Journal of Mathematical Physics, Analysis, Geometry
2013, vol. 9, No. 4, pp. 435–447

Surfaces Given with the Monge Patch in $\mathbb{E}^4$

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Received May 24, 2012

In the present paper we consider the surfaces in the Euclidean 4-space $\mathbb{E}^4$ given with a Monge patch $z = f(u, v), w = g(u, v)$ and study the curvature properties of these surfaces. We also give some special examples of these surfaces first defined by Yu. Aminov. Finally, we prove that every Aminov surface is a non-trivial Chen surface.

Key words: Monge patch, translation surface, Chen surface.

Mathematics Subject Classification 2010: 53C40, 53C42.

1. Introduction

In recent years there has been a tremendous increase in computer vision research with using range images (or depth maps) as sensor input data [3]. The most attractive feature of range images is the explicitness of the surface information. Many industrial and navigational robotic tasks will be more easily accomplished if such explicit depth information can be efficiently obtained and interpreted. Classical differential geometry provides a complete local description of smooth surfaces [4, 13]. The first and second fundamental forms of surfaces provide a set of differential-geometric shape descriptors that capture domain-independent surface information. Gaussian curvature is an intrinsic surface property which refers to an isometric invariant of a surface [4]. Both Gaussian and mean curvatures have the attractive characteristics of translational and rotational invariance. A depth surface is a range image observed from a single view which can be represented by a digital graph (Monge patch) surface. That is, a depth or a range value at a point (u, v) is given by a single valued function $z(u, v)$.

One interesting class of the surfaces in $\mathbb{E}^3$ is that of the translation surfaces, which can be parameterized locally as $z(u, v) = f(u) + g(v)$, where f and g are smooth functions. From the definition, it is clear that translation surfaces are double curved surfaces. Therefore, translation surfaces are made up of quadrilateral, that is, four sided facets. Because of this property, translation surfaces are used in architecture to design and construct free-form glass roofing structures, see [9]. Scherk's surface, obtained by H. Scherk in 1835, is the only non flat minimal surface that can be represented as a translation surface [16]. Translation surfaces have been studied from the various viewpoints by many differential geometers. L. Verstraelen, J. Walrave and S. Yaprak studied minimal translation surfaces in n-dimensional Euclidean spaces [17].

In [5], B.Y. Chen defined the allied vector field $a(v)$ of a normal vector field v. In particular, the allied mean curvature vector field is orthogonal to H. Further, B.Y. Chen defined the $\mathcal{A}$-surfaces to be the surfaces for which $a(H)$ vanishes identically. These surfaces are also called Chen surfaces [10]. The class of Chen surfaces contains all minimal and pseudo-umbilical surfaces and also all surfaces for which $dimN_1 \leq 1$, in particular, all hypersurfaces. These Chen surfaces are said to be trivial $\mathcal{A}$-surfaces [11]. In [15], B. Rouxel considered the ruled Chen surfaces in $\mathbb{E}^n$. For more details, see also [6] and [12].

The paper is organized as follows: Section 2 gives some basic concepts of the surfaces in $\mathbb{E}^4$. Section 3 says about the surfaces given with a Monge patch in $\mathbb{E}^4$. Further this section provides some basic properties of the surfaces in $\mathbb{E}^4$ and the structure of their curvatures. In the third section we consider Aminov surfaces given with the Monge patch in $\mathbb{E}^4$. We also present some examples of these surfaces. We obtain a few new interesting results. Namely, we obtain some equations on $r(u)$, when on M the equation $K + K_N = 0$ takes place. Then we obtain the condition for the case when M is a Wintgen ideal surface. We remark that on the Wintgen ideal surfaces the equation $K + K_N = \|H\|^2$ takes place. In the final section we obtain an important equation on the coefficients of the second quadratic form for Chen surfaces, when it is given at arbitrary parametrization. We also proved that every Aminov surface in $\mathbb{E}^4$ is a non-trivial Chen surface.

2. Basic Concepts

Let M be a smooth surface in $\mathbb{E}^4$ given with the patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M is spanned by $\{X_u, X_v\}$. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle, \tag{1}$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e., the surface patch $X(u,v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^4 = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of T_pM in $\mathbb{E}^4$. Let $\widetilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^4$. Given is any local vector field X_i, X_j tangent to M.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Consider the second fundamental map: $h : \chi(M) \times \chi(M) \to \chi^{\perp}(M)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j \ 1 \leq i, j \leq 2, \tag{2}$$

where ∇ is the induced Levi–Civita connection. The map is well-defined, symmetric and bilinear.

