RELATIVISTIC NEOCLASSICAL FLUXES IN HOT PLASMAS

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The radial fluxes of particles and energy with relativistic effects taken into account are represented in a form standard for neoclassical theory. All the formulations are based on the relativistic equations of motion and the relativistic drift-kinetic equation. As an illustration of the influence of relativistic effects, the radial neoclassical fluxes of electrons in 1/ν collisional regime are calculated and compared with those in the classical approach. The proposed formulation allows one to implement the relativistic effects in current transport codes.

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1. LINEAR DRIFT KINETIC EQUATION FOR RELATIVISTIC ELECTRONS

For calculation of the neoclassical fluxes in relativistic approach, we start from the linear drift kinetic equation (DKE) for relativistic electrons. Since we are interested in calculation of the radial fluxes in toroidal plasmas, we assume that such plasma parameters as density and temperatures are the functions of only the flux-surface label, ρ, i.e. $n_e(\rho)$ and $T_e(\rho)$.

Consideration of neoclassical fluxes is based on the assumption of smallness of deviation from the thermal equilibrium, which for the relativistic electrons is described by the relativistic Maxwellian [1] frequently called also the Jüttner distribution function [2]:

$$f_{eJM} = \frac{n_e}{\pi^{3/2} u_{th}^3} C_{JM}(\mu_r) e^{-\mu_r(\gamma-1)}, \tag{1}$$

where $\gamma = (1+u^2/c^2)^{1/2}$, $u_{\rm th} = p_{\rm th}/m_{\rm e0}$ is the thermal momentum per unit mass with $p_{\rm th} = (2m_{\rm e0}T_{\rm e})^{1/2}$ and $\mu_{\rm r} = m_{\rm e0}c^2/T_{\rm e}$, where $m_{\rm e0}$ and $T_{\rm e}$ are the rest-mass and the temperature of electrons, respectively. The Maxwellian is normalized by density, $\int d^3u\, f_{\rm eJM} = n_{\rm e}$, and

$$C_{JM} = \sqrt{\frac{\pi}{2\mu_r}} \frac{e^{-\mu_r}}{K_2(\mu_r)} \approx 1 - \frac{15}{8\mu_r} + \ldots \quad (\mu_r \gg 1). \tag{2}$$

Here and below, $K_{\rm n}(x)$ is the modified Bessel function of n-th order. Note also an important difference from the non-relativistic Maxwellian: since $\mu_{\rm r}(\gamma-1) = 2x^2/(\gamma+1)$ with $x = u/u_{\rm th}$, the "weight" of the tails with $\gamma > 1$ is increasing with growth of $T_{\rm e}$.

Now, distribution function can be approximated by $f_{\rm e} = f_{\rm eJM} + f_{\rm e1}$, and a linear drift kinetic equation for $f_{\rm e1}$ can be written as:

$$V(f_{e1}) - C_e(f_{e1}) = -\dot{\rho}\frac{\partial f_{eJM}}{\partial \rho} - \dot{u}\frac{\partial f_{eJM}}{\partial u}. \tag{3}$$

Here $\dot{\rho} = \dot{X}\cdot\nabla\rho$ is the radial component of the drift velocity.

Following the neoclassical ordering [3,4], the Vlasov operator can be approached as $V = \dot{\mathbf{X}}\cdot\nabla_s + \dot{u}(\partial/\partial u)$, where $\nabla_{\rm s}$ is the gradient within the magnetic surface and $\lambda = 0$. Here, $\lambda = (1-\xi^2)/b$ is the normalized magnetic moment, $\xi = u_{||}/u$ is the pitch and $b = B/B_0$ with B_0 as the reference magnetic field. The linearized Coulomb operator, $C_{\rm e}(f_{\rm e}) = C_{\rm ee}[f_{\rm e1}; f_{\rm eJM}] + C_{\rm ee}[f_{\rm eJM}, f_{\rm e1}] + C_{\rm ei}[f_{\rm e1}, f_{\rm iM}]$, must be taken in relativistic approach [1]. Here and below, the ion distribution function, $f_{\rm i} = f_{\rm iM}$, is assumed non-relativistic Maxwellian.

