\Omega\times[0,T]} = 0,& \\ u|_{t=0} = 0,& \end{aligned} \tag{$*$}$$

where $r, \gamma_1, \dots, \gamma_n \geq 2$, $q : \Omega \to (1, +\infty)$, $f : \Omega \times (0,T) \to \mathbb{R}^1$. Here,

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we prove that problem $(*)$ has a generalized solution. This solution is a limit with respect to the weak topology of the sequence of solutions to the following problem:

$$\begin{gathered}
-\varepsilon(|u^\varepsilon|^{r-2}u^\varepsilon_t)_t - \sum_{i=1}^n (|u^\varepsilon|^{\gamma_i-2}u^\varepsilon_{x_i})_{x_i} + |u^\varepsilon|^{r-2}u^\varepsilon_t \\
+|u^\varepsilon|^{q(x)-2}u^\varepsilon = f(x,t), \\
x\in\Omega,\quad t\in(0,T), \\
u^\varepsilon|_{\partial\Omega\times[0,T]} = 0, \\
u^\varepsilon|_{t=0} = 0,\quad u^\varepsilon_t|_{t=T} = 0,\quad \varepsilon>0.
\end{gathered} \tag{$**$}$$

Using the Galerkin method, we prove the existence of a generalized solution to this boundary-value problem. Note that Eqs. $(*)$ and $(**)$ contain the terms such that their degrees are some functions $q \not\equiv \text{const}$. Therefore, the solutons to $(*)$ and $(**)$ belong to a generalized Lebesgue space (see [18, 22]). If $r \neq 2$ and if $q \not\equiv \text{const}$, then the problems of types $(*)$ and $(**)$ were not earlier studied. The mixed problems for other types of the nonlinear parabolic equation with variable exponents of a nonlinearity were considered in [12]. The author and S. Lavrenyuk studied various initial boundary-value problems and problems without initial conditions for parabolic equations and variational inequalities with the variable exponent of a nonlinearity (see [10, 11, 13]). In [9], M. Bokalo and V. Sikorsky considered a problem without initial conditions for the parabolic equation in anisotropic Sobolev spaces. Note that the parabolic equations or inequalities in [9–13] contained a monotonous elliptic operator (unlike Eq. $(*)$).

In his paper [15], Yu. Dubinskii proved that the system of elliptic equations of type $(**)$ with $q(y) \equiv \text{const}$ and homogeneous Dirichlet boundary conditions has a solution. The mixed problem with homogeneous initial condition for equations of the form

$$|u|^{r-2}u_t - \alpha\Delta u - \beta\sum_{i=1}^n (|u|^{s-2}u_{x_i})_{x_i} + \gamma|u|^{h-2}u = f(x,t),$$

where $\alpha, \beta, \gamma > 0$, $h = s$, was considered in [1]. The variational inequalities, which correspond to the above-presented equations, were investigated in [5]. Using Schauder's theorem, the existence of the solution to a Dirichlet boundary-value problem for equations of type $(**)$ with $r = r(x,t)$, $\gamma_j = \gamma_j(x,t)$, $j = \overline{1,N}$, but without junior terms was investigated in [3]. The correspond mixed problem for equation $(*)$ for conditions $r = 2$ and without junior terms was studied in [4]. Some properties of solutions to the equation of type $(*)$ were proved in [2, 7, 8, 14, 23–25].

This paper is organized as follows. In Section 1, we give the statements of our problems. In Section 2, we consider some auxiliary facts, propositions, lemmas, and theorems. The third section involves the existence theorem of the solution to a problem of type $(**)$. In Section 4, we prove the existence theorem of the solution to problems of type $(*)$. The uniqueness of solutions to our problems is not studied.

Let us introduce the following notation. Let $\|\cdot; B\|$ be a norm of some Banach space B, B^k a Cartesian product of B, where $k \in \mathbb{N}$, B^* a conjugate space of B, $\langle \cdot, \cdot \rangle_B$ a scalar product of B^* and B. If $u : (0, T) \to B$, then $u(t) \stackrel{def}{=} u(\cdot, t)$ (see [17, p. 145]). The notation $B_1 \circlearrowleft B_2$ means that the space B_1 is continuously imbedded in B_2, $B_1 \overline{\circlearrowleft} B_2$ means that space B_1 is continuously and densely imbedded in B_2, and $B_1 \stackrel{K}{\circlearrowleft} B_2$ means that space B_1 is compactly imbedded in B_2. In addition, by C_i, we mark positive constants which depend only on the initial data.

1. Statement of problems

First, we assume that $G \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ is a domain such that the following condition is satisfied:

(G): $G = \Omega \times [0, \ell_{n+1}] \times \cdots \times [0, \ell_N]$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with the piecewise smooth boundary $\partial\Omega$, $n \in \{0, 1, \ldots, N\}$, $\ell_{n+1}, \ldots, \ell_N > 0$.

Note that the case $n = 0$ means that the domain Ω is absent, and $G = [0, \ell_1] \times \cdots \times [0, \ell_N]$. If $n = N$, then $G = \Omega$.

If condition **(G)** is satisfied, then the following assumptions are needed for the sequel. If $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$, then $y'_j = (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N) \in \mathbb{R}^{N-1}$, $dy = dy_1 \cdots dy_N$, $dy'_j = dy_1 \cdots dy_{j-1} dy_{j+1} \cdots dy_N$,

$$G_j = \Omega \times [0, \ell_{n+1}] \times \cdots \times [0, \ell_{j-1}] \times [0, \ell_{j+1}] \times \cdots \times [0, \ell_N] \quad \text{if} \quad j \geq n+1.$$

Suppose that G satisfies **(G)**, $a_1, \ldots, a_N, b_1, \ldots, b_N, g, f : G \to \mathbb{R}^1$, and the following conditions are satisfied:

(Q1): $q \in L^\infty(G)$, $1 < q_1 \equiv \operatorname{ess\,inf}_{y \in G} q(y) \leq \operatorname{ess\,sup}_{y \in G} q(y) \equiv q_2 < +\infty$;

(Γ1): $\gamma_1, \ldots, \gamma_N \in [2, +\infty)$.

We seek the function $u : G \to \mathbb{R}^1$ such that

$$-\sum_{j=1}^{N}(a_j(y)|u|^{\gamma_j-2}u_{y_j})_{y_j} + \sum_{j=1}^{N} b_j(y)|u|^{\gamma_j-2}u_{y_j} + g(y)|u|^{q(y)-2}u = f(y), \quad y \in G, \quad (1.1)$$

$$u|_{\partial\Omega\times[0,\ell_{n+1}]\times\cdots\times[0,\ell_N]} = 0, \quad (1.2)$$

$$\begin{matrix} 1) u|_{y_j=0} = 0, \\ 2) u_{y_j}|_{y_j=\ell_j} = 0, \end{matrix} \qquad j = \overline{n+1, N}. \quad (1.3)$$

Note that the case $n = 0$ means that condition (1.2) is absent. If $n = N$, then we have problem (1.1), (1.2). Using some additional conditions, we will prove the existence of a generalized solution to problem (1.1)–(1.3).

Further, we assume that $n \in \mathbb{N}$ and $N = n + 1$. In this case, it is convenient to use the following notation: $\ell_{n+1} = T$, where $T > 0$, $G = Q_{0,T} = \Omega \times (0,T)$, $y_1 = x_1, \ldots, y_n = x_n$, $y_{n+1} = t$. Finally, let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial\Omega \subset C^1$, $\Omega_\tau = \{(x,t) : x \in \Omega, t = \tau\}$, $Q_{t_1,t_2} = \Omega \times (t_1,t_2)$, $\tau \in [0,T]$, and $0 \le t_1 < t_2 \le T$. Suppose that

(Q2): $q \in L^\infty(\Omega)$, $1 < q_1 \equiv \operatorname{ess\,inf}_\Omega q(x) \le \operatorname{ess\,sup}_\Omega q(x) \equiv q_2 < +\infty$;

(Γ2): $r, \gamma_1, \ldots, \gamma_n \in [2, +\infty)$.

Using some additional conditions and the solution to problem (1.1)–(1.3), we prove the existence of the solution $u : Q_{0,T} \to \mathbb{R}^1$ to the following problem:

$$|u|^{r-2}u_t - \sum_{i=1}^{n}(a_i(x,t)|u|^{\gamma_i-2}u_{x_i})_{x_i} + \sum_{i=1}^{n} b_i(x,t)|u|^{\gamma_i-2}u_{x_i} + g(x,t)|u|^{q(x)-2}u = f(x,t), \quad (1.4)$$

for $(x,t) \in Q_{0,T}$,

$$u|_{\partial\Omega\times[0,T]} = 0, \quad (1.5)$$

$$u|_{t=0} = 0. \quad (1.6)$$

Note that our problems have their solutions in anisotropic Sobolev spaces and generalized Lebesgue spaces.

2. Notation and preliminary statements

Let us introduce the following notation. The generalized Lebesgue and Sobolev spaces were studied, in particular, in [18, 22]. Let $G \subset \mathbb{R}^N$ be a bounded domain with condition **(Q1)**, $1/q(y) + 1/q'(y) = 1$ for a.e. $y \in G$. By definition, we set $\rho_q(v, G) = \int_G |v(y)|^{q(y)} dy$, where v be some function. The generalized Lebesgue space is called the set of all measurable functions v such that $\rho_q(v, G) < +\infty$; we denote it by $L^{q(y)}(G)$. In [22, p. 616, 619, 621] I. Sharapudinov proved that $L^{q(y)}(G)$ is a reflexive space with respect to the norm

$$\|v; L^{q(y)}(G)\| = \inf\{\lambda > 0 \ : \ \rho_q(v/\lambda, G) \leq 1\}.$$

In [18, p. 594], O. Kovacik and J. Rakosnik noticed that if $\|v; L^{q(y)}(G)\| \leq 1$, then $\rho_q(v, G) \leq 1$. The reverse proposition follows from the definition of the norm of the space $L^{q(y)}(G)$. Note that $L^{q(y)}(G)$ is a Banach space, and if $r(y) \geq q(y)$, then $L^{r(y)}(G) \circlearrowleft L^{q(y)}(G)$ (see [18, p. 599, 600]). The dual space to $L^{q(y)}(G)$ is $L^{q'(y)}(G)$ (see [22, p. 619]). If condition **(Q2)** is fulfilled, then we similarly define the spaces $L^{q(x)}(\Omega)$ and $L^{q(x)}(Q_{0,T})$. The following propositions are needed for the sequel.

Statement 2.1 (Lemma 4.3 [19, p. 66]). *Let $P = (P_1, \dots, P_m) : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous function. If there exists $\rho > 0$ such that $(P(z), z)_{\mathbb{R}^m} \geq 0 \ \forall\, z \in \mathbb{R}^m$ $(|z| = \rho)$, then there exists $z^m \in \mathbb{R}^m$ $(|z| \leq \rho)$ such that $P(z^m) = 0$.*

Statement 2.2 (Lemma [15, p. 471]). *Suppose that $Z : G \times \mathbb{R}^1 \to \mathbb{R}^1$ satisfies the Caratheodory condition, and $\lambda \geq 0$. If the sequence $\{u^m\}_{m\in\mathbb{N}}$ satisfies the conditions*

1) $u^m \underset{m\to\infty}{\longrightarrow} u$ *a.e. in* G,

2) *for all $j \in \{1, \dots, N\}$, we have $|u^m|^\lambda u^m_{y_j} \underset{m\to\infty}{\longrightarrow} \frac{1}{1+\lambda}(|u|^\lambda u)_{y_j}$ slowly in $L^2(G)$,*

3) *if there exist $s > 2$ and $C_1 > 0$ such that $\|Z(y, u^m); L^s(G)\| \leq C_1$ for every $m \in \mathbb{N}$, then*

$$Z(y, u^m)|u^m|^\lambda u^m_{y_j} \underset{m\to\infty}{\longrightarrow} \frac{1}{1+\lambda} Z(y, u)(|u|^\lambda u)_{y_j} \quad \textit{slowly in} \quad L^1(G).$$

