AHARONOV-BOHM EFFECT: ANALYSIS OF SCATTERING IN THE FORWARD DIRECTION

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The long-standing problem of a divergent behaviour of the Aharonov-Bohm scattering amplitude in the forward direction is reconsidered. We show that this divergence has no physical consequences, being an artefact of the approximation that neglects the transverse size of a magnetic vortex. As long as the vortex transverse size is taken into account, this divergence is tamed but, however, in a certain sense manifests itself as a forward peak of the Fraunhofer diffraction. The peak is becoming more transparent in the limit of high energies of scattered particles.

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1. INTRODUCTION

The theoretical prediction of the Aharonov-Bohm effect in 1959 [1] was one of the most intriguing achievements in quantum theory. Now this effect has been long recognized for its crucial role in demonstrating that in addition to the usual local (classical) influence of electromagnetic field on charged particles, there exists the unusual nonlocal (purely quantum) influence of electromagnetic fluxes confined in the regions which are inaccessible to charged particles. A particular example is quantum-mechanical scattering of charged particles by an impermeable straight and infinitely long solenoid that encloses a magnetic flux: as was shown in [1], this process depends periodically on the value of the enclosed flux. However, the amplitude of this process, as was first obtained in [1], diverges in the strictly forward direction, and all previous attempts to eliminate this divergence failed unsuccessfully, see [2, 3]. Therefore, we reconsider this problem in the present paper.

Let us study scattering of a charged particle by an obstacle in the form of an impermeable tube which is filled with magnetic field of total flux Φ (magnetic vortex). The particle wave function out of the vortex obeys the Schrödinger equation

$$-\frac{\hbar^2}{2m}\left[\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\left(\partial_\varphi-\mathrm{i}\frac{\Phi}{\Phi_0}\right)^2\right]\psi_{\mathbf{k}}(\mathbf{r}) = \frac{\hbar^2k^2}{2m}\psi_{\mathbf{k}}(\mathbf{r}), \qquad (1)$$

where $\Phi_0=2\pi\hbar ce^{-1}$ is the London flux quantum, and our concern is in the motion in the plane which is orthogonal to the axis of the tube, since the motion along the axis is free; $\mathbf{r}$ and $\mathbf{k}$ are the two-dimensional vectors in this plane, φ is the angle between them. We impose condition

$$\lim_{r\to\infty} e^{\mathrm{i}kr}\psi_{\mathbf{k}}(\mathbf{r})|_{\varphi=\pm\pi}=1, \qquad (2)$$

signifying that the incident wave comes from the far left; the forward direction is $\varphi=0$, and the backward direction is $\varphi=\pm\pi$.

In the next section we consider scattering by an infinitely thin magnetic vortex. The transverse size of the vortex is taken into account in Section 3. The conclusions are drawn in Section 4.

2. INFINITELY THIN MAGNETIC VORTEX

If the transverse size of the tube is neglected, then it follows immediately that the finite solution to (1) satisfying (2) is

$$\psi_{\mathbf{k}}(\mathbf{r})=\sum_{n\in Z} e^{\mathrm{i}n\varphi}e^{\mathrm{i}(|n|-\frac{1}{2}|n-\mu|)\pi}J_{|n-\mu|}(kr), \qquad (3)$$

where $\mu=\Phi\Phi_0^{-1}$, and Z is the set of integer numbers.

The asymptotics of the wave function at large distances from the origin is

$$\psi_{\mathbf{k}}(\mathbf{r})=e^{ikr\cos\varphi}e^{\mathrm{i}\mu[\varphi-\mathrm{sgn}(\varphi)\pi]}+f(k,\varphi)\frac{e^{\mathrm{i}(kr+\pi/4)}}{\sqrt{r}} + O(r^{-3/2}), \qquad (4)$$

where it is implied that $-\pi<\varphi<\pi$,

$$f(k,\varphi)=\mathrm{i}\sqrt{\frac{2\pi}{k}}\sin(\mu\pi)\Gamma^{(\nu)}(\varphi), \qquad (5)$$

$$\Gamma^{(\nu)}(\varphi)=\frac{1}{2\pi\mathrm{i}}\sum_{n\in Z}\mathrm{sgn}(n-\mu)e^{\mathrm{i}n\varphi}, \qquad (6)$$