For any arbitrary orthonormal normal frame field $\{N_1, N_2\}$ of M, recall the shape operator $A : \chi^{\perp}(M) \times \chi(M) \to \chi(M)$;

$$A_{N_i} X_i = -(\widetilde{\nabla}_{X_i} N_i)^T, \quad X_i \in \chi(M). \tag{3}$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_k} X_i, X_j \rangle = \langle h(X_i, X_j), N_k \rangle = c_{ij}^k, \ 1 \leq i, j, k \leq 2. \tag{4}$$

Equation (2) is called the Gaussian formula, and

$$h(X_i, X_j) = \sum_{k=1}^{2} c_{ij}^k N_k, \qquad 1 \leq i, j \leq 2, \tag{5}$$

where c_{ij}^k are the coefficients of the second fundamental form.

Further, the *Gaussian curvature* and the *Gaussian torsion* of a regular patch $X(u, v)$ are given by

$$K = \frac{1}{W^2} \sum_{k=1}^{2} (c_{11}^k c_{22}^k - (c_{12}^k)^2) \tag{6}$$

and

$$K_N = \frac{1}{W^2} \left(E \left(c_{12}^1 c_{22}^2 - c_{12}^2 c_{22}^1 \right) - F \left(c_{11}^1 c_{22}^1 - c_{11}^2 c_{22}^1 \right) + G \left(c_{11}^1 c_{12}^2 - c_{11}^2 c_{12}^1 \right) \right), \tag{7}$$

respectively.

Further, the mean curvature vector of a regular patch $X(u, v)$ is defined by

$$\mathbf{H} = \frac{1}{2W^2} \sum_{k=1}^{2} (c_{11}^k G + c_{22}^k E - 2 c_{12}^k F) N_k. \tag{8}$$

Recall that a surface M is said to be *minimal* if its mean curvature vector vanishes identically [5].

The surface patch $X(u, v)$ is called *pseudo-umbilical* if the shape operator with respect to H is proportional to the identity (see [5]).

3. Surfaces Given with a Monge Patch in $\mathbb{E}^4$

The 2-dimensional surfaces in $\mathbb{E}^4$ are an interesting object for study. Here we have some difficult problems to be solved. For example, it is unknown whether there exists an isometric regular immersion of the whole Lobachevsky plane into $\mathbb{E}^4$. Hence the studying of various classes of surfaces in $\mathbb{E}^4$ from the point of view of the influence of the principal invariants — Gauss curvature K, Gauss torsion K_N and the vector of the mean curvature $\mathbf{H}$ on the behavior of surfaces, still remains actual.

In the paper, we use the representation of the surfaces in the explicit form

$$r(u,v) = (u, v, f(u,v), g(u,v)), \tag{9}$$

where f and g are some smooth functions. The parametrization (9) is called the *Monge patch in* $\mathbb{E}^4$.

First we obtain the following result.

Theorem 3.1. *Let M be a smooth surface given with the Monge patch (9). Then the mean curvature vector of M becomes*

$$\begin{aligned} \mathbf{H} &= \frac{1}{2\sqrt{A}W^2}(Gf_{uu} - 2Ff_{uv} + Ef_{vv})N_1 + \frac{1}{2\sqrt{A}W^3}\left(G(-Bf_{uu} + Ag_{uu})\right. \\ &\quad \left. -2F(-Bf_{uv} + Ag_{uv}) + E(-Bf_{vv} + Ag_{vv})\right) N_2, \end{aligned} \tag{10}$$

where

$$\begin{aligned} A &= 1 + (f_u)^2 + (f_v)^2, \\ B &= f_u g_u + f_v g_v \ , \\ C &= 1 + (g_u)^2 + (g_v)^2 \end{aligned} \tag{11}$$

such that $EG - F^2 = AC - B^2$.