Statement 2.3 (Aubin's Theorem [6] ([19, Theorem 5.1, p. 70])). *Suppose B_0, B, B_1 are Banach spaces, B_0, B_1 are reflexive spaces, $p_0, p_1 \in (1, +\infty)$, $Y_1 = \{v \in L^{p_0}(0,T;B_0) \mid v_t \in L^{p_1}(0,T;B_1)\}$, and $B_0 \overset{K}{\circlearrowleft} B \circlearrowleft B_1$; then $Y_1 \overset{K}{\circlearrowleft} L^{p_0}(0,T;B)$.*

Statement 2.4 (Lemma 2 and Theorems 1, 2 [16]). *Suppose that A_0, A_1 are linear normed spaces, M_1 is a seminormed set with respect to the seminorm $[\cdot]_{M_1}$, $M_1 \overset{K}{\circlearrowleft} A_0 \circlearrowleft A_1$, and $p, p_1 \geq 1$; then*

1) *$Y = \{u : (0,T) \to M_1 \mid \int_0^T [u(t)]_{M_1}^p \, dt + \int_0^T \|u_t(t)\|_{A_1}^{p_1} \, dt < +\infty\}$ is a seminormed set;*

2) *$Y \overset{K}{\circlearrowleft} L^p(0,T;A_0)$;*

3) *if $M_1 \overset{K}{\circlearrowleft} A_1$, then $Y \overset{K}{\circlearrowleft} C([0,T];A_1)$;*

4) *if $Y \circlearrowleft L^{p_0}(0,T;A_0)$, where $p_0 > 1$, then $Y \overset{K}{\circlearrowleft} L^q(0,T;A_0)$, where $q \in [1,p)$.*

Define the maps $h, \omega : \mathbb{R} \to \mathbb{R}$ by the rules

$$h(s) = \frac{|s|^\mu}{\mu}, \quad \omega(s) = |s|^{\mu-2}s, \quad s \in \mathbb{R}. \tag{2.1}$$

Remark 2.1. If $\mu > 1$, then $h'(s) = \omega(s)$, where $s \in \mathbb{R}$. If $\mu \geq 2$, then $\omega'(s) = (\mu-1)|s|^{\mu-2}$, where $s \in \mathbb{R}$.

The following lemmas are needed for the sequel.

Lemma 2.1. *Assume that $p, q, \mu \in (1, +\infty)$, and the function ω is defined in (2.1). If $u \in L^p(G)$ and if*

$$\mu \in (1, 1+p], \qquad q \in \Big[1, \frac{p}{\mu-1}\Big], \tag{2.2}$$

then $\omega(u) \in L^q(G)$ and $\|\omega(u); L^q(G)\| \leq C_2 \|u; L^p(G)\|^{\mu-1}$, where $C_2 > 0$ is independent on u.

Proof. Let $u \in L^p(G)$, $J \overset{\text{def}}{=} \int_G |\omega(u)|^q dx = \int_G |u|^{(\mu-1)q} dx$. If $q = \frac{p}{\mu-1}$, then $q(\mu-1) = p$ and $J = \|u; L^p(G)\|^{q(\mu-1)} < +\infty$. If $q < \frac{p}{\mu-1}$, then $\frac{p}{q(\mu-1)} > 1$. Using the Hölder inequality, we have

$$J \le C_2 \Bigg(\int_G |u|^p \, dx \Bigg)^{\frac{(\mu-1)q}{p}} = C_2 \|u; L^p(G)\|^{q(\mu-1)} < +\infty.$$

□

Lemma 2.2. *Assume that $\alpha_1 \ge 1$ and $\alpha_0 > 1 - \alpha_1$. If $u \in C^1(\overline{G})$, then*

$$\int_G |u|^{\alpha_0+\alpha_1} \, dy \le C_3 \Bigg(\int_G |u|^{\alpha_0} |u_{y_j}|^{\alpha_1} \, dy + \int_{\partial G} |u|^{\alpha_0+\alpha_1} \, dS \Bigg), \quad j \in \{1, \dots, N\}, \tag{2.3}$$

where $C_3 > 0$ is independent of u.

Proof. Using the condition $\alpha_0 + \alpha_1 > 1$ and Remark 2.1, we get

$$y_j \, (|u|^{\alpha_0+\alpha_1})'_{y_j} = (\alpha_0 + \alpha_1) \, y_j \, |u|^{\alpha_0+\alpha_1-2} u \, u_{y_j}.$$

On the other hand,

$$\int_G y_j \, (|u|^{\alpha_0+\alpha_1})'_{y_j} \, dy = \int_{\partial G} y_j \, |u|^{\alpha_0+\alpha_1} \cos(\nu, y_j) \, dS - \int_G |u|^{\alpha_0+\alpha_1} \, dy,$$

where ν is the unit vector of a normal which is external to G. Hence,

$$\begin{aligned} \int_G |u|^{\alpha_0+\alpha_1} \, dy &= - \int_G y_j \, (|u|^{\alpha_0+\alpha_1})'_{y_j} \, dy + \int_{\partial G} y_j \, |u|^{\alpha_0+\alpha_1} \cos(\nu, y_j) \, dS \\ &= -(\alpha_0 + \alpha_1) \int_G y_j \, |u|^{\alpha_0+\alpha_1-2} u \, u_{y_j} \, dy + \int_{\partial G} y_j \, |u|^{\alpha_0+\alpha_1} \cos(\nu, y_j) \, dS \\ &\le C_4 \Bigg(\int_G |u|^{\alpha_0+\alpha_1-1} \, |u_{y_j}| \, dy + \int_{\partial G} |u|^{\alpha_0+\alpha_1} \, dS \Bigg), \end{aligned} \tag{2.4}$$

where $C_4 > 0$ depends only on G and α_0, α_1. If $\alpha_1 = 1$, then inequality (2.4) is equal to (2.3). In the sequel, only the condition $\alpha_1 > 1$ is considered. By Young's inequality with the constant α_1, we get

$$\begin{aligned} |u|^{\alpha_0+\alpha_1-1} \, |u_{y_j}| &= |u|^{\frac{\alpha_0}{\alpha_1}} \, |u_{y_j}| \, |u|^{\alpha_0+\alpha_1-1-\frac{\alpha_0}{\alpha_1}} \\ &\le C_5(\varepsilon) |u|^{\alpha_0} \, |u_{y_j}|^{\alpha_1} + \varepsilon |u|^{(\alpha_0+\alpha_1-1-\frac{\alpha_0}{\alpha_1}) \frac{\alpha_1}{\alpha_1 - 1}}. \end{aligned}$$

Using (2.4), the equality

$$\Big(\alpha_0+\alpha_1-1-\frac{\alpha_0}{\alpha_1}\Big)\frac{\alpha_1}{\alpha_1-1}=\Big(\frac{\alpha_0(\alpha_1-1)}{\alpha_1}+\alpha_1-1\Big)\frac{\alpha_1}{\alpha_1-1}=\alpha_0+\alpha_1,$$

and the last inequality, we obtain

$$\int\limits_G |u|^{\alpha_0+\alpha_1}\,dy \le C_4\Bigg(C_5(\varepsilon)\int\limits_G |u|^{\alpha_0}\,|u_{y_j}|^{\alpha_1}\,dy + \varepsilon\int\limits_G |u|^{\alpha_0+\alpha_1}\,dy+\int\limits_{\partial G}|u|^{\alpha_0+\alpha_1}\,dS\Bigg).$$

If $\varepsilon>0$ is sufficiently small, then (2.3) holds. □

Lemma 2.3. *Assume that $\alpha_1\ge 0$, $\alpha_2\ge 0$, $\alpha_1+\alpha_2\ge 1$ and $\alpha_0>1-(\alpha_1+\alpha_2)$. If $u\in C^1(\overline{G})$, then*

$$\int\limits_G |u|^{\alpha_0+\alpha_1}|u_{y_j}|^{\alpha_2}\,dy \le C_6\Bigg(\int\limits_G |u|^{\alpha_0}|u_{y_j}|^{\alpha_1+\alpha_2}\,dy+\int\limits_{\partial G}|u|^{\alpha_0+\alpha_1+\alpha_2}\,dS\Bigg), \tag{2.5}$$

$j\in\{1,\dots,N\}$, where $C_6>0$ is independent of u.

Proof. For the case $\alpha_1=0$, there is nothing to prove. For $\alpha_2=0$, inequality (2.5) is equal to (2.3). In the sequel, only the conditions $\alpha_1>0$, $\alpha_2>0$ are considered. By Young's inequality with the constant $\frac{\alpha_1+\alpha_2}{\alpha_2}>1$, we obtain

$$|u|^{\alpha_0+\alpha_1}|u_{y_j}|^{\alpha_2}=|u|^{\frac{\alpha_0\alpha_2}{\alpha_1+\alpha_2}}|u_{y_j}|^{\alpha_2}|u|^{\alpha_0+\alpha_1-\frac{\alpha_0\alpha_2}{\alpha_1+\alpha_2}} \le C_7\Big(\Big[|u|^{\frac{\alpha_0\alpha_2}{\alpha_1+\alpha_2}}|u_{y_j}|^{\alpha_2}\Big]^{\frac{\alpha_1+\alpha_2}{\alpha_2}}+|u|^{\widetilde{\alpha}}\Big),$$

where

$$\begin{aligned}\widetilde{\alpha}&=\Big(\alpha_0+\alpha_1-\frac{\alpha_0\alpha_2}{\alpha_1+\alpha_2}\Big)\frac{\frac{\alpha_1+\alpha_2}{\alpha_2}}{\frac{\alpha_1+\alpha_2}{\alpha_2}-1}\\ &=\Big(\alpha_0+\alpha_1-\frac{\alpha_0\alpha_2}{\alpha_1+\alpha_2}\Big)\frac{\alpha_1+\alpha_2}{\alpha_1}\\ &=\frac{1}{\alpha_1}((\alpha_0+\alpha_1)(\alpha_1+\alpha_2)-\alpha_0\alpha_2)\\ &=\frac{1}{\alpha_1}(\alpha_0\alpha_1+\alpha_1^2+\alpha_1\alpha_2)=\alpha_0+\alpha_1+\alpha_2.\end{aligned}$$

Therefore,

$$\int_G |u|^{\alpha_0+\alpha_1}|u_{y_j}|^{\alpha_2}\,dx \le C_7\bigg(\int_G |u|^{\alpha_0}|u_{y_j}|^{\alpha_1+\alpha_2}\,dy + \int_G |u|^{\alpha_0+\alpha_1+\alpha_2}\,dy\bigg).$$

Thus, taking into account (2.3) with $\alpha_1+\alpha_2$ instead of α_1, we get (2.5). □

The proofs of Lemma 2.2 and Lemma 2.3 were found in [15, p. 459, 460]. To prove the following statement, we need these proofs and condition **(G)**.

Lemma 2.4. *Assume that condition* **(G)** *is satisfied and* $u \in C^1(\overline{G})$.

1) *Suppose that* $j \in \{n+1,\dots,N\}$; *then*

   a) *if* α_0,α_1,C_3 *are defined in Lemma* 2.2 *and if* u *satisfies the condition* $u|_{y_j=0}=0$ *or* $u|_{y_j=\ell_j}=0$, *then*

$$\int_G |u|^{\alpha_0+\alpha_1}\,dy \le C_3\int_G |u|^{\alpha_0}|u_{y_j}|^{\alpha_1}\,dy; \tag{2.6}$$

   b) *if* $\alpha_0,\alpha_1,\alpha_2,C_6$ *are defined in Lemma* 2.3 *and if* u *satisfies the condition* $u|_{y_j=0}=0$ *or* $u|_{y_j=\ell_j}=0$, *then*

$$\int_G |u|^{\alpha_0+\alpha_1}|u_{y_j}|^{\alpha_2}\,dy \le C_6\int_G |u|^{\alpha_0}|u_{y_j}|^{\alpha_1+\alpha_2}\,dy. \tag{2.7}$$

2) *Suppose that* u *satisfies the condition* $u|_{\partial\Omega\times[0,\ell_{n+1}]\times\cdots\times[0,\ell_N]}=0$; *then, for every* $j\in\{1,\dots,n\}$, *we also obtain estimates* (2.6) *and* (2.7).

Proof. Assume that condition **(G)** is satisfied and $u\in C^1(\overline{G})$.

