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Classical Solution of a Degenerate Elliptic-Parabolic Free Boundary Problem

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A free boundary problem describing a filtration process in a porous medium is considered. An unknown interface divides the filtration domain into elliptic and parabolic regions. In the parabolic region the governing equation is degenerate. The existence of a smooth solution in the weighted Hölder space is proved.

Key words: elliptic and parabolic equations, free boundary problems, classical solutions, a priori estimates.

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To the memory of our teacher I.I. Danyljuk

1. Statement of the Problem

In this paper we study the following problem. Let $\Omega = (0, l) \subset R^1$, $\Omega_T = \Omega \times (0, T)$,

$$\frac{\partial c(u)}{\partial t} - \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \quad \Omega_T, \tag{1.1}$$

$$u(0, t) = g_0(t) \quad \text{on} \quad [0, T], \tag{1.2}$$

$$u(l, t) = g(t) \quad \text{on} \quad [0, T], \tag{1.3}$$

$$c(u(y, 0)) = c(u_0(y)) \quad \text{on} \quad [0, l]. \tag{1.4}$$

Here $c(\eta)$ is a continuous function that is strictly increasing for $\eta > 0$ and $c(\eta) = 0$ for $\eta \leq 0$, $g_0(t)$, $g(t)$, $u_0(y)$ are given functions, $g_0(t) < 0$ and $g(t) > 0$ $\forall t \in [0, T]$.

Problem (1.1)–(1.4) arises in the theory of fluid flow through a partially saturated porous media (see [4–8, 10, 11, 13] and the references therein). The level set $\{u = 0\}$ splits the domain Ω_T in two regions in which equation (1.1) is respectively parabolic and elliptic. It was shown in [11, 7] that under appropriate conditions on the data there exists a function $y = s(t)$ for which

$$u(y,t) \le 0, \quad c(u(y,t)) = 0 \quad \text{for} \quad 0 \le y \le s(t),$$

and

$$u(y,t) > 0, \quad c(u(y,t)) > 0 \quad \text{for} \quad s(t) < y \le l.$$

In this problem the function $s(t)$ is the free (unknown) boundary, and our main concern is to describe the qualitative properties of $s(t)$. It was shown in [4] that under the conditions $c(\eta) \in C(R) \cap C^{2+\beta}(R^+)$, $c'(+0) = 0$, and $s(t) < l$ the function $s(t)$ is continuously differentiable. Contrary to [4], we will study the case when

$$c(\eta) = \begin{cases} \eta^{1-\alpha}, & \eta \ge 0, \quad \alpha \in (0,1), \\ 0, & \eta \le 0. \end{cases} \tag{1.5}$$

This situation leads to the additional singularity in the free boundary problem.

In the region where the medium is saturated, $y \in (0, s(t))$, we have

$$u(y,t) = -\frac{g_0(t)}{s(t)} y + g_0(t). \tag{1.6}$$

Note that the value $s(0)$ is defined by the initial function $u_0(y)$ and it is assumed that there is the only point $s(0) \in (0,l)$ that separates the saturated and unsaturated regions. In the unsaturated region, $y \in (s(t), l)$, the function $u(y,t)$ satisfies the equation

$$\frac{\partial u}{\partial t} - \frac{u^\alpha}{1-\alpha} \frac{\partial^2 u}{\partial y^2} = 0, \tag{1.7}$$

and since at $y = s(t)$ we have $u(s(t),t) = 0$, equation (1.7) is a degenerate parabolic equation.

On the free boundary we have the following conditions:

$$u(s(t) - 0, t) = u(s(t) + 0, t) = 0, \tag{1.8}$$

$$\frac{\partial u}{\partial y}(s(t) - 0, t) = \frac{\partial u}{\partial y}(s(t) + 0, t). \tag{1.9}$$

