EFFECTIVE EVALUATION OF THE EXACT RELATIVISTIC PLASMA DISPERSION FUNCTIONS

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A new effective evaluation method of the exact relativistic plasma dispersion functions in the real and complex regions is given on the basis of nonsingular forms of Cauchy or Cauchy-type integrals and the Euler-Maclaurin formula.

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INTRODUCTION

The computation of the exact relativistic plasma dispersion functions (PDFs) [1,2] is a necessary basis for both the analysis of the electron cyclotron waves in laboratory thermonuclear and hot astrophysical plasmas and the analysis of the ion cyclotron waves in extremely hot astrophysical plasmas. The full account of relativistic effects is especially important in the regimes of wave propagation in high-temperature plasma almost perpendicularly to the confining magnetic field in the vicinities of higher cyclotron harmonic resonances. These functions, as well as other PDFs (non-relativistic one and weakly relativistic ones), can be expressed in the form of Cauchy or Cauchy-type integrals defined on the real axis, provided that the densities of the corresponding integrals vanish at infinity, and hence can be computed for not very large $|z|$- values, being z their complex argument, by means of the direct numerical calculations of these singular integrals and using their asymptotic expansions for the remaining values of $|z|$ [2]. However in many numerical applications, PDFs must be routinely evaluated many times, therefore the efficiency of the numerical algorithm involved in their calculation is of primary importance.

For the simplest case of nonrelativistic PDF $w(z)=\exp(-z^2)\,\mathrm{erfc}(-iz)$, the use of continued fractions of Jacobi, which are the special diagonal case of Pade approximants, has been proved to provide such an efficient method for large-$|z|$ values in combination with the Taylor expansion of special kind for the remaining values of $|z|$ [3]. For given accuracy, these calculations are about two orders of magnitude faster than the direct computation of the Cauchy type integrals and one order of magnitude slower than the calculation of the exponential function. The same technique involving two approaches can be used for the weakly relativistic PDFs [4,5]. However, the technique [4], due to the use of recurrent relation for the weakly relativistic PDFs, lacks stability when $|z|$ becomes large.

The numerical technique [5], developed for the computation of the weakly relativistic PDFs for not very large $|z|$- values without the use of recurrent relations, can be also used for the most complicated case of the exact relativistic PDFs. But the main purpose of the present work is to present the new and more effective method to evaluate these functions in the same region.

1. EVALUATION OF THE EXACT RELATIVISTIC PDFs ON THE REAL AXIS

In plasma physics, solving boundary value problems requires computation of PDFs only on the real axis, so it makes sense to consider this case separately. On the real axis the exact relativistic PDFs can be defined by the means of the next formulae [1]

$$Z_{q+3/2}(a,x,\mu)=\frac{\sqrt{\pi\beta}\,e^{-\mu\sqrt{\beta}}}{\sqrt{2\mu}K_2(\mu)\left(\sqrt{a}\right)^{q+1/2}}\times$$

$$\int_0^{+\infty}\frac{\left(\sqrt{u(u/(2\mu)+1/\sqrt{\beta})}\right)^{q+1/2}I_{q+1/2}\left(2\beta a^{1/2}\sqrt{u(u/(2\mu)+1/\sqrt{\beta})}\right)e^{-\beta u}du}{u-a^*+x},$$

$$(0\le N_{\|}<1), \qquad (1)$$

$$Z_{q+3/2}(a,x,\mu)=\sqrt{-\beta}\,\frac{-\exp(-2\beta a-\mu)}{\sqrt{2\pi\mu}K_2(\mu)\left(\sqrt{a}\right)^{q+1/2}}\times$$

$$\int_{-\infty}^{+\infty}\frac{\left(\sqrt{a-t+t^2/(2\mu)}\right)^{q+1/2}K_{q+1/2}\left(-2\beta a^{1/2}\sqrt{a-t+t^2/(2\mu)}\right)\exp(\beta t)dt}{t-x},$$

$$(N_{\|}>1), \qquad (2)$$

where q is the number of harmonic $a=\mu N_{//}^2/2$, $x=\mu(1-q\Omega_c/\omega)$, $\mu=c^2 m_0/T$, c, m_0, T are the speed of light in vacuum, the rest mass of the particle and the temperature of particles, respectively, $\beta=\mu/(\mu-2a)$, $a^*=\mu\left(1-1/\sqrt{\beta}\right)$, $N_{\|}=k_{\|}c/\omega$ is the longitudinal refractive index, $\Omega_c=eB/(m_0 c)$ is the fundamental particle cyclotron resonance frequency, $K_2(x)$, $I_{q+1/2}(x)$, $K_{q+1/2}(x)$ are modified Bessel functions and the contour of integration is taken above the pole.