1) For every $j\in\{n+1,\dots,N\}$ and for every $t_0\in[0,\ell_j]$, we have

$$\int_G (y_j-t_0)\left(|u|^{\alpha_0+\alpha_1}\right)'_{y_j}\,dy = \int_{G_j}(y_j-t_0)\,|u|^{\alpha_0+\alpha_1}\,dy'_j\Bigg|_{y_j=0}^{y_j=\ell_j} - \int_G |u|^{\alpha_0+\alpha_1}\,dy.$$

Therefore, similarly as in Lemma 2.2, we get

$$\int_G |u|^{\alpha_0+\alpha_1}\,dy = \int_{G_j} (y_j - t_0)\,|u|^{\alpha_0+\alpha_1}\,dy_j'\Big|_{y_j=0}^{y_j=\ell_j} - \int_G (y_j - t_0)\,(\alpha_0+\alpha_1)\,|u|^{\alpha_0+\alpha_1-2}u\,u_{y_j}\,dy$$
$$\le (\ell_j - t_0)\int_{G_j} |u|^{\alpha_0+\alpha_1}|_{y_j=\ell_j}\,dy_j' + t_0\int_{G_j} |u|^{\alpha_0+\alpha_1}|_{y_j=0}\,dy_j' + \ell_j(\alpha_0+\alpha_1)\int_G |u|^{\alpha_0+\alpha_1-1}\,|u_{y_j}|\,dy.$$

For the case $u|_{y_j=0} = 0$, we put $t_0 = \ell_j$. For the case $u|_{y_j=\ell_j} = 0$, we put $t_0 = 0$. Hence,

$$\int_G |u|^{\alpha_0+\alpha_1}\,dy \le \ell_j(\alpha_0+\alpha_1)\int_G |u|^{\alpha_0+\alpha_1-1}\,|u_{y_j}|\,dy.$$

Continuing in the same way as in the proof of Lemma 2.2 (see (2.4) and down), we get (2.6).

If we replace (2.3) by (2.6) in the proof of Lemma 2.3, we obtain (2.7).

2) Substituting Ω for G in (2.3), (2.4) and integrating these inequalities with respect to $y_{n+1} \in (0, \ell_{n+1})$, ..., $y_N \in (0, \ell_N)$, we obtain (2.6), (2.7) for every $j \in \{1, \dots, n\}$, if $u|_{\partial\Omega\times[0,\ell_{n+1}]\times\cdots\times[0,\ell_N]} = 0$. □

Remark 2.2. It is easy to show that estimations (2.3), (2.5)–(2.7) hold for every functions u such that u is an element of Sobolev spaces such that the integrals in (2.3) or (2.5)–(2.7) are finite.

If condition **(G)** is satisfied, then, by definition, we introduce the notation

$$\Pi_0 = \{v \in C^1(\overline{G}) \mid v|_{\partial\Omega\times[0,\ell_{n+1}]\times\cdots\times[0,\ell_N]} = 0,\quad v|_{y_{n+1}=0} = \cdots = v|_{y_N=0} = 0\}.$$

Now we prove the following theorem.

Theorem 2.1. *Suppose condition* **(G)** *is satisfied,* $\beta_1, \dots, \beta_N$ *are real numbers such that*

$$\min_j \beta_j > -1, \tag{2.8}$$

$$\frac{1}{2}\max_j \beta_j \le \min_j \beta_j + 1, \tag{2.9}$$

and $\beta > 1$ *justifies the estimates*

$$\frac{1}{2}\max_j \beta_j + 2 \le \beta \le \min_j \beta_j + 3, \tag{2.10}$$

and $C_8 > 0$. *If, for every* $u \in \Pi_0$, *the estimate*

$$\int\limits_G |u|^{\beta_j} |u_{y_j}|^2 \, dy \le C_8 \tag{2.11}$$

holds for all $j \in \{1, \dots, N\}$, *then*

$$\big\| \, |u|^{\beta-2}u; W^{1,\frac{\min_j \beta_j+2}{\beta-1}}(G) \big\| \le C_9, \tag{2.12}$$

where $C_9 > 0$ *is independent of* u.

Proof. Assume that the function $u \in \Pi_0$ satisfies (2.11). Then the assumptions of Lemma 2.4 are fulfilled. First, let us prove that $|u|^{\beta-2}u \in L^{\frac{\min_j \beta_j+2}{\beta-1}}(G)$. Using estimate (2.6) with $\alpha_0 = \beta_j$, $\alpha_1 = 2$ (note that $\alpha_0 > 1 - \alpha_1 \iff \beta_j > 1 - 2 \iff$ (2.8)) and inequality (2.11), we get

$$\int\limits_G |u|^{\beta_j+2} \, dy \le C_3 \int\limits_G |u|^{\beta_j} |u_{y_j}|^2 \, dy \le C_3 C_8, \qquad j = \overline{1, N}.$$

Therefore, $u \in L^{\min_j \beta_j+2}(G)$. Take a point $\beta \in (1, 1 + (\min_j \beta_j + 2)]$. By Lemma 2.1, we get $|u|^{\beta-2}u \in L^q(G)$, where $q \in \big[1, \frac{\min_j \beta_j+2}{\beta-1}\big]$.

Assume that $j \in \{1, \dots, N\}$. Note that (2.10) yields $\beta - 1 \le \min_j \beta_j + 2 \le \beta_j + 2$. Hence, $\frac{\beta_j+2}{\beta-1} \ge 1$. Since $(|u|^{\beta-2}u)'_{y_j} = (\beta-1)|u|^{\beta-2}u_{y_j}$ and

$$\begin{aligned}(\beta-2)\,\frac{\beta_j+2}{\beta-1} &= \beta_j + 2 + (\beta-2)\,\frac{\beta_j+2}{\beta-1} - (\beta_j+2) \\ &= \beta_j + 2 + (\beta_j+2)\Big(\frac{\beta-2}{\beta-1} - 1\Big) = \beta_j + 2 - \frac{\beta_j+2}{\beta-1},\end{aligned}$$

we obtain

$$I_j \equiv \int\limits_G \left(|u|^{\beta-2}u_{y_j}\right)^{\frac{\beta_j+2}{\beta-1}}\,dy = \int\limits_G |u|^{\beta_j+2-\frac{\beta_j+2}{\beta-1}}|u_{y_j}|^{\frac{\beta_j+2}{\beta-1}}\,dy.$$

Using (2.11) and estimate (2.7) with $\alpha_0 = \beta_j$, $\alpha_1 = 2 - \frac{\beta_j+2}{\beta-1}$, $\alpha_2 = \frac{\beta_j+2}{\beta-1}$, we obtain

$$\begin{aligned} I_j = \int\limits_G |u|^{\alpha_0+\alpha_1}|u_{y_j}|^{\alpha_2}\,dy &\le C_6\int\limits_G |u|^{\alpha_0}|u_{y_j}|^{\alpha_1+\alpha_2}\,dy \\ &= C_6\int\limits_G |u|^{\beta_j}|u_{y_j}|^2\,dy \le C_6C_8. \end{aligned}$$

Finally, we make sure that all conditions of Lemma 2.4 are satisfied:

$$\begin{cases} \min\limits_j \frac{\beta_j+2}{\beta-1} \ge 1, \\ \alpha_1 \ge 0, \\ \alpha_2 \ge 0, \\ \alpha_1+\alpha_2 \ge 1, \\ \alpha_0+\alpha_1+\alpha_2 > 1, \end{cases} \iff \begin{cases} \min\limits_j \frac{\beta_j+2}{\beta-1} \ge 1, \\ \min\limits_j(2 - \frac{\beta_j+2}{\beta-1}) \ge 0, \\ \min\limits_j \frac{\beta_j+2}{\beta-1} \ge 0, \\ 2 \ge 1, \\ \min\limits_j(\beta_j+2) > 1, \end{cases}$$

$$\overset{\beta-1>0}{\iff} \begin{cases} \min\limits_j(\beta_j+2) \ge \beta-1, \\ \min\limits_j(2(\beta-1)-(\beta_j+2)) \ge 0, \\ \min\limits_j(\beta_j+2) > 1, \end{cases}$$

$$\iff \begin{cases} \min\limits_j \beta_j + 3 \ge \beta, \\ 2\beta - 2 - \max\limits_j(\beta_j+2) \ge 0, \\ \min\limits_j \beta_j + 2 > 1, \end{cases} \iff \begin{cases} \beta \le \min\limits_j \beta_j + 3, \\ 2\beta \ge \max\limits_j \beta_j + 4, \\ \min\limits_j \beta_j > -1. \end{cases}$$

The last inequalities follow from conditions (2.8)–(2.10). □

Note that we have proved Theorem 2.1 in same way as in [15], where a similar result was obtained for other $\beta_1,\dots,\beta_N,\beta$. By definition, we set

$$[v]_{M_1} = \sum_{j=1}^N \left(\int\limits_G |v|^{\beta_j}|v_{y_j}|^2\,dy\right)^{\frac{1}{\beta_j+2}}.$$

Let M_1 be a set of functions v such that

$$[v]_{M_1} < +\infty,$$

$$|v|^{\beta-2}v|_{\partial\Omega\times[0,\ell_{n+1}]\times\cdots\times[0,\ell_N]} = 0,$$

$$|v|^{\beta-2}v|_{y_j=0} = 0, \quad j = \overline{n+1, N},$$

where β satisfies (2.10). Note that M_1 is a seminormed nonlinear set (see the example of [16, p. 610]).

Theorem 2.2. *If conditions* **(G)**, (2.8), *and* (2.9) *are fulfilled, then*

$$M_1 \circlearrowleft L^s(G) \quad \text{and} \quad M_1 \overset{K}{\circlearrowleft} L^{s-\varepsilon}(G),$$

where

$$s = \begin{cases} \dfrac{N}{N-1}(\min_j \beta_j + 2), & \text{if } N > \frac{\min_j \beta_j + 2}{\beta - 1}, \\ \text{any } s_1 \text{ such that } s_1 \geq \beta - 1, & \text{if } N \leq \frac{\min_j \beta_j + 2}{\beta - 1}, \end{cases} \quad \varepsilon \in (0, s)$$

β *satisfies condition* (2.10).

Proof. Suppose that all assumptions of our theorem are satisfied. Using Theorem 2.1 and Sobolev's imbedding theorems for the case $N > \frac{\min_j \beta_j + 2}{\beta - 1}$, we have that, for any $u \in M_1$, the inequality

$$\begin{aligned} \|u; L^{\widetilde{s}}(G)\| &= \| \, |u|^{\beta-2}u; L^r(G)\|^{r/\widetilde{s}} \\ &\leq C_{10} \Big\| \, |u|^{\beta-2}u; W^{1,\frac{\min_j \beta_j + 2}{\beta - 1}}(G) \Big\|^{r/\widetilde{s}} \leq C_{11} \end{aligned} \tag{2.13}$$

holds, where $\widetilde{s} = (\beta - 1)r$,

$$r = \frac{N \frac{\min_j \beta_j + 2}{\beta - 1}}{N - (\frac{\min_j \beta_j + 2}{\beta - 1})} = \frac{N(\min_j \beta_j + 2)}{N(\beta - 1) - (\min_j \beta_j + 2)}.$$

For the case $\beta = \min_j \beta_j + 3$, we see that

$$\begin{aligned} \widetilde{s} = (\beta - 1)r = (\min_j \beta_j + 2) \frac{N(\min_j \beta_j + 2)}{N(\min_j \beta_j + 2) - (\min_j \beta_j + 2)} \\ = \frac{N}{N-1}(\min_j \beta_j + 2) = s. \end{aligned}$$

Hence, $M_1 \circlearrowleft L^s(G)$. If $N \le \frac{\min_j \beta_j + 2}{\beta - 1}$, then estimate (2.13) holds for every $r \in [1, +\infty)$. Therefore, $\widetilde{s} = (\beta-1)r \ge \beta - 1$.