We drop the unessential factor $1/(1-\alpha)$ in equation (1.7) and get the following free boundary problem for unknown functions $u(y,t)$ and $s(t)$:

$$\frac{\partial u}{\partial t} - u^{\alpha} u_{yy} = 0 \quad \text{in} \quad y \in (s(t), l), \quad t \in (0,T), \tag{1.10}$$

$$u(y,0) = u_0(y), \qquad y \in (s(0), l), \tag{1.11}$$

$$u(l,t) = g(t), \qquad t \in [0,T], \tag{1.12}$$

$$u(s(t),t) = 0, \qquad t \in [0,T], \tag{1.13}$$

$$\frac{\partial u}{\partial y}(s(t),t) = -\frac{g_0(t)}{s(t)}, \qquad t \in [0,T], \tag{1.14}$$

$$s(0) = s_0 \in (0,l). \tag{1.15}$$

The structure of the paper is as follows. In Sec. 2, we reduce the free boundary problem to a problem in a fixed domain and formulate our main result, Theorem 2.1. In Sec. 3, we reformulate the nonlinear free boundary problem as a nonlinear equation in Banach spaces by using Theorem 3.1. In Sec. 4, we formulate the results relating to the principal model problem for the degenerate parabolic equation. In Sec. 5, we finish the proof of Theorem 2.1. In Sec. 6, we study the properties of the model problem and derive the corresponding estimates by using an integral representation of its solution.

R e m a r k 1.1. The similar approach can be used for a free boundary problem in the case of the constitutive function of the form

$$c(\eta) = \begin{cases} \eta^{1+\alpha}, & \eta \geq 0, \quad \alpha \in (0,1), \\ 0, & \eta \leq 0. \end{cases}$$

2. Reduction of the Free Boundary Problem (1.10)–(1.15) to a Problem in a Fixed Domain and the Main Result

Let $s(t) = s_0 + \rho(t)$ and introduce a spatial variable

$$x = \frac{y - s_0 - \rho(t)}{l - s_0 - \rho(t)} = \frac{y - s(t)}{l - s(t)} \tag{2.1}$$

such that the segment $[s(t), l]$ is mapped onto $[0,1]$ and the free boundary $y = s(t)$ is mapped at $x = 0$. Denote $v(x,t) = u(y(x,t),t)$. In the new variables we get the following problem in the fixed domain $G_T = G \times (0,T)$, $G = [0,1]$:

$$\frac{\partial v}{\partial t} + \frac{x-1}{l - s_0 - \rho(t)} \frac{\partial v}{\partial x} \frac{d\rho}{dt} - v^{\alpha} \frac{v_{xx}}{(l - s_0 - \rho(t))^2} = 0 \quad \text{in} \quad G_T, \tag{2.2}$$

$$v(x,0) = v_0(x), \quad x \in [0,1], \tag{2.3}$$

$$v(1,t) = g(t), \quad t \in [0,T], \tag{2.4}$$

$$v(0,t) = 0, \quad t \in [0,T], \tag{2.5}$$

$$\frac{\partial v}{\partial x}|_{x=0} = -g_0(t)\frac{l - s_0 - \rho(t)}{s_0 + \rho(t)}, \quad t \in [0,T], \tag{2.6}$$

$$\rho(0) = 0. \tag{2.7}$$

We will use the anisotropic Hölder spaces $C^{\beta,\gamma}(\overline{G}_T)$ of smooth functions $u(x,t)$ with the norm

$$|u|_{G_T}^{(\beta,\gamma)} = |u|_{G_T}^{(0)} + \langle u\rangle_{x,G_T}^{(\beta)} + \langle u\rangle_{t,G_T}^{(\gamma)}, \quad \beta,\gamma \in (0,1),$$

where

$$|u|_{G_T}^{(0)} = \max_{\overline{G}_T} |u(x,t)|,$$

$$\langle u\rangle_{x,G_T}^{(\beta)} = \sup_{(x_1,t),(x_2,t)\in\overline{G}_T} \frac{|u(x_1,t) - u(x_2,t)|}{|x_1 - x_2|^\beta},$$

$$\langle u\rangle_{t,G_T}^{(\gamma)} = \sup_{(x,t_1),(x,t_2)\in\overline{G}_T} \frac{|u(x,t_1) - u(x,t_2)|}{|t_1 - t_2|^\gamma},$$

and the Hölder space $C^{1+\gamma}([0,T])$ with the norm $|u(t)|_{[0,T]}^{(1+\gamma)} = |u|_{[0,T]}^{(0)} + |u_t|_{[0,T]}^{(\gamma)}$ with $|u_t|_{[0,T]}^{(\gamma)} = |u_t|_{[0,T]}^{(0)} + \langle u_t\rangle_{t,[0,T]}^{(\gamma)}$. We will also use the standard Hölder spaces $C^\gamma([0,T])$, $0 < \gamma < 1$, with the norm $|u|_{[0,T]}^{(\gamma)} = |u|_{[0,T]}^{(0)} + \langle u\rangle_{t,[0,T]}^{(\gamma)}$.