Relativistic drift velocity, $\dot{\mathbf{X}}$, is derived by formal gyro-ordering [5,6] and written in a form traditional for the transport theory [3,4], where the hamiltonian incompressibility (important for gyrokinetic) is neglected and, additionally, all the terms proportional to $\nabla\times\boldsymbol{B}$ which do not contribute to the radial transport are omitted:

$$\dot{\boldsymbol{X}} = \frac{u\xi}{\gamma}\boldsymbol{h} + \frac{c}{B^2}[\boldsymbol{E}\times\boldsymbol{B}] + \frac{mcu^2(1+\xi^2)}{2e\gamma B^3}[\boldsymbol{B}\times\nabla B]. \tag{4}$$

Here e is the particle charge. In the same approach, the equation for $\dot{u}$ can be written (in assumption that $\partial\Phi/\partial t = \partial B/\partial t = 0$):

$$\dot{u} = \frac{e\gamma}{m_{e0}u}\dot{\boldsymbol{X}}\cdot(E_{||}\boldsymbol{h} + \boldsymbol{E}_\perp), \tag{5}$$

where $\boldsymbol{E}_\perp = -\nabla\Phi$ is the radial electric field, $E_{||} = \boldsymbol{E}\cdot\boldsymbol{h}$ and $\boldsymbol{h} = \boldsymbol{B}/B$. Here, the first term describes the acceleration of electrons due to the longitudinal electric field (in this paper, this term is omitted from the consideration, i.e. $E_{||}=0$) and the second term corresponds to a work of the radial electric field due to the ∇B-drift.

Since only the radial fluxes are under consideration, u in Eq. [5] can be replaced by the relation with ρ. Futhermore, it is convenient to redefine the Maxwellian as $F_{\rm eJM}(\rho,u) = \exp(-e\Phi/T_{\rm e})f_{\rm eJM}$. Then the right-hand-side of Eq. [3] can be represented as the standard set of thermodynamic forces [4],

$$-\dot{\rho}\frac{\partial F_{eJM}}{\partial \rho} = -\dot{\rho}F_{eJM}[A_1(\rho) + \kappa A_2(\rho)], \tag{6}$$

where $\kappa \equiv K/T_{\rm e} = \mu_r(\gamma-1)$ is the relativistic kinetic energy normalized by T_e, and the thermodynamical forces A_1 and A_2 are defined as

$$A_1(\rho) = \frac{n_e'}{n_e} - \left(\frac{3}{2} + \Re\right)\frac{T_e'}{T_e} - \frac{e\Phi'}{T_e}, \quad A_2(\rho) = \frac{T_e'}{T_e}. \tag{7}$$

Here $n_e' \equiv dn_e/d\rho$, $T_e' \equiv dT_e/d\rho$, and $\Phi' \equiv d\Phi/d\rho$. Note that A_1 contains the additional relativistic correction, $\Re(\mu_{\rm r})$, given by

$$\Re = \mu_r\left(\frac{K_3}{K_2} - 1\right) - \frac{5}{2} \approx \frac{15}{8\mu_r} + \ldots \quad (\mu_r \gg 1). \tag{8}$$

Since kinetic equation is extremely complex to solve, simplifications are required. In particular, the mono-energetic approach [3,4] can be applied for calculations of the radial fluxes induced by only the radial gradients.