Let us show the compact imbedding. Take a point $\varepsilon \in (0, s-1)$. Therefore,

$$\int\limits_G |u|^{s-\varepsilon}\, dy \le C_{12} \Bigg(\int\limits_G (|u|^{s-\varepsilon})^{\frac{s}{s-\varepsilon}}\, dy \Bigg)^{\frac{s-\varepsilon}{s}}$$
$$= C_{12} \|u; L^s(G)\|^{s-\varepsilon} \le C_{13} \quad \forall\, u \in B_R,$$

where $B_R = \{u \in M_1 : [u]_{M_1} \le R\}$, $R > 0$. Hence, there exists a sequence $\{u^m\}_{m\in\mathbb{N}} \subset B_R$ such that $u^m \underset{m\to\infty}{\longrightarrow} u$ slowly in $L^{s-\varepsilon}(G)$. Taking the imbedding $W^{1,\frac{\min_j \beta_j+2}{\beta-1}}(G) \overset{K}{\circlearrowleft} L^r(G)$ and Lemma 1.18 [17, p. 39] into account, we obtain $|u^m|^{\beta-2}u^m \underset{m\to\infty}{\longrightarrow} |u|^{\beta-2}u$ a.e. in G. If $f_m = |u^m - u|^{s-\varepsilon}$, then the sequence $\{f_m\}_{m\in\mathbb{N}}$ is bounded in $L^{\frac{s}{s-\varepsilon}}(G)$. Without loss of generality, we can assume that $f_m \underset{m\to\infty}{\longrightarrow} 0$ slowly in $L^{\frac{s}{s-\varepsilon}}(G)$ and a.e. in G. Thus, $\int_G f_m\, dy = \int_G |u^m - u|^{s-\varepsilon}\, dy \underset{m\to\infty}{\longrightarrow} 0$, and the theorem is proved. □

Corollary 2.1. *It is easy to see that* $M_1 \overset{K}{\circlearrowleft} L^{\min_j \beta_j + 2}(G)$ (*see* [16, *p.* 619]).

3. The boundary-value problem for nonlinear degenerate elliptic equations

Let us show that problem (1.1)–(1.3) has a solution. Assume that $N \in \mathbb{N}$. Under conditions **(G)**, **(Q1)**, and **(Γ1)** of Section 1, we define

$\gamma_{\max} = \max\{\gamma_1, \dots, \gamma_N\}$, $\gamma_{\min} = \min\{\gamma_1, \dots, \gamma_N\}$,

- $S_1 = \{1, \dots, N\} \setminus S_2$, where S_2 is a collection of numbers $j \in \{1, \dots, N\}$ such that $\gamma_j = \gamma_{\max}$.

We also consider the case $S_1 = \varnothing$, i.e. $\gamma_1 = \ldots = \gamma_N$. Suppose that $a_1, \ \ldots, \ a_N, b_1, \ \ldots, \ b_N, g, f : G \to \mathbb{R}^1$ are functions such that

(A1): $a_j \in L^\infty(G)$, $a_j(y) \ge a_0 > 0$ for a.e. $y \in G$, where $j \in \{1, \dots, N\}$;

(B1): for every $j \in \{1, \dots, N\}$, we have that the function $b_j \in L^\infty(G)$ satisfies one of the following conditions:

a) if $j \in S_1$ and if $j \le n$, then

$$(b_j)_{y_j} \in L^\infty(G), \quad |(b_j(y))_{y_j}| \le b^1 \quad \text{for a.e. } y \in G; \qquad (3.1)$$

b) if $j \in S_1$ and if $j \ge n+1$, then condition (3.1) is satisfied, and $b_j(y) \ge b_0 > 0$ for a.e. $y \in G$;

c) if $j \in S_2$ and if $j \le n$, then $b_j(y) \equiv \widetilde{b}_j \in \mathbb{R}^1$;

d) if $j \in S_2$ and if $j \ge n+1$, then $b_j(y) \equiv \text{const} \ge b_0 > 0$;

(D1): $g \in L^\infty(G)$, $0 < g_0 \le g(y) \le g^0 < +\infty$ for a.e. $y \in G$;

(F1): $f \in L^{\frac{\gamma_{\max}}{\gamma_{\max}-1}}(G)$.

By definition, put $Tr_0^\alpha = \{w : G \to \mathbb{R} \mid \exists \{w^m\}_{m\in\mathbb{N}} \subset \Pi_0 : (|w^m|^{\alpha_j-1}w^m)_{y_j} \underset{m\to\infty}{\longrightarrow} (|w|^{\alpha_j-1}w)_{y_j}$ slowly in $L^2(G)$ for every $j \in \{1,\dots,N\}\}$, where $\alpha = (\alpha_1,\dots,\alpha_N)$.

Definition 3.1. *A function u is called the generalized solution to problem* (1.1)–(1.3) *if the following conditions hold: $u \in L^{q(y)}(G) \cap L^{\gamma_{\max}}(G)$; $|u|^{\frac{\gamma_j}{2}-1}u$, $(|u|^{\frac{\gamma_j}{2}-1}u)_{y_j} \in L^2(G)$ for every $j \in \{1,\dots,N\}$;*

$$\int_G \Bigg[\sum_{j=1}^N a_j|u|^{\frac{\gamma_j}{2}-1}\frac{2}{\gamma_j}(|u|^{\frac{\gamma_j}{2}-1}u)_{y_j}v_{y_j} + \sum_{j=1}^N b_j|u|^{\frac{\gamma_j}{2}-1}\frac{2}{\gamma_j}(|u|^{\frac{\gamma_j}{2}-1}u)_{y_j}v + g|u|^{q(y)-2}uv\Bigg]dy = \int_G fv\,dy \qquad (3.2)$$

for every $v \in \Pi_0$; u satisfies (1.2), (1.3) 1), *i.e.* $u \in Tr_0^\alpha$ *for some* α.

Remark 3.1. It is easy to show (see, e.g., [20, p. 181]) that the boundary condition (1.3) 2) is involved in (3.2).

By definition, put

$$\mathcal{L}(w,v) = \int_G \Bigg[\sum_{j=1}^N a_j(y)|w|^{\gamma_j-2}w_{y_j}v_{y_j} + \sum_{j=1}^N b_j(y)|w|^{\gamma_j-2}w_{y_j}v + g(y)|w|^{q(y)-2}wv\Bigg]dy,$$

where $w, v : G \to \mathbb{R}^1$.

Lemma 3.1. *If conditions* **(G)**, **(Q1)**, **(Γ1)**, **(A1)**, **(B1)**, **(D1)** *are satisfied, then, for every* $u \in \Pi_0$ *and* $i \in S_2$*, we get*

$$\begin{aligned}\mathcal{L}(u,u) &\\ \geq \int\limits_G &\Big[a_0 \sum_{j \neq I} |u|^{\gamma_j-2}|u_{y_j}|^2 + (a_0 - \varepsilon N C_3)|u|^{\gamma_i-2}|u_{y_i}|^2 + g_0|u|^{q(y)}\Big]\,dy \\ &+ \sum_{j=n+1}^{N} c^j \int\limits_{G_j} |u|^{\gamma_j}|_{y_j=\ell_j}\,dy_j' - C14(\varepsilon), \quad (3.3)\end{aligned}$$

where $C_{14} > 0$*,*

$$c^j = \begin{cases} \frac{b_0}{\gamma_j}, & \text{if } b_j \text{ satisfies } b) \text{ or } d) \text{ of } \mathbf{(B1)}, \\ 0, & \text{for the other case,} \end{cases}$$

for all $j \in \{n+1, \ldots, N\}$*.*

Proof. Suppose $u \in \Pi_0$, and $i \in S_2 \neq \varnothing$. It is easy to show that

$$\mathcal{L}(u,u) \geq \int\limits_G a_0 \sum_{j=1}^{N} |u|^{\gamma_j-2}|u_{y_j}|^2\,dy + \sum_{j=1}^{N} I_j + g_0 \int\limits_G |u|^{q(y)}\,dy, \qquad (3.4)$$

where $I_j = \int_G b_j(y)|u|^{\gamma_j-2}u_{y_j}u\,dy$, $j = \overline{1,N}$. Using estimate (2.6) with $\alpha_0 = \gamma_{\max} - 2$ and $\alpha_1 = 2$, we get

$$\int\limits_G |u|^{\gamma_{\max}}\,dy = \int\limits_G |u|^{\gamma_{\max}-2+2}\,dy \leq C_3 \int\limits_G |u|^{\gamma_{\max}-2}|u_{y_i}|^2\,dy. \qquad (3.5)$$

Take a point $j \in \{1, \ldots, N\}$. Then $I_j = \int_G \frac{b_j}{\gamma_j}\frac{\partial}{\partial y_j}(|u|^{\gamma_j})\,dy = I_j^* - I_j^{**}$, where

$$I_j^* = \int\limits_{\partial G} \frac{b_j}{\gamma_j}|u|^{\gamma_j}\cos(\nu, y_j)\,dS, \qquad I_j^{**} = \frac{1}{\gamma_j}\int\limits_G (b_j)_{y_j}|u|^{\gamma_j}\,dy.$$

a) If $j \in S_1$ and if $j \leq n$, then, using condition (1.2), we have

$$I_j^* = \int\limits_{\partial\Omega\times[0,\ell_{n+1}]\times\cdots\times[0,\ell_N]} \frac{b_j}{\gamma_j}|u|^{\gamma_j}\cos(\nu, y_j)\,dS = 0.$$

By estimate (3.5) and Young's inequality with the constant $\frac{\gamma_{\max}}{\gamma_j} > 1$, we get

$$I_j^{**} \le \frac{b^1}{\gamma_j} \int\limits_G |u|^{\gamma_j}\, dy \le \varepsilon \int\limits_G |u|^{\gamma_{\max}}\, dy + C_{15}(\varepsilon)$$

$$\le \varepsilon C_3 \int\limits_G |u|^{\gamma_{\max}-2} |u_{y_I}|^2\, dy + C_{15}(\varepsilon). \quad (3.6)$$

b) If $j \in S_1$ and if $j \ge n+1$, then, taking condition (1.3) into account, we have

$$I_j^* = \int\limits_{G_j} \frac{b_j}{\gamma_j} |u|^{\gamma_j}\, dy_j' \Big|_{y_j=0}^{y_j=\ell_j} = \int\limits_{G_j} \frac{b_j}{\gamma_j} |u|^{\gamma_j}|_{y_j=\ell_j}\, dy_j' \ge \frac{b_0}{\gamma_j} \int\limits_{G_j} |u|^{\gamma_j}|_{y_j=\ell_j}\, dy_j'. \quad (3.7)$$

The integral I_j^{**} is estimated with the help of (3.6).

c) If $j \in S_2$ and if $j \le n$, we have again $I_j^* = 0$ (see (1.2)) and $I_j^{**} = 0$.

d) If $j \in S_2$ and if $j \ge n+1$, then I_j^* satisfies (3.7). In addition, $I_j^{**} = 0$.

Thus, using (3.4), we get (3.3). □

Let us prove the following theorem.

Theorem 3.1. *If conditions* **(G)**, **(Q1)**, **(Γ1)**, and **(A1)**–**(F1)** *are satisfied and if*

$$\frac{\gamma_{\max}}{2} \le \gamma_{\min}, \quad (3.8)$$

then there exists the solution to the boundary-value problem (1.1)–(1.3), *and the vector* α (*see Definition* 3.1) *is equal to* $(\frac{\gamma_1}{2}, \dots, \frac{\gamma_N}{2})$.

Proof. Now we use the Galerkin method. Let $\{w^1, \dots, w^m, \dots\}$ be a basis for the set Π_0. By definition, put $u^m(y) = \sum_{\mu=1}^m z_\mu^m\, w^\mu(y)$, $y \in G$, where the constants $z_1^m, \dots, z_m^m \in \mathbb{R}$ are the solutions to the system of equations

$$\mathcal{L}(u^m, w^\mu) = \int\limits_G f w^\mu\, dy, \qquad \mu = \overline{1,m}. \quad (3.9)$$

Assume that $P = (P_1, \dots, P_m)$, $P_\mu(z) = \mathcal{L}(h^m, w^\mu) - \int_G f w^\mu\, dy$, $z = (z_1, \dots, z_m) \in \mathbb{R}^m$, $\mu = \overline{1,m}$, where $h^m(y) = \sum_{i=1}^m z_i\, w^i(y)$, $y \in G$. Since [see (3.5) for the case $u = h^m \in \Pi_0$]

$$\int\limits_G f h^m \, dy \le \varepsilon \int\limits_G |h^m|^{\gamma_{\max}} \, dy + C_{16}(\varepsilon) \int\limits_G |f|^{\frac{\gamma_{\max}}{\gamma_{\max}-1}} \, dy$$
$$\le \varepsilon C_3 \int\limits_G |h^m|^{\gamma_{\max}-2} |h^m_{y_I}|^2 \, dy + C_{17}(\varepsilon), \quad (3.10)$$

where $i \in S_2$, we see that Lemma 3.1 yields

$$\mathcal{L}(h^m, h^m) - \int\limits_G f h^m \, dy$$
$$\ge \int\limits_G \Big[a_0 \sum_{j \ne I} |h^m|^{\gamma_j - 2} |h^m_{y_j}|^2 + (a_0 - \varepsilon N C_3 - \varepsilon C_3) |h^m|^{\gamma_i - 2} |h^m_{y_i}|^2$$
$$+ g_0 |h^m|^{q(y)} \Big] \, dy + \sum_{j=n+1}^{N} c^j \int\limits_{G_j} |h^m|^{\gamma_j}|_{y_j = \ell_j} \, dy'_j - C_{18}(\varepsilon), \quad (3.11)$$

where $\varepsilon > 0$. Therefore, if $\varepsilon > 0$ is chosen to be sufficiently small, then

$$(P(z), z)_{\mathbb{R}^m}$$
$$= \sum_{\mu=1}^{m} \Big(\mathcal{L}(h^m, w^\mu) - \int\limits_G f w^\mu \, dy \Big) z_\mu = \mathcal{L}(h^m, h^m) - \int\limits_G f h^m \, dy$$
$$\ge \frac{a_0}{2} \int\limits_G \sum_{j=1}^{N} |h^m|^{\gamma_j - 2} |h^m_{y_j}|^2 \, dy - C_{18}(\varepsilon) \underset{|z| \to +\infty}{\longrightarrow} +\infty.$$

Hence, using Statement 2.1, we obtain that there exist the constants $z_1^m, \dots, z_m^m$ such that (3.9) holds.