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ν is the integer part of μ, and

$$\mathrm{sgn}(u) = \begin{cases} 1, & u > 0, \\ -1, & u < 0. \end{cases}$$

Correspondingly, the S-matrix in this case takes form

$$S(k,\, \varphi;\, k',\, \varphi') = \cos(\mu\pi)\frac{1}{k}\delta(k-k')\Delta(\varphi-\varphi') + \delta(k-k')\frac{\mathrm{i}}{\sqrt{2\pi k}}f(k,\, \varphi-\varphi'), \quad (7)$$

where

$$\Delta(\varphi) = \frac{1}{2\pi}\sum_{n\in Z} e^{in\varphi} \quad (8)$$

is the delta-function for the azimuthal angle, $\Delta(\varphi + 2\pi) = \Delta(\varphi)$.

The condition of the unitarity of the S-matrix, $S^\dagger S = SS^\dagger = \mathrm{I}$, results, in view of equality

$$\Gamma^{(\nu)}(\varphi) + \left[\Gamma^{(\nu)}(-\varphi)\right]^* = 0, \quad (9)$$

in the following relation:

$$\sin^2(\mu\pi)\Delta(\varphi'-\varphi'') = \frac{k}{2\pi}\int\limits_{-\pi}^{\pi} d\varphi\, f^*(k,\, \varphi-\varphi')f(k,\, \varphi-\varphi''). \quad (10)$$

Thus, we see that the optical theorem which should be derived from (10) by putting $\varphi' = \varphi'' = 0$ is hardly informative, being a relation between infinite quantities, $\Delta(0)$.

It is instructive to derive the explicit form of $\Gamma^{(\nu)}(\varphi)$ (6) here. Using elementary trigonometric relation

$$\cot(\varphi/2)\left\{\sin[(n+1)\varphi] - \sin(n\varphi)\right\} = \cos[(n+1)\varphi] + \cos(n\varphi),$$

one can get

$$\int\limits_0^{\pi} d\varphi \cot(\varphi/2)\sin(N\varphi) = \pi, \qquad N = 1,\, 2,\, 3,\, \ldots,$$

which results in relation

$$\cot\frac{\varphi}{2} = 2\sum_{\substack{n\in Z\\ n\geq 1}} \sin(n\varphi).$$

The use of the latter along with the definition (8) yields

$$\sum_{\substack{n\in Z\\ n\geq N}} e^{\mathrm{i}n\varphi} = \pi\Delta(\varphi) - e^{\mathrm{i}N\varphi}(e^{\mathrm{i}\varphi}-1)^{-1}, \quad N = 1,\, 2,\, 3,\, \ldots, \quad (11)$$

whence it follows that

$$\Gamma^{(\nu)}(\varphi) = \frac{e^{\mathrm{i}(\nu+\frac{1}{2})\varphi}}{2\pi}\frac{1}{\sin(\varphi/2)}, \quad (12)$$

where the divergence at $\varphi = 2\pi l$ ($l \in Z$) in (12), as well as in the second term in (11), is to be understood in the principal-value sense.

Although amplitude f (5) with $\Gamma^{(\nu)}$ (12) (first obtained more than half a century by Aharonov and Bohm [1] and then rederived in the framework of different approaches; perhaps the one presented here is the simplest and the most straigtforward) diverges in the forward direction, this divergence has no physical consequences, because there is a crossover to another regime in the strictly forward direction: amplitude f, instead of being proportional to $k^{-1/2}$, becomes, formally, proportional to $r^{1/2}$. This is most easily seen from the following expression which is valid for all scattering angles and has been obtained in [4]:

$$f(k,\varphi)\frac{e^{\mathrm{i}(kr+\pi/4)}}{\sqrt{r}} = \mathrm{i}\sin(\mu\pi)e^{\mathrm{i}kr\cos\varphi}e^{\mathrm{i}(\nu+\frac{1}{2})\varphi} \times \mathrm{sgn}\left[\sin(\varphi/2)\right]\mathrm{erfc}\left[e^{-\mathrm{i}\pi/4}\sqrt{2kr}|\sin(\varphi/2)|\right], \quad (13)$$

where $\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int\limits_z^\infty du\, e^{-u^2}$ is the complementary error function. In the strictly forward direction one gets a discontinuity,

$$\lim_{\varphi\to\pm 0} f(k,\varphi)\frac{e^{\mathrm{i}(kr+\pi/4)}}{\sqrt{r}} = \pm\mathrm{i}\sin(\mu\pi)e^{\mathrm{i}kr}, \quad (14)$$

which cancels the discontinuity of the incident wave (first term in (4)). Consequently, wave function (4) is finite and continuous in the forward direction:

$$\psi_{\mathbf{k}}(\mathbf{r})|_{\varphi=0} = \cos(\mu\pi)e^{\mathrm{i}kr}, \quad (15)$$

that is consistent with its exact expression (3). The appearance of factor $\cos(\mu\pi)$ in the transmitted wave (15) can be intuitively understood as a result of self-interference from different sides of the vortex [5, 6].

The divergence of the scattering amplitude and the total cross section, as well as the failure with the optical theorem, has no physical meaning, being an artefact of the approximation which neglects the vortex thickness: this is certainly an excessive idealization, whereas any realistic vortex is of finite nonzero thickness. As long as the vortex thickness is taken into account, all these drawbacks are eliminated.

3. MAGNETIC VORTEX OF NONZERO TRANSVERSE SIZE

We consider the wave function out of the vortex of finite thickness $2r_c$ and impose the Robin boundary condition at the edge of the vortex

$$\left.\left\{\left[\cos(\rho\pi) + r_c\sin(\rho\pi)\frac{\partial}{\partial r}\right]\psi_{\mathbf{k}}(\mathbf{r})\right\}\right|_{r=r_c} = 0; \quad (16)$$

hence, $\rho = 0$ corresponds to the Dirichlet condition (perfect conductivity of the boundary)

$$\psi_{\mathbf{k}}(\mathbf{r})|_{r=r_c} = 0, \quad (17)$$

and $\rho = 1/2$ corresponds to the Neumann condition (absolute rigidity of the boundary)

$$\left[\frac{\partial}{\partial r}\psi_{\mathbf{k}}(\mathbf{r})\right]\Big|_{r=r_c} = 0. \tag{18}$$

Then (3) is changed to

$$\psi_{\mathbf{k}}(\mathbf{r}) = \sum_{n\in Z} e^{in\varphi} e^{\mathrm{i}\left(|n|-\frac{1}{2}|n-\mu|\right)\pi}\Big[J_{|n-\mu|}(kr) - \Upsilon^{(\rho)}_{|n-\mu|}(kr_c) H^{(1)}_{|n-\mu|}(kr)\Big], \tag{19}$$

and (5) is changed to

$$f(k,\,\varphi) = \mathrm{i}\sqrt{\frac{2\pi}{k}}\sin(\mu\pi)\Gamma^{(\nu)}(\varphi) + \mathrm{i}\sqrt{\frac{2}{k\pi}}\sum_{n\in Z} e^{in\varphi} e^{\mathrm{i}(|n|-|n-\mu|\pi)}\Upsilon^{(\rho)}_{|n-\mu|}(kr_c), \tag{20}$$

where

$$\Upsilon^{(\rho)}_{\alpha}(u) = \frac{\cos(\rho\pi)J_\alpha(u) + \sin(\rho\pi)u\frac{d}{du}J_\alpha(u)}{\cos(\rho\pi)H^{(1)}_\alpha(u) + \sin(\rho\pi)u\frac{d}{du}H^{(1)}_\alpha(u)}. \tag{21}$$