P r o o f. The tangent space of M is spanned by the vector fields

$$\begin{aligned} \frac{\partial X}{\partial u} &= (1, 0, f_u, g_u), \\ \frac{\partial X}{\partial v} &= (0, 1, f_v, g_v). \end{aligned}$$

Hence the coefficients of the first fundamental form of the surface are:

$$\begin{aligned} E &= \langle X_u(u,v), X_u(u,v)\rangle = 1 + (f_u)^2 + (g_u)^2, \\ F &= \langle X_u(u,v), X_v(u,v)\rangle = f_u f_v + g_u g_v, \\ G &= \langle X_v(u,v), X_v(u,v)\rangle = 1 + (f_v)^2 + (g_v)^2, \end{aligned} \tag{12}$$

where $\langle , \rangle$ is the standard scalar product in $\mathbb{R}^4$.

The second partial derivatives of $X(u, v)$ are expressed as follows:

$$\begin{aligned} X_{uu}(u,v) &= (0,0,f_{uu},g_{uu}), \\ X_{uv}(u,v) &= (0,0,f_{uv},g_{uv}), \\ X_{vv}(u,v) &= (0,0,f_{vv},g_{vv}). \end{aligned} \tag{13}$$

Further, the normal space of M is spanned by the vector fields

$$\begin{aligned} N_1 &= \frac{1}{\sqrt{A}}(-f_u,-f_v,1,0), \\ N_2 &= \frac{1}{W\sqrt{A}}(Bf_u - Ag_u, Bf_v - Ag_v, -B, A). \end{aligned} \tag{14}$$

Using (4), (13) and (14), we can calculate the coefficients of the second fundamental form h as follows:

$$\begin{aligned} c_{11}^1 &= \langle X_{uu}(u,v), N_1\rangle = \frac{f_{uu}}{\sqrt{A}}, \\ c_{12}^1 &= \langle X_{uv}(u,v), N_1\rangle = \frac{f_{uv}}{\sqrt{A}}, \\ c_{22}^1 &= \langle X_{vv}(u,v), N_1\rangle = \frac{f_{vv}}{\sqrt{A}}, \\ c_{11}^2 &= \langle X_{uu}(u,v), N_2\rangle = \frac{-Bf_{uu}+Ag_{uu}}{W\sqrt{A}}, \\ c_{12}^2 &= \langle X_{uv}(u,v), N_2\rangle = \frac{-Bf_{uv}+Ag_{uv}}{W\sqrt{A}}, \\ c_{22}^2 &= \langle X_{vv}(u,v), N_2\rangle = \frac{-Bf_{vv}+Ag_{vv}}{W\sqrt{A}}. \end{aligned} \tag{15}$$

Then, substituting (12) and (15) into (8), we get (10). This completes the proof of the theorem. ■

In [2], Yu. Aminov proved the following result.

Theorem 3.2. [2] *Let M be a smooth surface given with the Monge patch* (9). *Then the Gaussian curvature K and the Gaussian torsion K_N of M become*

$$K = \frac{C(f_{uu}f_{vv} - f_{uv}^2) - B(f_{uu}g_{vv} + g_{uu}f_{vv} - 2f_{uv}g_{uv}) + A(g_{uu}g_{vv} - g_{uv}^2)}{W^4} \tag{16}$$

and

$$K_N = \frac{E(f_{uv}g_{vv} - g_{uv}f_{vv}) - F(f_{uu}g_{vv} - g_{uu}f_{vv}) + G(f_{uu}g_{uv} - g_{uu}f_{uv})}{W^4}, \tag{17}$$

respectively.

Proposition 3.1. *Let M be a smooth surface given with the Monge patch of the form*

$$\begin{aligned} f(u,v) &= \phi_u(u,v), \\ g(u,v) &= \phi_v(u,v). \end{aligned} \tag{18}$$

Then the Gaussian curvature K coincides with the Gaussian torsion K_N of M.

P r o o f. Suppose M is a smooth surface given with the Monge patch (9). Then, using (11) and (12), we get

$$\begin{aligned} E &= A = 1 + (\phi_{uu})^2 + (\phi_{uv})^2, \\ F &= B = \phi_{uu}\phi_{uv} + \phi_{uv}\phi_{vv} \ , \\ G &= C = 1 + (\phi_{uv})^2 + (\phi_{vv})^2. \end{aligned} \tag{19}$$

Furthermore, substituting (18) into (16), (17) and using partial derivatives of the functions given in (18), we obtain $K = K_N$. ■

E x a m p l e 3.1. For the surface M given with the Monge patch

$$\begin{aligned} f(u,v) &= \phi_u(u,v) = e^u \cos v, \\ g(u,v) &= \phi_v(u,v) = -e^u \sin v, \end{aligned}$$

the Gaussian curvature K coincides with the Gaussian torsion K_N of M [1].