Multiplying both sides of (3.9) by z_μ^m and summing these equalities over μ, we get $\mathcal{L}(u^m, u^m) = \int_G f u^m \, dy$. Using (3.11) with u^m instead of h^m, we obtain

$$\int\limits_G \Big[a_0 \sum_{j \ne I} |u^m|^{\gamma_j - 2} |u^m_{y_j}|^2$$
$$+ (a_0 - \varepsilon N C_4 - \varepsilon C_3) |u^m|^{\gamma_i - 2} |u^m_{y_i}|^2 + g_0 |u^m|^{q(y)} \Big] \, dy$$
$$+ \sum_{j=n+1}^{N} c^j \int\limits_{G_j} |u^m|^{\gamma_j}|_{y_j = \ell_j} \, dy'_j \le C_{18}(\varepsilon). \quad (3.12)$$

If $\varepsilon > 0$ it sufficiently small, then (3.12) yields estimate (2.11), where $u = u^m$, $\beta_j = \gamma_j - 2$, $j = \overline{1, N}$. Therefore (see Theorem 2.1),

$$\| \, |u^m|^{\beta-2} u^m; W^{1, \frac{\gamma_{\min}}{\beta-1}}(G) \| \le C_{19}, \tag{3.13}$$

where $C_{19} > 0$ is independent of m. From the inequalities $\gamma_1, \dots, \gamma_N \ge 2 > 1$, it follows that conditions (2.8) are satisfied. Condition (2.9) follows from (3.8). The constant $\beta \ge 2$ satisfies the condition

$$\frac{\gamma_{\max}}{2} + 1 \le \beta \le \gamma_{\min} + 1. \tag{3.14}$$

Inequality (3.8) implies that $\frac{\gamma_{\min}}{\beta-1} \ge 1$. For the case $\frac{\gamma_{\max}}{2} = \gamma_{\min}$, we can choose a constant β such that $\frac{\gamma_{\min}}{\beta-1} > 1$.

Since $| \, |u^m|^{\frac{\gamma_j}{2}-1} u^m|^2 = |u^m|^{\gamma_j}$ and $|u^m|^{\frac{\gamma_j}{2}-1} u^m_{y_j} = \frac{2}{\gamma_j} (|u^m|^{\frac{\gamma_j}{2}-1} u^m)_{y_j}$, we have

$$|u^m|^{\gamma_j - 2} |u^m_{y_j}|^2 = [|u^m|^{\frac{\gamma_j}{2}-1} u^m_{y_j}]^2 = \frac{4}{\gamma_j^2} [(|u^m|^{\frac{\gamma_j}{2}-1} u^m)_{y_j}]^2.$$

Then it follows from estimates (3.5) and (3.12) that

$$\int_G \Big[| \, |u^m|^{\frac{\gamma_j}{2}-1} u^m|^2 + | \, (|u^m|^{\frac{\gamma_j}{2}-1} u^m)_{y_j}|^2 \Big] \, dy \le C_{20}. \tag{3.15}$$

Therefore, there exists a subsequence $\{u^{m_k}\}_{k\in\mathbb{N}} \subset \{u^m\}_{m\in\mathbb{N}}$ such that

$$|u^{m_k}|^{\frac{\gamma_j}{2}-1} u^{m_k} \underset{k\to\infty}{\longrightarrow} \chi_0^j, \quad (|u^{m_k}|^{\frac{\gamma_j}{2}-1} u^{m_k})_{y_j} \underset{k\to\infty}{\longrightarrow} \chi^j \quad \text{slowly in } L^2(G), \tag{3.16}$$

where $j \in \{1, \dots, N\}$.

Using estimate (3.13), Rellich–Kondrashov theorem, Lemma 1.28, and Lemma 1.18 [17, p. 47, 39], we obtain that if β satisfies (3.14), then there exists a subsequence (we call it $\{u^{m_k}\}_{k\in\mathbb{N}}$ again) such that

$$|u^{m_k}|^{\beta-2} u^{m_k} \underset{k\to\infty}{\longrightarrow} |u|^{\beta-2} u \ \text{ strongly in } L^{\frac{\gamma_{\min}}{\beta-1}}(G) \text{ and a.e. in } G.$$

Therefore, for every $j \in \{1, \dots, N\}$, we obtain that $\chi_0^j = |u|^{\frac{\gamma_j}{2}-1} u$. Thus, using the distributional convergence in $D^*(G)$, we have that $\chi^j = (\chi_0^j)_{y_j}$, $j = \overline{1, N}$.

Now we make passage to the limit with $m = m_k$ in (3.9). Take points $j \in \{1, \dots, N\}$, $k \in \mathbb{N}$, $\mu = \overline{1, m_k}$. First, consider the expression

$$J_k^2 = \int_G a_j |u^{m_k}|^{\gamma_j - 2} u_{y_j}^{m_k} w_{y_j}^{\mu} \, dy.$$

If $\gamma_j = 2$, then (see (3.16)) $u_{y_j}^{m_k} \underset{k\to\infty}{\longrightarrow} u_{y_j}$ slowly in $L^2(G)$. Hence,

$$J_k^2 = \int_G a_j u_{y_j}^{m_k} w_{y_j}^{\mu} \, dy \underset{k\to\infty}{\longrightarrow} \int_G a_j u_{y_j} w_{y_j}^{\mu} \, dy.$$

If $\gamma_j > 2$, then $|u^{m_k}|^{\gamma_j - 2} u_{y_j}^{m_k} = Z(y, u^{m_k}) |u^{m_k}|^{\frac{\gamma_j}{2} - 1} u_{y_j}^{m_k}$, where $Z(y, u^{m_k}) = |u^{m_k}|^{\frac{\gamma_j}{2} - 1}$. For this case, we have that the function Z is continuous. Using (3.15), we get

$$\int_G |Z(y, u^{m_k})|^s \, dy = \int_G |u^{m_k}|^{\gamma_j} \, dy \le C_{20},$$

where $s = \frac{2\gamma_j}{\gamma_j - 2} = 2 + \frac{4}{\gamma_j - 2} > 2$. Therefore (see Statement 2.2),

$$J_k^2 = \int_G Z(y, u^{m_k}) |u^{m_k}|^{\frac{\gamma_j}{2} - 1} u_{y_j}^{m_k} a_j w_{y_j}^{\mu} \, dy$$
$$\underset{k\to\infty}{\longrightarrow} \int_G |u|^{\frac{\gamma_j}{2} - 1} \frac{2}{\gamma_j} (|u|^{\frac{\gamma_j}{2} - 1} u)_{y_j} a_j w_{y_j}^{\mu} \, dy.$$

Consider the junior terms. If $\gamma_j \ge 2$, then, like for the integral J_k^2, we get

$$J_k^1 \equiv \int_G b_j |u^{m_k}|^{\gamma_j - 2} u_{y_j}^{m_k} w^{\mu} \, dy \underset{k\to\infty}{\longrightarrow} \int_G b_j |u|^{\frac{\gamma_j}{2} - 1} \frac{2}{\gamma_j} (|u|^{\frac{\gamma_j}{2} - 1} u)_{y_j} w^{\mu} \, dy.$$

Using (3.12) and the almost everywhere convergence in the domain G, we see that $|u^{m_k}|^{q(y) - 2} u^{m_k} \underset{k\to\infty}{\longrightarrow} |u|^{q(y) - 2} u$ slowly in $L^{\frac{q(y)}{q(y) - 1}}(G)$. Therefore,

$$J_k^0 = \int_G g |u^{m_k}|^{q(y) - 2} u^{m_k} w^{\mu} \, dy \underset{k\to\infty}{\longrightarrow} \int_G g |u|^{q(y) - 2} u w^{\mu} \, dy.$$

Thus, equality (3.9) tends to (3.2) with the replacement of v by w^μ. It is easy to show that (3.2) holds. This completes the proof of Theorem 3.1. □

Remark 3.2. Taking Sobolev's imbedding theorems into account (see, for example [17, p. 47]), we obtain

$$W^{1,\frac{\gamma_{\min}}{\beta-1}}(G) \circlearrowleft L^{\frac{N\frac{\gamma_{\min}}{\beta-1}}{N-\frac{\gamma_{\min}}{\beta-1}}}(G) = L^{\frac{N\gamma_{\min}}{N(\beta-1)-\gamma_{\min}}}(G),$$

if $\frac{\gamma_{\min}}{\beta-1} < N$. Therefore, estimate (3.13) implies that

$$\int\limits_G |u^m|^{\frac{N\gamma_{\min}(\beta-1)}{N(\beta-1)-\gamma_{\min}}}\,dy = \int\limits_G \||u^m|^{\beta-2}u^m|^{\frac{N\gamma_{\min}}{N(\beta-1)-\gamma_{\min}}}\,dy \le C_{21}.$$

If β is chosen so that $\beta = \gamma_{\min}+1$, then the sequence $\{u^m\}_{m\in\mathbb{N}}$ is bounded in the space $L^{\frac{N\gamma_{\min}}{N-1}}(G)$. Thus, the generalized solution to problems (1.1)–(1.3) belongs to $L^{\frac{N\gamma_{\min}}{N-1}}(G)$.

4. The initial boundary-value problem for the nonlinear degenerate parabolic equation

Now we use the previous results and prove the existence of the solution to problem (1.4)–(1.6). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary $\partial\Omega$, $Q_{t_1,t_2} = \Omega \times (t_1, t_2)$, $0 \le t_1 < t_2 \le T$. Suppose conditions **(Q2)** and **(Γ2)** are satisfied. We will need the following notation:

$\gamma_{\max} = \{\gamma_1, \dots, \gamma_n\}$, $\gamma_{\min} = \{\gamma_1, \dots, \gamma_n\}$,

$r_{\max} = \max\{r, \gamma_{\max}\}$, $r_{\min} = \min\{r, \gamma_{\min}\}$,

$q_{\max} = \max\{q_2, \gamma_{\max}\}$, $q_{\min} = \min\{q_1, \gamma_{\min}\}$,

$S_1 = \{1, \dots, n\} \setminus S_2$, where S_2 is a set of numbers $i \in \{1, \dots, n\}$ such that $\gamma_i = r_{\max}$.