In the limit of high energies of scattered particles, $kr_c \gg 1$, the S-matrix unitarity condition becomes

$$\frac{1}{\mathrm{i}}\sqrt{\frac{k}{2\pi}}\cos(\mu\pi)\left[f_c(k,\,\varphi'-\varphi'') - f_c^*(k,\,\varphi''-\varphi')\right] + 2\sin^2(\mu\pi)\Delta^{(\nu)}_{kr_c}(\varphi'-\varphi'') + O(\sqrt{kr_c}) = \frac{k}{2\pi}\int\limits_{-\pi}^{\pi} d\varphi\, f_c^*(k,\,\varphi-\varphi')f_c(k,\,\varphi-\varphi''), \tag{22}$$

where f_c is given by the second line in (20), and

$$\Delta^{(\nu)}_x(\varphi) = \frac{1}{2\pi}\sum_{|n-\mu|\le x} e^{in\varphi} \tag{23}$$

is the regularized (smoothed) angular delta-function,

$$\lim_{x\to\infty}\Delta^{(\nu)}_x(\varphi) = \Delta(\varphi), \qquad \Delta^{(\nu)}_x(0) = \frac{x}{\pi}.$$

The optical theorem in this limit takes form

$$2\sqrt{\frac{2\pi}{k}}\cos(\mu\pi)\mathrm{Im}\, f_c(k,\,0) + \frac{4\pi}{k}\sin^2(\mu\pi)\Delta^{(\nu)}_{kr_c}(0) + O(k^{-1}) = \sigma, \tag{24}$$

where

$$\sigma = \int\limits_{-\pi}^{\pi} d\varphi |f_c(k,\,\varphi)|^2 \tag{25}$$

is the total cross section in the high-energy limit, $kr_c \gg 1$.

The scattering amplitude in the high-energy limit is shown to be given by expression (see also [7]):

$$\begin{aligned} f_c(k,\,\varphi) = {} & \mathrm{i}\sqrt{\frac{2\pi}{k}}\left[\cos(\mu\pi)\Delta^{(\nu)}_{kr_c}(\varphi) - \sin(\mu\pi)\Gamma^{(\nu)}_{kr_c}(\varphi)\right] \\ & - \sqrt{\frac{r_c}{2}|\sin(\varphi/2)|}\exp\{-2\mathrm{i}kr_c|\sin(\varphi/2)| \\ & + \mathrm{i}\mu\left[\varphi\ - \mathrm{sgn}(\varphi)\pi\right]\}\exp\left\{-\mathrm{i}\left[2\chi^{(\rho)}(kr_c,\varphi) + \pi/4\right]\right\} \\ & + \sqrt{r_c}O\left[(kr_c)^{-1/6}\right],\ kr_c \gg 1, \end{aligned} \tag{26}$$

where

$$\Gamma^{(\nu)}_x(\varphi) = \frac{1}{2\pi\mathrm{i}}\sum_{|n-\mu|\le x}\mathrm{sgn}(n-\mu)e^{\mathrm{i}n\varphi}, \tag{27}$$

and

$$\chi^{(\rho)}(kr_c,\varphi) = \arctan\left[\frac{2kr_c|\sin^3(\varphi/2)|}{2\cot(\rho\pi)\sin^2(\varphi/2) - 1}\right]. \tag{28}$$

The explicit form of $\Delta^{(\nu)}_{kr_c}(\varphi)$ (23) and $\Gamma^{(\nu)}_{kr_c}(\varphi)$ (27) is as follows:

$$\Delta^{(\nu)}_{kr_c}(\varphi) = \frac{e^{\mathrm{i}(\nu+\frac{1}{2})\varphi}}{2\pi}\frac{\sin(s_c\varphi)}{\sin(\varphi/2)} \tag{29}$$

and

$$\Gamma^{(\nu)}_{kr_c}(\varphi) = \frac{e^{\mathrm{i}(\nu+\frac{1}{2})\varphi}}{2\pi}\frac{1-\cos(s_c\varphi)}{\sin(\varphi/2)} \tag{30}$$