In this case, linearized Coulomb operator can be approached by only pitch-angle scattering taken into account as dominating process for electrons,

$$C_e(f_{e1}) = \nu_D(u)L(f_{e1}), \tag{9}$$

where L is the Lorentz operator,

$$L = \frac{2|\xi|}{b}\frac{\partial}{\partial\lambda}\left(\lambda|\xi|\frac{\partial}{\partial\lambda}\right), \tag{10}$$

and $\nu_D(u) = \nu_D^{ee}(u) + \nu_D^{ei}(u)$ is the deflection frequency. The explicit expressions for ν_D^{ee} and ν_D^{ei} in the relativistic approach are given in [1].

It is generally accepted that for calculation of the radial fluxes, the drift-kinetic equation Eq. (3) can be considered in the mono-energetic approach [3,4] with omitted the acceleration term in the Vlasov operator. In this case, Eq. (3) can be written as

$$\left(\xi\boldsymbol{h} + \frac{c}{B}\frac{\gamma E_\rho}{u}\nabla\rho\times\boldsymbol{h}\right)\cdot\nabla_s f_{e1} - \frac{\gamma\nu_D(u)}{u}L(f_{e1}) = -\frac{\gamma\dot{\rho}}{u}\frac{\partial F_{eJM}}{\partial\rho}, \tag{11}$$

where similar to the non-relativistic formulation, the energy enters only as parameter in values $\gamma E_\rho/u$ and $\gamma\nu_D(u)/u$.

2. EVALUATION OF NEOCLASSICAL TRANSPORT IN 1/ν REGIME

Now, let us calculate the radial fluxes of particles and energy for 1/ν collisional regime which dominates in stellarators. In this section we follow the paper Ref. [7] introducing the relativistic corrections when necessary. Since 1/ν regime is interesting for us only as an example of the relativistic consideration, we do not discuss any applicability of the results to any concrete experiment.

Assuming that $\boldsymbol{E}\times\boldsymbol{B}$ drift of electrons on the magnetic surface does not produce any significant contribution in transport and neglecting this term in Eq. (11), this equation can be solved by integration along the field-line. Here, only the trapped electrons, $B_0/B_{max} < \lambda < B_0/B_{min}$, are considered ($B_{max}$ and B_{min} are the absolute maximum and minimum of B at the given magnetic surface, respectively).

Enumerating the local minima of B along the magnetic field-line by k and integrating Eq. (11) over the bounce trajectory (assumed be closed), one can obtain

$$\frac{2\gamma\nu_D(u)}{u}\frac{\partial}{\partial\lambda}\left(\lambda I^{(k)}\frac{\partial f_{e1}^{(k)}}{\partial\lambda}\right) = \delta\rho^{(k)}\frac{\partial F_{eJM}}{\partial\rho} \tag{12}$$

with

$$I^{(k)} = \oint_k \frac{ds}{b}\xi, \quad \delta\rho^{(k)} = \frac{\gamma}{u}\oint_k \frac{ds}{\xi}\dot{\rho},$$

where $\delta\rho^{(k)}$ is the radial displacement of electron due to the ∇B-drift after one bounce period. After series of transformations the order of Eq. (12) can be reduced,

$$\frac{\partial f_{e1}^{(k)}}{\partial\lambda} = -\frac{H^{(k)}}{6\lambda I^{(k)}}\frac{u^2}{\gamma\nu_D(u)}\frac{\partial F_{eJM}}{\partial\rho}, \tag{13}$$

with $H^{(k)} = \oint_k ds\xi(3+\xi^2)|\nabla\rho|k_G/(b\omega_c)$. Here, $\omega_c = eB/(m_{e0}c)$ is the cyclotron frequency, $k_G = \boldsymbol{n}_\rho\cdot(\boldsymbol{h}\times(\boldsymbol{h}\cdot\nabla)\boldsymbol{h})$ is the geodesic curvature of the magnetic field line and $\boldsymbol{n}_\rho = \nabla\rho/|\nabla\rho|$ is the unit vector normal to the magnetic surface.