Definition 3.1. *The surface given by the parametrization* (9),

$$f(u,v) = f_3(u) + g_3(v), \ \ g\ (u,v) = f_4(u) + g_4(v), \tag{20}$$

is called a translation surface in the Euclidean 4-space $\mathbb{E}^4$ [7].

In the case (20) we obtain simple expressions for K, K_N and $\mathbf{H}$. As a consequence of Theorems 1 and 2, we get the following results.

Corollary 3.1. *Let M be a translation surface given with the Monge patch* (20)*. Then the Gaussian curvature K and the Gaussian torsion K_N of M become*

$$K = \frac{f_3''(u)g_3''(v)C - (f_3''(u)g_4''(v) + f_4''(u)g_3''(v))B + f_4''(u)g_4''(v)A}{W^4}$$

and

$$K_N = \frac{F(f_4''(u)g_3''(v) - f_3''(u)g_4''(v))}{W^4},$$

respectively, where

$$\begin{aligned} E &= 1+(f_3'(u))^2+(f_4'(u))^2, \\ F &= f_3'(u)g_3'(v)+f_4'(u)g_4'(v), \\ G &= 1+(g_3'(v))^2+(g_4'(v))^2 \end{aligned}$$

and

$$\begin{aligned} A &= 1+(f_3'(u))^2+(g_3'(v))^2, \\ B &= f_3'(u)f_4'(u)+g_3'(v)g_4'(v), \\ C &= 1+(f_4'(u))^2+(g_4'(v))^2. \end{aligned}$$

Corollary 3.2. *Let M be a translation surface given with the Monge patch* (20)*. Then the mean curvature vector of M becomes*

$$\mathbf{H}=\frac{f_3''(u)G+g_3''(v)E}{2\sqrt{A}W^2}N_1+\frac{G(f_4''(u)A-f_3''(u)B)+E(g_4''(v)A-g_3''(v)B)}{2\sqrt{A}W^3}N_2.$$

E x a m p l e 3.2. The translation surface given with the surface patch of

$$X(u,v)=(u,v,u^2+v^2,u^2-v^2)$$

has the vanishing Gaussian curvature and the Gaussian torsion [2].

Theorem 3.3. [7] *Let M be a translation surface in $\mathbb{E}^4$. Then M is minimal if and only if either M is a plane or*

$$\begin{aligned} f_k(u) &= \frac{c_k}{c_3^2+c_4^2}\left(\log\left|\cos(\sqrt{a}u)\right|+cu\right)+e_k u, \\ g_k(v) &= \frac{c_k}{c_3^2+c_4^2}\left(-\log\left|\cos(\sqrt{b}v)\right|+dv\right)+p_k v,\ \ k=3,4, \end{aligned}$$

where c_k, e_k, p_k, $a>0, b>0, c, d$ are real constants.

4. Aminov Surfaces in $\mathbb{E}^4$

In the present section we consider the surfaces M with

$$f(u,v)=r(u)\cos v,\ \ g\ (u,v)=r(u)\sin v, \tag{21}$$

earlier considered in [1]. We call these surfaces *Aminov surfaces* in the Euclidean 4-space $\mathbb{E}^4$.

As a consequence of Theorem 2, we get the following result.

Corollary 4.1. *Let M be an Aminov surface given with the Monge patch* (21)*. Then the Gaussian curvature K and the Gaussian torsion K_N of M become*

$$K = -\frac{r(u)r''(u)(1+r^2(u)) + (r'(u))^2(1+(r'(u))^2)}{(1+r^2(u))^2(1+(r'(u))^2)^2} \tag{22}$$

and

$$K_N = \frac{r'(u)r''(u)(1+r^2(u)) + r(u)r'(u)(1+(r'(u))^2)}{(1+r^2(u))^2(1+(r'(u))^2)^2}, \tag{23}$$

respectively.