These functions belong to a rather broad and important class of analytic functions defined by Cauchy and type of Cauchy-type integrals, with the integral density tending to 0 at infinity. Their anti-hermitian parts equal to the density of the integral multiplied by $-\pi i$ and their hermitian parts equal to the principal values of those integrals. For large $|x|$ values, where the integral density is less than, say, exp(-36), these principal values can be evaluated on the base of the asymptotic expansion in the infinite point [6]

$$Z_{q+1/2} = \frac{K_q(\mu)}{K_2(\mu)z} - \mu\frac{K_{q+1}(\mu) - K_q(\mu)}{K_2(\mu)z^2} + ... = \frac{1}{z}\left(A_0^q + \frac{A_1^q}{z} + \frac{A_2^q}{z^2} + ...\right). \quad (3)$$

For the remaining $|x|$ values, the singular integrals can be evaluated by means of direct numerical integration on the base of the following nonsingular integral forms:

$$\mathrm{P}\int_a^{+\infty}\frac{f(t)}{t-b}dt = \int_a^b\frac{f(t)-f(2b-t)}{t-b}dt - \int_{-\infty}^a\frac{f(2b-t)}{t-b}dt, (0 \le N_{\parallel} < 1), \quad (4)$$

$$\mathrm{P}\int_{-\infty}^{+\infty}\frac{f(t)}{t-b}dt = \int_{-\infty}^b\frac{f(t)-f(2b-t)}{t-b}dt, \quad (N_{\parallel} > 1), \quad (5)$$

since integrands in (1), (2) can be expressed in terms of exponentials, and therefore, can be estimated up to the required precision. However, the speed of the calculations, as noted in the introduction, is rather far from ideal.

Fortunately, the formulae (4), (5) have an extra interesting property, which is manifested only in their use in conjunction with the Euler-Maclaurin formula

$$\int_a^b f(x)dx = h(y_1 + ... + y_{n-1}) - \frac{B_1 h}{1!}[f(b) + f(a)] - \frac{B_2 h^2}{2!}[f'(b) - f'(a)] - \frac{B_4 h^4}{4!}[f^{(3)}(b) - f^{(3)}(a)] - ... \quad (6)$$

where $y_0 = f(a)$, $y_n = f(b)$, $y_1,...y_{n-1}$ are the values of function $f(x)$ in successive equally spaced points with a step h and B_m are the Bernoulli coefficients.

In the case of (5), it is necessary to take the limit with a and b in (6) tending to $-\infty$ and $+\infty$, respectively. Then we have

$$\int_{-\infty}^{+\infty} f(x)dx = h[... + f(-h) + f(0) + f(h) + ...], \quad (7)$$

since it is easy to see from the expansion (3) that the reminder of the terms with the odd derivatives of $f(x)$ in (6) vanish due to the conditions of the problem at infinity.

In the case of (5), it is necessary to take $a \to -\infty$ in (6). Then, instead of (7), we have

$$\int_{-\infty}^{b} f(x)dx = h\left[... + f(b-2h) + f(b-h) + \frac{1}{2}f(b)\right], \quad (8)$$

since can be demonstrated that all the odd derivatives of $f(x)$ at the point b vanish, i.e., the function $f(x)$ is even with respect to the point b.

It is interest to note that the rather simple formulae (7), (8), being practically the trapezoidal quadrature, are nevertheless exact for all the values of h such that the series in (6) converges. Since the limiting behavior of Bernoulli coefficients is $B_m = O(m!/(2\pi)^m)$, these formulae are exact for $h < 1$. In applications the series in (7) and (8) should be cut if the integral density will be less than, say, exp(-36). Direct calculations along those formulae show that the value h should lie within the range, for parameters corresponding to thermonuclear plasma, $h = 0.2...0.6$ to preserve 8 significant digits. This allows one performing computations about ten times faster in comparison with the direct numerical integration on the base of nonsingular forms only.

2. EVALUATION OF THE EXACT RELATIVISTIC PDFs IN THE COMPLEX REGION

In plasma electrodynamics, solving of initial value problems requires computation of PDFs in the whole Riemann sphere of the complex region. In the case of (2) the exact relativistic PDFs have two separate analytic branches in the complex region, similarly to the nonrelativistic PDF, and in the case of (1) these branches are analytically connected in the same way as those of the weakly relativistic PDFs [2]. The values of both branches are calculated in a similar way, and it is sufficient to estimate the values of each branch only in one semi-plane since the values in the upper and lower semi-planes are connected through the Sohotski-Plemelj formulae. Therefore it is sufficient to consider only the computation of the main branch in the upper semi-plane. Moreover, as mentioned in the introduction, it is sufficient as well to calculate these functions for not very large $|z|$- values.