Note that if $s > 1$, then $s' = \frac{s}{s-1}$, that is, $\frac{1}{s} + \frac{1}{s'} = 1$. In the same way, we define the function $q' : \Omega \to \mathbb{R}$ and the numbers $r', \gamma_1', \dots, \gamma_n', q_1', q_2', \gamma_{\max}'$, $\gamma_{\min}', r_{\max}', r_{\min}', q_{\max}', q_{\min}' > 1$. By definition, put

$$[v]_M = \sum_{i=1}^n \left(\int\limits_\Omega |v|^{\gamma_i-2}|v_{x_i}|^2\,dx \right)^{\frac{1}{\gamma_i}}.$$

By $W_M^\gamma(\Omega)$, we denote the set of functions $v : \Omega \to \mathbb{R}^1$ such that $[v]_M < +\infty$ and $|v|^{\beta-2}v|_{\partial\Omega} = 0$, where β satisfies (3.14). Suppose $V = W_M^\gamma(\Omega) \cap L^{q(x)}(\Omega)$,

$$U(Q_{0,T}) = \Bigg\{ u : (0,T) \to V \;\Bigg|\; \int\limits_{Q_{0,T}} \Bigg[\sum_{i=1}^n \big[|u|^{\frac{\gamma_i}{2}-1} (|u|^{\frac{\gamma_i}{2}-1}u)_{x_i} \big]^{\gamma_i'} + |u|^{q(x)} \Bigg] dx\, dt < +\infty \Bigg\},$$

$W_0^{1,\overline{\gamma}}(\Omega) = \{u \in W_0^{1,1}(\Omega) \mid u_{x_i} \in L^{\gamma_i}(\Omega),\ i = \overline{1,n}\}$, $Z = W_0^{1,\overline{\gamma}}(\Omega) \cap L^{q(x)}(\Omega)$, and $\mathcal{Z}(Q_{0,T}) = \{v : (0,T) \to Z \mid \|v; \mathcal{Z}(Q_{0,T})\| < +\infty\}$, where

$$\|v; \mathcal{Z}(Q_{0,T})\| = \sum_{i=1}^n \|v_{x_i}; L^{\gamma_i}(Q_{0,T})\| + \|v; L^{q(x)}(Q_{0,T})\|, \quad v \in \mathcal{Z}(Q_{0,T}).$$

Definition 4.1. *The generalized solution to problem* (1.4)–(1.6) *is called the function u if the following conditions hold: $u \in U(Q_{0,T}) \cap L^1(0,T; W_0^{1,1}(\Omega))$; $|u|^{r-2}u \in C([0,T]; W^{-1,q'_{\max}}(\Omega))$; u satisfies the initial condition* (1.6)*;*

$$\begin{aligned} \int\limits_{Q_{0,T}} \Bigg[|u|^{\frac{r}{2}-1} \frac{2}{r} (|u|^{\frac{r}{2}-1}u)_t v + \sum_{i=1}^n a_i |u|^{\frac{\gamma_i}{2}-1} \frac{2}{\gamma_i} (|u|^{\frac{\gamma_i}{2}-1}u)_{x_i} v_{x_i} \\ + \sum_{i=1}^n b_i |u|^{\frac{\gamma_i}{2}-1} \frac{2}{\gamma_i} (|u|^{\frac{\gamma_i}{2}-1}u)_{x_i} v + g|u|^{q(x)-2}uv \Bigg] dx\, dt \\ = \int\limits_{Q_{0,T}} fv\, dx\, dt \quad (4.1) \end{aligned}$$

for all functions $v \in \Pi_1$, where $\Pi_1 = \{v \in C^1(\overline{Q_{0,T}}) \mid v|_{\partial\Omega\times[0,T]} = 0, v|_{t=0} = 0\}$.

We assume that the following assumptions hold:

(A2): $a_i \in L^\infty(Q_{0,T})$, $a_i(x,t) \geq a_0 > 0$ for a.e. $(x,t) \in Q_{0,T}$, where $i = \overline{1,n}$;

(B2): for every $i \in \{1,\dots,n\}$, the function $b_i \in L^\infty(Q_{0,T})$ satisfies one of the following conditions:

a) if $i \in S_1$, then $(b_i)_{x_i} \in L^\infty(Q_{0,T})$;

b) if $i \in S_2$, then $b_i(x,t) \equiv \widetilde{b}_i \in \mathbb{R}^1$;

(D2): $g \in L^{\infty}(Q_{0,T})$, $0 < g_0 \le g(x,t) \le g^0 < +\infty$ for a.e. $(x,t) \in Q_{0,T}$;

(F2): $f \in L^{\frac{\gamma_{\max}}{\gamma_{\max}-1}}(Q_{0,T})$.

Note that $Z^* = W^{-1,\overline{\gamma}'}(\Omega) + L^{q'(x)}(\Omega)$, where $W^{-1,\overline{\gamma}'}(\Omega) = [W_0^{1,\overline{\gamma}}(\Omega)]^*$,

$$W_0^{1,q_{\max}}(\Omega) \overline{\circlearrowleft} Z \overline{\circlearrowleft} W_0^{1,q_{\min}}(\Omega), \quad W^{-1,q'_{\min}}(\Omega) \overline{\circlearrowleft} Z^* \overline{\circlearrowleft} W^{-1,q'_{\max}}(\Omega).$$

By Theorem 1 [10, p. 311], we have

$$L^{q_2}(0,T;L^{q(x)}(\Omega)) \overline{\circlearrowleft} L^{q(x)}(Q_{0,T}) \overline{\circlearrowleft} L^{q_1}(0,T;L^{q(x)}(\Omega)).$$

Therefore,

$$L^{q_{\max}}(0,T;W_0^{1,q_{\max}}(\Omega)) \overline{\circlearrowleft} \mathcal{Z}(Q_{0,T}) \overline{\circlearrowleft} L^{q_{\min}}(0,T;W_0^{1,q_{\min}}(\Omega))$$

and

$$L^{q'_{\min}}(0,T;W^{-1,q\min'}(\Omega)) \overline{\circlearrowleft} [\mathcal{Z}(Q_{0,T})]^* \overline{\circlearrowleft} L^{q'_{\max}}(0,T;W^{-1,q'_{\max}}(\Omega)).$$

Remark 4.1. If $\widetilde{W} = \{z \mid z, z_t \in L^{q'_{\max}}(0,T;W^{-1,q'_{\max}}(\Omega))\}$, then it is easy to show that $\widetilde{W} \circlearrowleft C([0,T];W^{-1,q'_{\max}}(\Omega))$, $\overline{C^{\infty}([0,T];W^{-1,q'_{\max}}(\Omega))} = \widetilde{W}$, and, for every $w \in \widetilde{W}$, $\varphi \in C^1([0,T])$, we have

$$\int_0^T w_t(t)\varphi(t)\,dt = w(T)\varphi(T) - w(0)\varphi(0) - \int_0^T w(t)\varphi'(t)\,dt. \tag{4.2}$$

Further let us prove the following theorem.

Theorem 4.1. *Suppose that conditions* **(Q2)**, **(Γ2)**, and **(A2)–(F2)** *are satisfied. If*

$$1)\ \frac{r_{\max}}{2} \le r_{\min}, \quad 2)\ \frac{\gamma_{\min}}{r-1} > 1, \quad 3)\ q_{\max} \ge \frac{\gamma_{\min}}{\gamma_{\min}-(r-1)}, \tag{4.3}$$

$$\frac{\gamma_{\max}}{r-1} \le 2, \tag{4.4}$$

then the initial boundary-value problem (1.4)–(1.6) *has a generalized solution u such that*

$$\int_{Q_{0,T}} \Big[\sum_{i=1}^n \big[(|u|^{\frac{\gamma_i}{2}-1}u)_{x_i} \big]^2 + \sum_{i=1}^n \big[|u|^{\frac{\gamma_i}{2}-1}u \big]^2 + |u|^{q(x)} \Big]\,dx\,dt < +\infty, \tag{4.5}$$

$$\||u|^{r-2}u; L^{\frac{\gamma_{\min}}{r-1}}(0,T; W_0^{1,\frac{\gamma_{\min}}{r-1}}(\Omega))\| \\ + \|(|u|^{r-2}u)_t; L^{q'_{\max}}(0,T; W^{-1,q'_{\max}}(\Omega))\| < +\infty. \quad (4.6)$$

Proof. Take $\varepsilon > 0$. Consider the boundary-value problem

$$-\varepsilon(|u^\varepsilon|^{r-2}u_t^\varepsilon)_t + |u^\varepsilon|^{r-2}u_t^\varepsilon - \sum_{i=1}^n (a_i|u^\varepsilon|^{\gamma_i-2}u_{x_i}^\varepsilon)_{x_i} \\ + \sum_{i=1}^n b_i|u^\varepsilon|^{\gamma_i-2}u_{x_i}^\varepsilon + g|u^\varepsilon|^{q(x)-2}u^\varepsilon = f, \quad (4.7)$$

$$u^\varepsilon|_{\partial\Omega\times[0,T]} = 0, \quad (4.8)$$

$$u^\varepsilon|_{t=0} = 0, \quad u_t^\varepsilon|_{t=T} = 0. \quad (4.9)$$

By (4.3) 1) and Theorem 3.1, we obtain the existence of the solution u^ε to problem (4.7)–(4.9). The function u^ε satisfies all estimates of Theorem 3.1 and

$$\int\limits_{Q_{0,T}} \Big[\varepsilon|u^\varepsilon|^{\frac{r}{2}-1}\frac{2}{r}(|u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon)_t v_t + |u^\varepsilon|^{\frac{r}{2}-1}\frac{2}{r}(|u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon)_t v \\ + \sum_{i=1}^n a_i|u^\varepsilon|^{\frac{\gamma_i}{2}-1}\frac{2}{\gamma_i}(|u^\varepsilon|^{\frac{\gamma_i}{2}-1}u^\varepsilon)_{x_i}v_{x_i} + \sum_{i=1}^n b_i|u^\varepsilon|^{\frac{\gamma_i}{2}-1}\frac{2}{\gamma_i}(|u^\varepsilon|^{\frac{\gamma_i}{2}-1}u^\varepsilon)_{x_i}v \\ + g|u^\varepsilon|^{q(x)-2}u^\varepsilon v\Big]\,dxdt = \int\limits_{Q_{0,T}} fv\,dxdt, \quad (4.10)$$

where $v \in \Pi_1$. Using estimates (3.12) and (3.15), we get

$$\int\limits_{Q_{0,T}} \Big[\sum_{i=1}^n \big[\,(|u^\varepsilon|^{\frac{\gamma_i}{2}-1}u^\varepsilon)_{x_i}\big]^2 + \sum_{i=1}^n |u^\varepsilon|^{\gamma_i} + |u^\varepsilon|^{q(x)}\Big]\,dx\,dt \le C_{22} \quad (4.11)$$

Now we note that condition **(F2)** implies that (see (3.10))

$$\int\limits_{Q_{0,T}} fu^\varepsilon\,dx\,dt \le \delta \int\limits_{Q_{0,T}} |u^\varepsilon|^{\gamma_{i_0}}\,dx\,dt + C_{23}(\delta)\int\limits_{Q_{0,T}} |f|^{\frac{\gamma_{\max}}{\gamma_{\max}-1}}\,dx\,dt, \quad (4.12)$$

where $\delta > 0$, $i_0 \in \{1,\dots,n\}$, and $\gamma_{\max} = \gamma_{i_0}$. First, we evaluate the term with b_i (see, for comparison, Lemma 3.1 and Theorem 3.1) with the help of the integral $\delta \int_{Q_{0,T}} |u^\varepsilon|^{\gamma_{i0}}\,dx\,dt$. Further we use estimate (3.5). Finally, we can choose a constant $\delta > 0$ such that $a_0 - \delta > 0$, and we get estimate (4.11), where $C_{22} > 0$ is independent of $\varepsilon > 0$. Note that we cannot replace $\gamma_{\max}$ by $r_{\max}$ in estimate (4.12). If $r = r_{\max} > \gamma_{\max}$, then the constant $\delta > 0$ must satisfies the condition $\varepsilon - \delta > 0$. For this case, the constant $C_{22} > 0$ depends on $\varepsilon > 0$. This condition is the insurmountable obstacle in our next consideration.

It is easy to prove that

$$\varepsilon \int\limits_{Q_{0,T}} |(|u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon)_t|^2\,dx\,dt \le C_{24},$$

where $C_{24} > 0$ is independent of $\varepsilon > 0$. Using this inequality and condition (4.9), we get

$$\varepsilon \int\limits_{Q_{0,T}} |u^\varepsilon|^r\,dx\,dt = \varepsilon \int\limits_{Q_{0,T}} ||u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon|^2\,dx\,dt$$
$$\le \varepsilon C_{25} \int\limits_{Q_{0,T}} |(|u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon)_t|^2\,dx\,dt \le C_{26},$$

where $C_{26} > 0$ is independent of $\varepsilon > 0$ if, for instance, $\varepsilon \le 1$.