Proposition 4.1. *Let M be an Aminov surface given with the Monge patch* (21)*. If $K + K_N = 0$, then the equality*

$$(r(u) - (r'(u))\left((r'(u)(1+(r'(u))^2) - r''(u)(1+r^2(u))\right) = 0 \tag{24}$$

holds.

P r o o f. Using (22) and (23), we get the result. ■

As a consequence of Proposition 8, we can give the following example.

E x a m p l e 4.1. The Aminov surface given with the surface patch of

$$X(u,v) = (u, v, \lambda e^u \cos v, \lambda e^u \sin v) \tag{25}$$

satisfies the relation $K + K_N = 0$.

As a consequence of Theorem 1, we get the following results.

Proposition 4.2. *Let M be an Aminov surface given with the Monge patch* (21)*. Then the mean curvature vector of M becomes*

$$\mathbf{H} = \frac{(Gr''(u) - Er(u))}{2W^2\sqrt{A}}\left\{\cos v N_1 + \left(\frac{A\sin v - B\cos v}{W}\right)N_2\right\}, \tag{26}$$

where

$$\begin{aligned} A &= 1 + (r'(u))^2\cos^2 v + r^2(u)\sin^2 v, \\ B &= \left((r'(u))^2 - r^2(u)\right)\cos v \sin v, \\ C &= 1 + (r'(u))^2\sin^2 v + r^2(u)\cos^2 v \end{aligned}$$

and

$$\begin{aligned} E &= 1 + (r'(u))^2, \\ F &= 0, \\ G &= 1 + r^2(u) \end{aligned}$$

such that $EG - F^2 = AC - B^2$.

Corollary 4.2. *Let M be an Aminov surface given with the Monge patch* (21). *Then the mean curvature of M becomes*

$$H = \frac{r''(u)(1 + r^2(u)) - r(u)(1 + (r'(u))^2)}{2(1 + r^2(u))(1 + (r'(u))^2)^{3/2}}. \tag{27}$$

Corollary 4.3. *Let M be an Aminov surface given with the Monge patch* (21). *If M is minimal, then*

$$r(u) = \frac{1}{2a}\left(a^2 e^{\pm\frac{2(u+b)}{a}} + a^2 - 1\right) e^{\pm\frac{(u+b)}{a}}, \tag{28}$$

where a and b are real constants.

P r o o f. Suppose that M is minimal, then using equality (27), we get

$$r''(u)(1 + r^2(u)) - r(u)(1 + (r'(u))^2) = 0. \tag{29}$$

By using Maple calculations, it is easy to show that (28) is a non-trivial solution of (29). ■

Definition 4.1. *A surface M is said to be the Wintgen ideal surface in $\mathbb{E}^4$ if the equality*

$$K + |K_N| = \|\mathbf{H}\|^2 \tag{30}$$

holds [18].

We obtain the following result.

Theorem 4.1. *Let M be an Aminov surface given with the Monge patch (21). If M is the Wintgen ideal surface, then the equality*

$$2r''(1+r^2)(1+(r')^2)(2r'-r)+(1+(r')^2)^2(4rr'-4(r')^2-r^2)-(r'')^2(1+r^2)^2 = 0 \tag{31}$$

holds.

P r o o f. Substituting (22), (23) and (27) into (30), we get (31). ■

5. Chen Surfaces in $\mathbb{E}^4$

Let M be a smooth surface in $\mathbb{E}^4$ given with the patch $X(u, v) : (u, v) \in D \subset \mathbb{E}^2$. If we chose an orthonormal tangent frame field $\{X, Y\}$

$$\begin{aligned} X &= \frac{X_u}{\sqrt{E}}, \\ Y &= \frac{\sqrt{E}}{W}\left(X_v - \frac{F X_u}{E}\right), \end{aligned} \tag{32}$$

then the coefficients of the second fundamental form are given by

$$\begin{aligned} h_{11}^{\alpha} &= \langle h(X,X), N_{\alpha}\rangle = \frac{c_{11}^{\alpha}}{E},\ 1 \leq \alpha \leq 2, \\ h_{12}^{\alpha} &= \langle h(X,Y), N_{\alpha}\rangle = \frac{1}{W}\left(c_{12}^{\alpha} - \frac{F}{E}c_{11}^{\alpha}\right), \\ h_{22}^{\alpha} &= \langle h(Y,Y), N_{\alpha}\rangle = \frac{1}{W^2}\left(Ec_{22}^{\alpha} - 2Fc_{12}^{\alpha} + \frac{F^2}{E}c_{11}^{\alpha}\right). \end{aligned} \tag{33}$$