Let us apply this solution to the radial fluxes of particles and energy, respectively, defined as

$$\Gamma_e^\rho = \langle \Gamma_e\cdot\nabla\rho\rangle = \left\langle \int d^3u\,\dot{\rho} f_{e1}\right\rangle, \tag{14}$$

$$Q_e^\rho = \langle \mathbf{Q}_e\cdot\nabla\rho\rangle = T_e\left\langle \int d^3u\,\kappa\dot{\rho} f_{e1}\right\rangle. \tag{15}$$

Producing in both integrals the integration by parts over λ, considering the averaging over the flux-surface as the limit of integration along the field-line (here, the same technique as was applied in [7]) and using Eq. (14), the final expressions for radial fluxes are

$$I_i = -G_0 C_M \int_0^\infty d\kappa \frac{e^{-\kappa}\kappa^{5/2}}{\gamma\hat{\nu}_D(u)}\left(\frac{\gamma+1}{2}\right)^{5/2} h_i \frac{\partial \ln F_{eJM}}{\partial\rho}, \tag{16}$$

where $I_1 \equiv \Gamma^\rho$ and $I_2 \equiv Q_e^\rho/T_e$ with $h_1 = 1$ and $h_2 = \mu_r(\gamma-1)$; $\nu^*_D(u) \equiv \nu_D(u)/\nu_{e0}$ with $\nu_{e0} = 4\pi n_e e^4 \ln\Lambda/(m_{e0}^2 u_{th}^3)$.

The coefficient G_0 (the same for both relativistic and non-relativistic formulations), accumulates all the parameters for plasmas and magnetic configuration which are specific for the considered 1/ν regime,

$$G_0 = \frac{4\sqrt{2}}{9\pi^{3/2}}\frac{u_{th}^4}{R^2\omega_{e0}^2}\frac{n_e}{\nu_{e0}}\langle|\nabla\rho|\rangle^2 \varepsilon_{eff}^{3/2},$$

where R is the major radius, $\omega_{e0} = eB_0/m_{e0}c$, and $\varepsilon_{eff}^{3/2}$ is the effective ripple amplitude (not shown here; for details see Ref. [7]).

Substituting the derivative of Maxwellian Eq. (6), the fluxes can be expressed in a standard manner,

$$I_i = -n_e \sum_{j=1,2} L_{ij} A_j, \tag{17}$$

where the thermodynamic forces A_1 and A_2 are defined by Eq. (7). The transport coefficients, L_{ij}, can be easily obtained from Eq. (17). It can be checked also that this definition satisfies to the Onsager symmetry and $L_{12} = L_{21}$.

Note, that since the thermodynamic force A_1 contain the relativistic correction $\Re$, a comparison of only the transport coefficients L_{ij} will not be sufficient to estimate the role of relativistic corrections and it is meaningful to compare a full fluxes I_i in both relativistic and non-relativistic approaches. In order to simplify a comparison, let us consider two cases.

First is the case with $n_e' = \Phi' = 0$, when the fluxes can be written as

$$I_i^{(1)} = -n_e\left(-L_{i,1}\left(\frac{3}{2}+\Re\right)+L_{i,2}\right)\frac{T_e'}{T_e}, \tag{18}$$

and the second is the case with $T_e' = 0$, when

$$I_i^{(2)} = -n_e L_{i,1}\left(\frac{n_e'}{n_e} - \frac{e\Phi'}{T_e}\right). \tag{19}$$

In both these cases, the ratio of fluxes $I_i^{(1)}$ and $I_i^{(2)}$ does not contain the gradients and any factors specific for the magnetic equilibrium, and can be easily calculated.

In Fig. 1, the ratios of $\Gamma_e^\rho/\Gamma_e^{\rho(nr)}$ and $Q_e^\rho/Q_e^{\rho(nr)}$ are shown for both cases. Additionally, the ratio for conductive heat flux, $q_e^\rho/q_e^{\rho(nr)}$, is also represented. Here, the radial conductive heat flux of relativistic electrons is defined as $q_e^\rho = \langle\mathbf{q}_e\cdot\nabla\rho\rangle = Q_e^\rho - (5/2+\Re)T_e\Gamma_e^\rho$. This definition has the same meaning as a classical one but includes the relativistic effects.