It is of great interest to note that the method of evaluation of (1) and (2) described above is proved to be applicable also in the upper semi-plane near the real axis by replacing x by $z = x + iy$. Numerical calculations show that, up to 8 significant digits, this method is applicable in the whole band $0 \le y \le 2.5$ with the same h values as for the real axis. Such an unusual ease of analytic continuation of exact PDFs from the real axis into the upper semiplane confirms the fact of accuracy of the formulas (7) and (8) with the h values used in the previous section.

Similarly, for $2.5 \le y \le 10$ there was applied Euler-Maclaurin formula, though not for the principal values of Cauchy integrals (4) and (5), but for the whole exact PDFs (1) and (2), which are also nonsingular in this region just like (4) and (5) at the real axis. Such a method was used for estimation of the nonrelativistic

PDF [7] and the possibility to use this method for Cauchy integrals was in first pointed out in [8].

In this region, to preserve 8 significant digits, we chose a step h within the range $h = 0.4...0.8$ (with the increase of h when y increases), which allows 20-percent gain in computation speed, compared with the lower region ($0 \le y \le 2.5$).

It worth to note that in the case of extremely hot astrophysical plasma the region of the real axis, in which antihermitian parts of the exact relativistic PDFs are significantly larger than zero, can greatly expand in comparison with the case of the laboratory thermonuclear (for example ITER-like) plasma. This peculiarity can lead for such extreme plasmas to somewhat lowering in the speed of computations using this method,. But this case of extremely hot astrophysical plasmas can be the subject of a separate work.

CONCLUSIONS

1. On the base of nonsingular forms for Cauchy and Cauchy-type integrals and the Euler-Maclaurin formula, a new efficient method to compute the exact relativistic PDFs for real and complex argument is given.
2. Comparison of this method with the method of direct numerical evaluation of Cauchy and Cauchy-type integral shows that the present method for laboratory thermonuclear plasmas is an order of magnitude more efficient.
3. This method can be used for evaluation of any Cauchy and Cauchy-type integral defined on the real axis, provided that the density of the corresponding integral vanishes at infinity and, in particular, for evaluation of the nonrelativistic and weakly relativistic PDFs.

REFERENCES

1. F. Castejón and S.S. Pavlov. Relativistic plasma dielectric tensor evaluation based on the exact plasma dispersion function concept // *Physics of Plasmas*. 2006, № 13, p. 072105.
2. F. Castejón and S.S. Pavlov. The exact plasma dispersion functions in complex region // *Nuclear Fusion*. 2008, № 48, p. 054003.
3. W. Gautschi. Efficient computation of the complex error function // *SIAM J. Numer. Anal*. 1970, № 7, p. 187.
4. V. Krivenski and A. Orefice. Weakly relativistic dielectric tensor and dispersion functions of a Maxwellian plasma // *J. Plasma Physics.* 1983, v. 30, part. 1, p. 125.
5. S.S. Pavlov, F. Castejón and M. Tereshchenko. Weakly relativistic plasma dispersion functions computation using superasymptotic and hyperasymptotic series // *Problems of Atomic Science and Technology. Series «Plasma Physics» (16).* 2010, № 6, p. 73.
6. S.S. Pavlov, F. Castejón, A. Cappa, M. Tereshchenko. Fast computation of the exact plasma dispersion functions // *Problems of Atomic Science and Technology. Series «Plasma Physics» (15)*. 2009, № 1, p. 69.
7. V.N. Faddeeva and N.M. Terentjev. *Tabulation of the function* $W(z) = \exp(-z^2)[1 + 2i/\sqrt{\pi}\int_0^z \exp(t^2)dt]$ *of complex argument.* State Press of Tech.-Theor. Literature. Moscow, 1954 (in Russian).
8. V.A. Fock. *Theory of definition the resistance of rocks by carottage*. 1933 (in Russian).

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ЭФФЕКТИВНОЕ ВЫЧИСЛЕНИЕ ТОЧНЫХ РЕЛЯТИВИСТСКИХ ПЛАЗМЕННЫХ ДИСПЕРСИОННЫХ ФУНКЦИЙ

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Предлагается новый эффективный метод вычисления точных релятивистских плазменных дисперсионных функций в реальной и комплексной областях на основе интегральных форм Коши или типа Коши и формулы Эйлера-Маклорена.

ЕФЕКТИВНЕ ОБЧИСЛЕННЯ ТОЧНИХ РЕЛЯТИВІСТСЬКИХ ПЛАЗМОВИХ ДИСПЕРСІЙНИХ ФУНКЦІЙ

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Пропонується новий ефективний метод обчислення точних релятивістських плазмових дисперсійних функцій в реальній та комплексній областях на основі інтегральних форм Коші або типу Коші та формули Ейлера-Маклорена.