Further, the shape operator matrix of the surface $M \subset \mathbb{E}^4$ becomes

$$A_{N_{\alpha}} = \begin{pmatrix} h_{11}^{\alpha} & h_{12}^{\alpha} \\ h_{12}^{\alpha} & h_{22}^{\alpha} \end{pmatrix}.$$

Hence, the mean curvature vector of a regular patch $X(u,v)$ is defined by

$$\begin{aligned} \mathbf{H} &= \frac{1}{2}(tr(A_{N_1}) + tr(A_{N_2})) \\ &= H_1N_1 + H_2N_2, \end{aligned} \tag{34}$$

where the functions

$$H_1 = \frac{1}{2}(h_{11}^1 + h_{22}^1),\ H_2 = \frac{1}{2}(h_{11}^2 + h_{22}^2) \tag{35}$$

are called the *first and second harmonic curvatures of* M, respectively.

Chose an orthonormal normal frame field N_1, N_2 of M such that the vector field N_1 is parallel to the mean curvature vector $\mathbf{H}$. In [5], B.-Y. Chen defined the allied vector field $a(\mathbf{H})$ of the mean curvature vector field $\mathbf{H}$ by the formula

$$a(\mathbf{H}) = \frac{\|\mathbf{H}\|}{2}\left\{tr(A_{N_1}A_{N_2})\right\}N_2. \tag{36}$$

In particular, the allied mean curvature vector field of the mean curvature vector $\mathbf{H}$ is a well-defined normal vector field orthogonal to $\mathbf{H}$. If the allied mean vector $a(\mathbf{H})$ vanishes identically, then the surface M is called the *A-surface* of $\mathbb{E}^4$. Furthermore, $\mathcal{A}$-surfaces are also called *Chen surfaces* [10]. The class of Chen surfaces contains all minimal and pseudo-umbilical surfaces and also all surfaces for which $dim N_1 \leq 1$, in particular, all hypersurfaces. These Chen surfaces are said to be trivial $\mathcal{A}$-surfaces [11].

Theorem 5.1. *Let M be a smooth surface in $\mathbb{E}^4$ given with the patch $X(u,v)$: $(u,v) \in D \subset \mathbb{E}^2$. Then M is a non-trivial Chen surfaces if and only if*

$$\begin{aligned} &\left((h_{11}^1)^2 - (h_{11}^2)^2 + (h_{22}^1)^2 - (h_{22}^2)^2 + 2(h_{12}^1)^2 - 2(h_{12}^2)^2\right)H_1H_2 \\ &\quad + \left(h_{11}^1h_{11}^2 + h_{22}^1h_{22}^2 + 2h_{12}^1h_{12}^2\right)(H_2^2 - H_1^2) = 0 \end{aligned} \tag{37}$$

holds, where H_1 and H_2 are the first and second harmonic curvatures of M as defined before.

P r o o f. Suppose M is a non-minimal surface in $\mathbb{E}^4$. Then we can construct another orthonormal normal frame field

$$\widetilde{N}_1 = \frac{H_1 N_1 + H_2 N_2}{\sqrt{H_1^2 + H_2^2}}, \ \widetilde{N}_2 = \frac{H_2 N_1 - H_1 N_2}{\sqrt{H_1^2 + H_2^2}} \tag{38}$$

such that $\widetilde{N}_1$ is parallel to $\mathbf{H}$.