Therefore, there exists a sequence $\{u^{\varepsilon_m}\}_{m\in\mathbb{N}} \subset \{u^\varepsilon\}_{\varepsilon>0}$ such that

$$(|u^{\varepsilon_m}|^{\frac{\gamma_i}{2}-1}u^{\varepsilon_m})_{x_i} \underset{\varepsilon_m\to 0}{\longrightarrow} \chi^i_{\frac{\gamma_i}{2}}, \quad |u^{\varepsilon_m}|^{\frac{\gamma_i}{2}-1}u^{\varepsilon_m} \underset{\varepsilon_m\to 0}{\longrightarrow} \chi_{\frac{\gamma_i}{2}}$$
$$\text{slowly in } L^2(Q_{0,T}),\ i = \overline{1,n},$$

$$|u^{\varepsilon_m}|^{q(x)-2}u^{\varepsilon_m} \underset{\varepsilon_m\to 0}{\longrightarrow} \chi_q \quad \text{slowly in } L^{q'(x)}(Q_{0,T}),$$

$$\sqrt{\varepsilon_m}(|u^{\varepsilon_m}|^{\frac{r}{2}-1}u^{\varepsilon_m})_t \underset{\varepsilon_m\to 0}{\longrightarrow} \chi^0_{\frac{r}{2}} \quad \text{slowly in } L^2(Q_{0,T}).$$

Let us prove the additional estimates of the functions $\{u^\varepsilon\}_{\varepsilon>0}$. By definition, put

$$\langle F(t)v, z\rangle_Z = \int\limits_{\Omega_t} \Big[fz - \sum_{i=1}^n a_i |v^\varepsilon|^{\frac{\gamma_i}{2}-1} \frac{2}{\gamma_i} (|v|^{\frac{\gamma_i}{2}-1}v)_{x_i} z_{x_i}$$
$$- \sum_{i=1}^n b_i |v|^{\frac{\gamma_i}{2}-1} \frac{2}{\gamma_i} (|v|^{\frac{\gamma_i}{2}-1}v)_{x_i} z - g|v|^{q(x)-2}vz \Big]\,dx, \quad t \in (0,T),$$

$\langle \mathcal{F}u, w\rangle_{\mathcal{Z}(Q_{0,T})} = \int_0^T \langle F(t)u(t), w(t)\rangle_Z \, dt$. If $i \in \{1, \dots, n\}$, then

$$\frac{1}{\frac{2\gamma_i}{(\gamma_i-2)\gamma_i'}} + \frac{1}{\frac{2}{\gamma_i'}} = \frac{(\gamma_i - 2)\gamma_i'}{2\gamma_i} + \frac{\gamma_i'}{2} = \frac{\gamma_i'}{2}\Big(\frac{\gamma_i - 2}{\gamma_i} + 1\Big)$$
$$= \frac{\gamma_i'}{2}\Big(\frac{2\gamma i - 2}{\gamma_i}\Big) = \gamma_i' \frac{\gamma_i - 1}{\gamma_i} = 1.$$

Hence, using Young's inequality with the constants $\frac{2\gamma_i}{(\gamma_i-2)\gamma_i'} > 1$ and $\frac{2}{\gamma_i'} > 1$, we obtain

$$\big| |u^\varepsilon|^{\frac{\gamma_i}{2}-1}(|u^\varepsilon|^{\frac{\gamma_i}{2}-1}u^\varepsilon)_{x_i}\big|^{\gamma_i'}$$
$$= |u^\varepsilon|^{\frac{(\gamma_i-2)\gamma_i'}{2}} |(|u^\varepsilon|^{\frac{\gamma_i}{2}-1}u^\varepsilon)_{x_i}|^{\gamma_i'}$$
$$\le C_{27}(|u^\varepsilon|^{\gamma_i} + |(|u^\varepsilon|^{\frac{\gamma_i}{2}-1}u^\varepsilon)_{x_i}|^2).$$

By these estimate and (4.11), we get that $F(t)u^\varepsilon(t) \in Z^*$. Note that if $z \in \mathcal{Z}(Q_{0,T})$, then

$$\langle \mathcal{F}u^\varepsilon, z\rangle_{\mathcal{Z}(Q_{0,T})} \le C_{28}\Bigg\{ \sum_{i=1}^n \Bigg(\int\limits_{Q_{0,T}} |z_{x_i}|^{\gamma_i} \, dx\, dt\Bigg)^{1/\gamma_i}$$
$$+ \sum_{i=1}^n \Bigg(\int\limits_{Q_{0,T}} |z|^{\gamma_i} \, dx\, dt\Bigg)^{1/\gamma_i} + \|z; L^{q(x)}(Q_{0,T})\| \Bigg\},$$

where $C_{28} > 0$ is independent of ε. Hence, $\|\mathcal{F}u^\varepsilon; [\mathcal{Z}(Q_{0,T})]^*\| \le C_{29}$ and

$$\|\mathcal{F}u^\varepsilon; L^{q'_{\max}}(0, T; W^{-1,q_{\max}}(\Omega))\| \le C_{30}, \tag{4.13}$$

where $C_{30} > 0$ is independent of ε.

Substituting $\varphi(t)z(x)$ for $v(x, t)$ in (4.10), we get

$$\int\limits_{Q_{0,T}} \Big[\varepsilon |u^\varepsilon|^{\frac{r}{2}-1}\frac{2}{r}(|u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon)_t\varphi' z + |u^\varepsilon|^{\frac{r}{2}-1}\frac{2}{r}(|u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon)_t \varphi z\Big] \, dx\, dt$$
$$= \int\limits_0^T \langle F(t)u^\varepsilon(t), z\rangle_Z \, \varphi(t) \, dt,$$

where $\varphi \in C^\infty([0, T])$, $\varphi(0) = 0$, $z \in Z$. By definition, put $\widehat{u}^\varepsilon = |u^\varepsilon|^{\frac{r}{2}-1}\frac{2}{r}(|u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon)_t$. Then,

$$\varepsilon \int\limits_0^T \langle \widehat{u}^\varepsilon(t), z\rangle_Z \, \varphi'(t) \, dt + \int\limits_0^T \langle \widehat{u}^\varepsilon(t), z\rangle_Z \, \varphi(t) \, dt = \int\limits_0^T \langle F(t)u^\varepsilon(t), z\rangle_Z \, \varphi(t) \, dt,$$

$$\Big\langle \varepsilon \int\limits_0^T \widehat{u}^\varepsilon(t)\,\varphi'(t)\,dt, z\Big\rangle_Z + \Big\langle \int\limits_0^T \widehat{u}^\varepsilon(t)\,\varphi(t)\,dt, z\Big\rangle_Z = \Big\langle \int\limits_0^T F(t)u^\varepsilon(t)\,\varphi(t)\,dt, z\Big\rangle_Z,$$

where $z \in Z$. Therefore, we get the equality in the space Z^*:

$$\varepsilon \int\limits_0^T \widehat{u}^\varepsilon(t)\varphi'(t)\,dt + \int\limits_0^T \widehat{u}^\varepsilon(t)\varphi(t)\,dt = \int\limits_0^T F(t)u^\varepsilon(t)\varphi(t)\,dt. \tag{4.14}$$

Taking $\varphi = \psi$, where $\psi \in C_0^\infty((0,T))$, we get the equation in the space Z^*,

$$-\varepsilon\widehat{u}_t^\varepsilon(t) + \widehat{u}^\varepsilon(t) = F(t)u^\varepsilon(t), \qquad t \in (0,T), \tag{4.15}$$

where $\widehat{u}_t^\varepsilon$ is a distributional derivative of the function $\widehat{u}^\varepsilon \in D^*(0,T;Z^*)$. Since $\widehat{u}^\varepsilon \in \widetilde{W}$ (see Remark 4.1), we see that $\widehat{u}^\varepsilon \in C([0,T];W^{-1,q'_{\max}}(\Omega))$. Hence, using (4.2) and (4.14), we obtain

$$\varepsilon\widehat{u}^\varepsilon(T)\varphi(T) - \varepsilon \int\limits_0^T \widehat{u}_t^\varepsilon(t)\varphi(t)\,dt + \int\limits_0^T \widehat{u}^\varepsilon(t)\varphi(t)\,dt = \int\limits_0^T F(t)u^\varepsilon(t)\varphi(t)\,dt.$$

If we combine this with (4.15), we get $\varepsilon\widehat{u}^\varepsilon(T)\varphi(T) = 0$. Taking $\varphi(T) = \frac{1}{\varepsilon}$ gives

$$\widehat{u}^\varepsilon(T) = 0. \tag{4.16}$$

It is easy to show that the function

$$\widehat{u}^\varepsilon(t) = \frac{1}{\varepsilon} \int\limits_0^{T-t} e^{-\frac{T-t-\eta}{\varepsilon}} F(T-\eta)u^\varepsilon(T-\eta)\,d\eta, \qquad t \in (0,T), \tag{4.17}$$

is a solution to problem (4.15), (4.16). Since the function $\varphi(s) = |s|^{q'_{\max}}$, $s \in \mathbb{R}$, is a convex function if $q'_{\max} > 1$, we obtain (see [21, p. 59] with $V = H = \mathbb{R}$)

$$\left|\frac{1}{\varepsilon}\int\limits_0^\tau e^{-\frac{\tau-s}{\varepsilon}}\xi(s)\,ds\right|^{q'_{\max}} \le \frac{1}{\varepsilon}\int\limits_0^\tau e^{-\frac{\tau-s}{\varepsilon}}|\xi(s)|^{q'_{\max}}\,ds, \qquad \tau \in (0,T), \tag{4.18}$$

where $\xi : (0,T) \to \mathbb{R}$.

By definition, put $\|\cdot\| = \|\cdot; W^{-1,q'_{\max}}(\Omega)\|$. Using (4.17) and (4.18), we get

$$\int_0^T \|\widehat{u}^\varepsilon(t)\|^{q'_{\max}}\,dt \le \int_0^T \left(\frac{1}{\varepsilon}\int_0^{T-t} e^{-\frac{T-t-\eta}{\varepsilon}} \|F(T-\eta)u^\varepsilon(T-\eta)\|\,d\eta\right)^{q'_{\max}} dt$$
$$\le \int_0^T dt\,\frac{1}{\varepsilon}\int_0^{T-t} e^{-\frac{T-t-\eta}{\varepsilon}} \|F(T-\eta)u^\varepsilon(T-\eta)\|^{q'_{\max}}\,d\eta$$
$$= \int_0^T \|F(T-\eta)u^\varepsilon(T-\eta)\|^{q'_{\max}} \left(\frac{1}{\varepsilon}\int_0^{T-\eta} e^{-\frac{T-t-\eta}{\varepsilon}}\,dt\right) d\eta.$$

Since

$$\frac{1}{\varepsilon}\int_0^{T-\eta} e^{-\frac{T-t-\eta}{\varepsilon}}\,dt = -\frac{1}{\varepsilon}\int_{T-\eta}^0 e^{-\frac{\tau}{\varepsilon}}\,d\tau = \frac{1}{\varepsilon}\int_0^{T-\eta} e^{-\frac{\tau}{\varepsilon}}\,d\tau \le \frac{1}{\varepsilon}\int_0^{\infty} e^{-\frac{\tau}{\varepsilon}}\,d\tau = 1,$$

we get (see the previous inequality)

$$\int_0^T \|\widehat{u}^\varepsilon(t)\|^{q'_{\max}}\,dt \le \int_0^T \|F(T-\eta)u^\varepsilon(T-\eta)\|^{q'_{\max}}\,d\eta = \int_0^T \|F(t)u^\varepsilon(t)\|^{q'_{\max}}\,dt.$$

Using (4.13) and the equality

$$\widehat{u}^\varepsilon = |u^\varepsilon|^{\frac{r}{2}-1}\frac{2}{r}(|u^\varepsilon|^{\frac{r}{2}-1}u^\varepsilon)_t = \frac{1}{r-1}(|u^\varepsilon|^{r-2}u^\varepsilon)_t,$$

we have

$$\|(|u^\varepsilon|^{r-2}u^\varepsilon)_t; L^{q'_{\max}}(0,T;W^{-1,q'_{\max}}(\Omega))\| \le C_{31}, \tag{4.19}$$

where $C_{31} > 0$ is independent of ε.