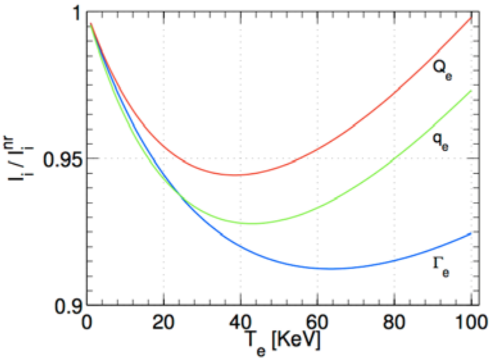

Fig. 1. The relativistic radial fluxes normalized by the non-relativistic values for the case 1 (Eq. 18)

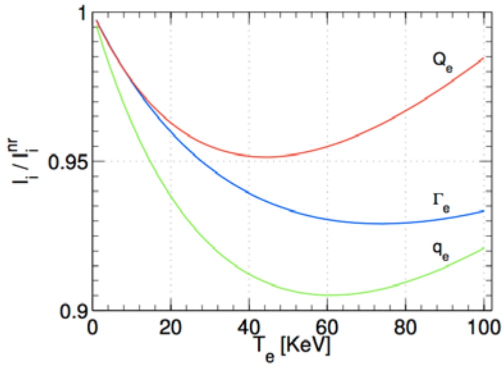

Fig. 2. The relativistic radial fluxes normalized by the non-relativistic values for the case (Eq. 19)

Here we observe two effects. First is related to the integral "weight" of the relativistic Maxwellian bulk, which leads to reduction of fluxes with growth of temperature (see expression for $C_M(\mu_r)$, Eq. (2)), and the second is related to increasing contribution from the tails which appears at higher temperatures. The latter is more pronounced for the fluxes for which the integrand in Eq. (16) has a higher degree of κ. The most surprising is that the obtained results are counterintuitive, i.e. the relativistic effects are found to be more significant for the regimes where the contribution from thermal electrons dominates.

Note that in this example, devoted to only the preliminary check of the relativistic effects in $1/\nu$ regime, we do not analyze a validity of this collisional regime and assume only that it is the same for all temperatures.

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РЕЛЯТИВИСТСКИЕ РАДИАЛЬНЫЕ ПОТОКИ В ГОРЯЧЕЙ ПЛАЗМЕ

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Радиальные потоки частиц и энергии, включающие релятивистские эффекты, представлены в форме, стандартной для неоклассической теории. Все формулировки основаны на релятивистских уравнениях движения и релятивистском дрейфово-кинетическом уравнении. В качестве иллюстрации влияния релятивистских эффектов на процессы переноса в плазме предлагается сравнительная оценка релятивистских и нерелятивистских радиальных неоклассических потоков электронов, посчитанных для стеллараторного режима $1/\nu$. Предложенная формулировка позволяет включить релятивистские эффекты в существующие транспортные коды.

РЕЛЯТИВІСТСЬКІ РАДІАЛЬНІ ПОТОКИ У ГОРЯЧІЙ ПЛАЗМІ

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Радіальні потоки частинок та енергії, що ураховують релятивістські ефекти, запропоновано у формі, яка є стандартною для неокласичної теорії. Усі формулювання базуються на релятивістських рівняннях руху та релятивістському дрейфово-кінетичному рівнянні. В якості ілюстрації впливу релятивістських ефектів на процеси переносу в плазмі запропоновано порівняння релятивістських та нерелятивістських радіальних неокласичних потоків електронів, разрахованих для стелараторного режиму $1/\nu$. Запропоноване формулювання дозволяє ураховувати релятивістські ефекти в існуючих транспортних кодах.