Furthermore, with respect to this frame we can obtain

$$\begin{aligned}
\widetilde{h}_{11}^1 &= \left\langle h(X,X), \widetilde{N}_1 \right\rangle = \frac{H_1 h_{11}^1 + H_2 h_{11}^2}{\sqrt{H_1^2 + H_2^2}}, \\
\widetilde{h}_{12}^1 &= \left\langle h(X,Y), \widetilde{N}_1 \right\rangle = \frac{H_1 h_{12}^1 + H_2 h_{12}^2}{\sqrt{H_1^2 + H_2^2}}, \\
\widetilde{h}_{22}^1 &= \left\langle h(Y,Y), \widetilde{N}_1 \right\rangle = \frac{H_1 h_{22}^1 + H_2 h_{22}^2}{\sqrt{H_1^2 + H_2^2}}, \\
\widetilde{h}_{11}^2 &= \left\langle h(X,X), \widetilde{N}_2 \right\rangle = \frac{H_2 h_{11}^1 - H_1 h_{11}^2}{\sqrt{H_1^2 + H_2^2}}, \\
\widetilde{h}_{12}^2 &= \left\langle h(X,Y), \widetilde{N}_2 \right\rangle = \frac{H_2 h_{12}^1 - H_1 h_{12}^2}{\sqrt{H_1^2 + H_2^2}}, \\
\widetilde{h}_{22}^2 &= \left\langle h(Y,Y), \widetilde{N}_2 \right\rangle = \frac{H_2 h_{22}^1 - H_1 h_{22}^2}{\sqrt{H_1^2 + H_2^2}}.
\end{aligned} \tag{39}$$

So, the shape operator matrices of M with respect to $\widetilde{N}_1$ and $\widetilde{N}_2$ become

$$A_{\widetilde{N}_1} = \begin{pmatrix} \frac{H_1 h_{11}^1 + H_2 h_{11}^2}{\sqrt{H_1^2+H_2^2}} & \frac{H_1 h_{12}^1 + H_2 h_{12}^2}{\sqrt{H_1^2+H_2^2}} \\ \frac{H_1 h_{12}^1 + H_2 h_{12}^2}{\sqrt{H_1^2+H_2^2}} & \frac{H_1 h_{22}^1 + H_2 h_{22}^2}{\sqrt{H_1^2+H_2^2}} \end{pmatrix}, \ A_{\widetilde{N}_2} = \begin{pmatrix} \frac{H_2 h_{11}^1 - H_1 h_{11}^2}{\sqrt{H_1^2+H_2^2}} & \frac{H_2 h_{12}^1 - H_1 h_{12}^2}{\sqrt{H_1^2+H_2^2}} \\ \frac{H_2 h_{12}^1 - H_1 h_{12}^2}{\sqrt{H_1^2+H_2^2}} & \frac{H_2 h_{22}^1 - H_1 h_{22}^2}{\sqrt{H_1^2+H_2^2}} \end{pmatrix}, \tag{40}$$

respectively.

Suppose M is a non-trivial Chen surface. Then, by the definition, $tr\left(A_{\widetilde{N}_1} A_{\widetilde{N}_2}\right) = 0$. Using (40), we get the result, that is, equality (37).

Conversely, if equality (37) holds, then $tr\left(A_{\widetilde{N}_1} A_{\widetilde{N}_2}\right) = 0$. So, M is a non-trivial Chen surface. ∎

We obtain the following result.

Theorem 5.2. *Let M be an Aminov surface in $\mathbb{E}^4$ given with the Monge patch* (21)*. Then M is a non-trivial Chen surface.*

P r o o f. Suppose M is an Aminov surface in $\mathbb{E}^4$ given with the parametrization (21). By the use of (15) with (33) a simple calculation gives

$$\begin{aligned} h_{11}^1 &= \frac{r''(u)\cos v}{\varphi\psi^2}, \ h_{12}^1 = \frac{-r'(u)\sin v}{\varphi\psi\omega}, \\ h_{22}^1 &= \frac{-r(u)\cos v}{\varphi\omega^2}, \ h_{11}^2 = \frac{\omega r''(u)\sin v}{\varphi\psi^3}, \\ h_{12}^2 &= \frac{r'(u)\cos v}{\varphi\omega^2}, \ h_{22}^2 = \frac{-r(u)\sin v}{\varphi\psi\omega}, \end{aligned} \tag{41}$$

where φ, ψ and ω are differentiable functions defined by

$$\begin{aligned} \varphi &= \sqrt{1+(r'(u))^2\cos^2 v+(r(u))^2\sin^2 v}, \\ \psi &= \sqrt{1+(r'(u))^2}, \\ \omega &= \sqrt{1+(r(u))^2}. \end{aligned} \tag{42}$$

Substituting (41) into (37), we get the result. ∎

Acknowledgements. We thank Prof. Dr. Yuriy Aminov, B. Verkin Institute for Low Temperature Physics and Engineering of National Academy of Sciences of Ukraine, for his useful comments and for encouragement during the study.

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