Now we obtain the estimate of the set $\{|u^\varepsilon|^{r-2}u^\varepsilon\}_{\varepsilon>0}$. Take $i \in \{1,\dots,n\}$. By (4.3), we obtain $\frac{\gamma_i}{r-1} > 1$ (see condition (4.3) 2)). Therefore, using (3.7) with $G = Q_{0,T}$ and the equalities

$$\alpha_0 = (r-2)\frac{\gamma_i}{r-1} - 2 + \frac{\gamma_i}{r-1} = \gamma_i - 2,$$
$$\alpha_1 = 2 - \frac{\gamma_i}{r-1}, \qquad \alpha_2 = \frac{\gamma_i}{r-1},$$

we get

$$\int\limits_{Q_{0,T}} \left|\frac{1}{r-1}(|u^\varepsilon|^{r-2}u^\varepsilon)_{x_i}\right|^{\frac{\gamma_i}{r-1}} dx\,dt = \int\limits_{Q_{0,T}} \left|\,|u^\varepsilon|^{r-2}u^\varepsilon_{x_i}\right|^{\frac{\gamma_i}{r-1}} dx\,dt$$

$$= \int\limits_{Q_{0,T}} |u^\varepsilon|^{(r-2)\frac{\gamma_i}{r-1}-2+\frac{\gamma_i}{r-1}+2-\frac{\gamma_i}{r-1}}|u^\varepsilon_{x_i}|^{\frac{\gamma_i}{r-1}} dx\,dt$$

$$= \int\limits_{Q_{0,T}} |u^\varepsilon|^{\alpha_0+\alpha_1}|u^\varepsilon_{x_i}|^{\alpha_2} dx\,dt \le C_6 \int\limits_{Q_{0,T}} |u^\varepsilon|^{\alpha_0}|u^\varepsilon_{x_i}|^{\alpha_1+\alpha_2} dx\,dt$$

$$= C_6 \int\limits_{Q_{0,T}} |u^\varepsilon|^{\gamma_i-2}|u^\varepsilon_{x_i}|^2 dx\,dt.$$

Note that

$$\begin{cases} \alpha_1 \ge 0, \\ \alpha_2 \ge 0, \\ \alpha_1+\alpha_2 \ge 1, \\ \alpha_0+\alpha_1+\alpha_2 > 1, \end{cases} \iff \begin{cases} 2-\frac{\gamma_i}{r-1} \ge 0, \\ \frac{\gamma_i}{r-1} \ge 0, \\ 2-\frac{\gamma_i}{r-1}+\frac{\gamma_i}{r-1} \ge 1, \\ \gamma_i-2+2-\frac{\gamma_i}{r-1}+\frac{\gamma_i}{r-1} > 1, \end{cases}$$

$$\iff \begin{cases} \frac{\gamma_i}{r-1} \le 2, \\ \frac{\gamma_i}{r-1} \ge 0, \\ 2 \ge 1, \\ \gamma_i > 1, \end{cases}$$

and estimate (2.7) holds. Therefore,

$$\int\limits_{Q_{0,T}} |(|u^\varepsilon|^{r-2}u^\varepsilon)_{x_i}|^{\frac{\gamma_i}{r-1}} dx\,dt \le C_{32},$$

where $C_{32} > 0$ is independent of ε. Thus,

$$\|\,|u^\varepsilon|^{r-2}u^\varepsilon; L^{\frac{\gamma_{\min}}{r-1}}(0,T;W_0^{1,\frac{\gamma_{\min}}{r-1}}(\Omega))\| \le C_{33}, \tag{4.20}$$

where $C_{33} > 0$ is independent of ε.

If $s_1 \le s_2$, then the following imbedding is well known: $W_0^{1,s_2}(\Omega) \overset{\to}{\circlearrowleft} W_0^{1,s_1}(\Omega)$. Therefore, $W^{-1,s_1'}(\Omega) \overset{\to}{\circlearrowleft} W^{-1,s_2'}(\Omega)$ (here, $\frac{1}{s_2} \le \frac{1}{s_1}$, that is, $1-\frac{1}{s_2'} \le 1-\frac{1}{s_1'}$, $\frac{1}{s_1'} \le \frac{1}{s_2'}$, $s_2' \le s_1'$). Note that

$$|u^\varepsilon|^{r-2}u^\varepsilon \in L^{\frac{\gamma_{\min}}{r-1}}(0,T;W_0^{1,\frac{\gamma_{\min}}{r-1}}(\Omega)) \circlearrowleft L^{\frac{\gamma_{\min}}{r-1}}(Q_{0,T}).$$

By (4.3), we get

$$q_{\max} \geq \frac{\gamma_{\min}}{\gamma_{\min} - (r-1)}, \qquad \frac{q'_{\max}}{q'_{\max} - 1} \geq \frac{\gamma_{\min}}{\gamma_{\min} - (r-1)},$$

$$q'_{\max} \geq \frac{\gamma_{\min}}{\gamma_{\min} - (r-1)}(q'_{\max} - 1),$$

$$q'_{\max}\Big(\frac{\gamma_{\min}}{\gamma_{\min} - (r-1)} - 1\Big) \leq \frac{\gamma_{\min}}{\gamma_{\min} - (r-1)},$$

$$q'_{\max}(\gamma_{\min} - (\gamma_{\min} - (r-1)) \leq \gamma_{\min},$$

i.e. $q'_{\max} \leq \frac{\gamma_{\min}}{r-1}$. Therefore, $L^{\frac{\gamma_{\min}}{r-1}}(\Omega) \;\overline{\circlearrowleft}\; L^{q'_{\max}}(\Omega) \;\overline{\circlearrowleft}\; W^{-1,q'_{\max}}(\Omega)$.

Finally, we note that

$$W_0^{1,\frac{\gamma_{\min}}{r-1}}(\Omega) \overset{K}{\circlearrowleft} L^{\frac{\gamma_{\min}}{r-1}}(\Omega) \circlearrowleft W^{-1,q'_{\max}}(\Omega), \tag{4.21}$$

and estimates (4.20) and (4.19) hold. By Proposition 2.3 and Lemma 1.18 [17, p. 39], we obtain

$$|u^{\varepsilon_m}|^{r-2}u^{\varepsilon_m} \underset{m\to\infty}{\longrightarrow} |u|^{r-2}u \;\text{ strongly in } L^{\frac{\gamma_{\min}}{r-1}}(Q_{0,T}) \text{ and a.e. in } Q_{0,T}. \tag{4.22}$$

Hence, $\chi_q = |u|^{q(x)-2}u$, $\chi_{\frac{\gamma_i}{2}} = |u|^{\frac{\gamma_i}{2}-1}u$, $\chi^i_{\frac{\gamma_i}{2}} = (|u|^{\frac{\gamma_i}{2}-1}u)_{x_i}$, $i = \overline{1,n}$. Thus, taking $\varepsilon = \varepsilon_m$ in (4.10) and letting $m \to \infty$ give (4.1).

Imbedding (4.21) implies that $W_0^{1,\frac{\gamma_{\min}}{r-1}}(\Omega) \overset{K}{\circlearrowleft} W^{-1,q'_{\max}}(\Omega)$. Indeed, if $\{z^m\}_{m\in\mathbb{N}}$ is a bounded sequence in the space $W_0^{1,\frac{\gamma_{\min}}{r-1}}(\Omega)$, then we can choose a subsequence (we call it $\{z^m\}_{m\in\mathbb{N}}$ again) strongly converging to some function z in the space $L^{\frac{\gamma_{\min}}{r-1}}(\Omega)$. Hence,

$$\|z - z^m; W^{-1,q'_{\max}}(\Omega)\| \leq C_3\|z - z^m; L^{\frac{\gamma_{\min}}{r-1}}(\Omega)\| \underset{m\to\infty}{\longrightarrow} 0.$$

Consequently, using item 3) of Statement 2.4 for $M_1 = W_0^{1,\frac{\gamma_{\min}}{r-1}}(\Omega)$ (note that any normed space is a seminormed set) and $A_1 = W^{-1,q'_{\max}}(\Omega)$, we get $Y \overset{K}{\circlearrowleft} C([0,T]; A_1)$. Thus, $|u|^{r-2}u \in C([0,T]; W^{-1,q'_{\max}}(\Omega))$, and the function u satisfies the initial condition (1.6), i.e., this function is a limit of the sequence of functions which satisfy (1.6). The theorem is proved. □

Note that we can get rid of condition (4.4).

Theorem 4.2. *Suppose conditions* **(Q2)**, **(Γ2)**, **(A2)–(F2)**, *and* (4.3) *are satisfied; then the initial boundary-value problem* (1.4)–(1.6) *has the generalized solution* u *such that* (4.5) *and* (4.6) *hold.*

Proof. Let u^ε be a solution to problem (4.7)–4.9. We repeat the proof of Theorem 4.1 from the beginning to formula (4.19). However, we replace (4.20) by another estimate. Let M_1 be a seminormed set of numbers from Theorem 2.2 for $N = n$, $G = \Omega$, and $\beta_i = \frac{\gamma_i - 2(r-1)}{r-1}$. Using the Hölder inequality with the constant $\frac{\beta_i+2}{\min\limits_i \beta_i+2} \geq 1$, we obtain

$$\int_0^T [|u^\varepsilon(t)|^{r-2}u^\varepsilon(t)]_{M_1}^{\min\limits_i \beta_i+2}\,dt$$

$$\leq C_{35}\int_0^T \sum_{i=1}^n \left(\int_\Omega \big|\,|u^\varepsilon|^{r-2}u^\varepsilon|^{\beta_i}|(|u^\varepsilon|^{r-2}u^\varepsilon)_{x_i}|^2\,dx\right)^{\frac{\min\limits_i \beta_i+2}{\beta_i+2}} dt$$

$$\leq C_{36}\int_{Q_{0,T}} \sum_{i=1}^n \big|\,|u^\varepsilon|^{r-2}u^\varepsilon|^{\beta_i}|(|u^\varepsilon|^{r-2}u^\varepsilon)_{x_i}|^2\,dx\,dt + C_{37}$$

$$= C_{36}\int_{Q_{0,T}} \sum_{i=1}^n |u^\varepsilon|^{(r-1)\beta_i+(r-2)2}|u^\varepsilon_{x_i}|^2\,dx\,dt + C_{37}$$

$$= C_{36}\int_{Q_{0,T}} \sum_{i=1}^n |u^\varepsilon|^{\gamma_i-2}|u^\varepsilon_{x_i}|^2\,dx\,dt + C_{37} \leq C_{38}.$$

Consequently, $\{|u^\varepsilon|^{r-2}u^\varepsilon\}_{\varepsilon>0}$ is a bounded set in Y (see Statement 2.4) with $A_1 = W^{-1,q'_{\max}}(\Omega)$, $p_1 = q'_{\max}$, $p = \min\limits_i \beta_i + 2 = \frac{\gamma_{\min}-2(r-1)}{r-1} + 2 = \frac{\gamma_{\min}}{r-1}$. Take $A_0 = L^{\frac{\gamma_{\min}}{r-1}}(\Omega)$. Corollary 2.1 implies that $M_1 \overset{K}{\circlearrowleft} L^{\min\limits_I \beta_i+2}(\Omega) = L^{\frac{\gamma_{\min}}{r-1}}(\Omega) = A_0$. In addition, $A_0 \circlearrowleft A_1$. Therefore, similarly to Theorem 4.1 (we replace Statement 2.3 by Statement 2.4), we get (4.22). Finally, we completes the proof of our theorem in the same way as that of Theorem 4.1. □

Example. Take $\gamma_1 = \cdots = \gamma_n = 2$ and $r \geq 2$. Condition (4.3) implies that

$$\begin{cases} \frac{r}{2} \leq 2, \\ \frac{2}{r-1} > 1, \end{cases} \Leftrightarrow \begin{cases} r \leq 4, \\ 2 > r-1, \end{cases} \Leftarrow r < 3.$$

Consequently, if $r \in [2,3)$ and if $q_2 \geq \frac{2}{3-r}$, then Theorem 4.1 implies the existence of a generalized solution to the following problem:

$$|u|^{r-2}u_t - \Delta\, u + |u|^{q(x)-2}u = f(x,t), \tag{4.23}$$

$$u|_{\partial\Omega\times[0,T]} = 0, \qquad u|_{t=0} = 0. \tag{4.24}$$

Note that, for this simple case, condition (4.4) is satisfied automatically.

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