in the case

$$[\![kr_c+\mu]\!] - \nu = [\![kr_c-\mu]\!] + \nu + 1 = s_c, \tag{31}$$

or

$$\Delta^{(\nu)}_{kr_c}(\varphi) = \frac{e^{\mathrm{i}(\nu+\frac{1}{2}\mp\frac{1}{2})\varphi}}{2\pi}\frac{\sin\left[\left(s_c+\frac{1}{2}\right)\varphi\right]}{\sin(\varphi/2)} \tag{32}$$

and

$$\Gamma^{(\nu)}_{kr_c}(\varphi) = \frac{e^{\mathrm{i}(\nu+\frac{1}{2}\mp\frac{1}{2})\varphi}}{2\pi}\left\{\frac{1-\cos\left[\left(s_c+\frac{1}{2}\right)\varphi\right]}{\sin(\varphi/2)} - \tan(\varphi/4) \pm \mathrm{i}\right\} \tag{33}$$

in the case

$$[\![kr_c+\mu]\!] - \nu - \frac{1}{2} \pm \frac{1}{2} = [\![kr_c-\mu]\!] + \nu + \frac{1}{2} \mp \frac{1}{2} = s_c. \tag{34}$$

In the strictly forward direction we get:

$$f_c(k,\,0) = \mathrm{i}\sqrt{\frac{2k}{\pi}}\,r_c\cos(\mu\pi) + O(k^{-1/2}). \tag{35}$$

The Fraunhofer diffraction on the vortex in the forward direction is described by the first line in (26), while the classical reflection from the vortex in other directions is described by the second two lines; evidently, (35) is due to the Fraunhofer diffraction, since the classical reflection in the strictly forward direction is absent.

It should be noted that the left-hand side of (24) in the nonvanishing order involves the contribution of the diffraction peak only, whereas the right-hand side of (24) includes the contribution of the classical reflection as well. We show that the total cross section in the high-energy limit (25) is independent of the vortex flux, as well as of the choice of the boundary condition from the variety of the Robin ones:

$$\sigma = 4r_c + O(k^{-1}), \tag{36}$$

this is twice the classical total cross section, the latter been equal to $2r_c$. Thus, the contribution of the diffraction peak to the total cross section is flux independent and is equal to that of classical reflection.

4. CONCLUSION

We conclude that the flux of an impermeable magnetic vortex serves as a gate for the propagation of high-energy (almost classical) particles: the penetration of the particles in the strictly forward direction, see (15) and (35), is maximally possible at integer values of μ (i.e. for the flux of even number of the Abrikosov vortices) and is completely absent at half-integer values of μ (i.e. for the flux of odd number of the Abrikosov vortices).

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ЭФФЕКТ ААРОНОВА-БОМА: АНАЛИЗ РАССЕЯНИЯ В НАПРАВЛЕНИИ ВПЕРЕД

Н.Д. Власий

Рассматривается давно существующая проблема расходимости амплитуды рассеяния Ааронова-Бома в направлении вперед. Мы показываем, что эта расходимость не имеет физических последствий, являясь искусственным следствием приближения, которое не учитывает поперечных размеров магнитного вихря. Как только поперечные размеры вихря принимаются во внимание, эта расходимость исчезает, но, однако, в некотором смысле проявляется как пик фраунгоферовской дифракции в направлении вперед. Пик становится более выраженным в пределе высоких энергий рассеиваемых частиц.

ЕФЕКТ ААРОНОВА-БОМА: АНАЛІЗ РОЗСІЯННЯ В НАПРЯМКУ ВПЕРЕД

Н.Д. Власій

Розглядається давно існуюча проблема розбіжності амплітуди розсіяння Ааронова-Бома в напрямку вперед. Ми показуємо, що ця розбіжність не має фізичних наслідків, оскільки сама є штучним наслідком наближення, котре нехтує поперечними розмірами магнітного вихора. Як тільки поперечні розміри вихора враховуються, ця розбіжність зникає, проте, все ж таки, певним чином проявляється як пік фраунгоферової дифракції в напрямку вперед. Пік стає більш виразним в границі високих енергій розсіюваних частинок.