For our purposes the weighted Hölder space $C_\alpha^{2+\beta,\beta/q}(\overline{G}_T)$ is appropriate, where $\beta \in (0,1)$, $q = 2 - \alpha$, and

$$\|u\|_{C_\alpha^{2+\beta,\beta/q}(\overline{G}_T)} \equiv \|u\|_{\alpha,G_T}^{(2+\beta)} = |u|_{G_T}^{(0)} + |u_x|_{G_T}^{(0)} + \langle u_x\rangle_{t,G_T}^{\left(\frac{\beta+1-\alpha}{q}\right)}$$
$$+ |x^\alpha u_{xx}|_{G_T}^{(\beta,\ \beta/q)} + |u_t|_{G_T}^{(\beta,\beta/q)}.$$

We define the space $C_\alpha^{2+\gamma}(\overline{G})$ in the similar way.

We denote by $\dot{C}_\alpha^{2+\beta,\beta/q}(\overline{G}_T)$ the subspace of $C_\alpha^{2+\beta,\beta/q}(\overline{G}_T)$ such that $u(x,t) \in \dot{C}_\alpha^{2+\beta,\beta/q}(\overline{G}_T)$ if $u(x,0) = u_t(x,0) = 0$ and similarly define the spaces $\dot{C}^{\beta,\gamma}(\overline{G}_T)$ for the functions such that $u(x,0) = 0$ and the space $\dot{C}^\gamma([0,T])$.

In problem (2.2)–(2.7), we assume that

$$x^\alpha v_{0xx}(x) \in C^{\frac{2\gamma}{q}}([0,1]),\ v_0(x) \in C^1([0,1]),\ \frac{\partial v_0}{\partial x}(0) \geq \nu = \text{const} > 0, \tag{2.8}$$

$$g_0(t), g(t) \in C^{1+\gamma/q}([0,T_1]), \quad T_1 > 0, \quad \gamma > 2\beta/q$$

and that the consistency conditions of the first order are fulfilled. It means that

$$v_0(0) = 0, \quad v_0(1) = g(0),$$
$$\frac{1}{l - s_0}\frac{\partial v_0}{\partial x}(0)\left(\frac{d\rho}{dt}|_{t=0}\right) + \frac{1}{(l - s_0)^2}(v_0^\alpha(x)v_{0xx}(x))|_{x=0} = 0, \tag{2.9}$$
$$\frac{\partial g}{\partial t}(0) - \frac{1}{(l - s_0)^2}v_0^\alpha(1)v_{0xx}(1) = 0.$$

Theorem 2.1. *Let the conditions* (2.8), (2.9) *be fulfilled with* $q = 2 - \alpha$, $0 < 2\beta/q < \gamma$, $\alpha + 2\gamma/q < 1$. *Then there exists a unique solution of problem* (2.2)–(2.7) *for some* $0 < T \le T_1$ *such that* $v(x,t) \in C_\alpha^{2+\beta,\beta/q}(\overline{G}_T)$, $\rho(t) \in C^{1+\beta/q}([0,T])$.

This theorem asserts the existence and uniqueness of the classical solution to the elliptic-parabolic degenerate problem locally in time.

The similar result is valid if we change the boundary conditions (1.2), (1.3) for the Neumann or mixed boundary conditions.

3. The Nonlinear Functional Equation

Introduce the notations

$$\rho^{(1)} \equiv \frac{d\rho}{dt}(0), \quad v^{(1)}(x) \equiv \frac{\partial v}{\partial t}(x, 0).$$

Then from (2.9)

$$\rho^{(1)} = -\frac{(v_0^\alpha(x)v_{0xx}(x))|_{x=0}}{(l - s_0)v_{0x}(0)}, \tag{3.1}$$

and from equation (2.2)

$$v^{(1)}(x) = \frac{1 - x}{l - s_0}v_{0x}(x)\rho^{(1)} + \frac{v_0^\alpha(x)v_{0xx}(x)}{(l - s_0)^2}. \tag{3.2}$$

Now we construct a function $w(x,t)$ such that $w(x,t) \in C_\alpha^{2+\gamma,\gamma/q}(\overline{G}_T)$, $\gamma > 2\beta/q$, $w(x,0) = v_0(x)$, $w_t(x,0) = v^{(1)}(x)$, $w(0,t) \equiv 0$. Consider a model problem

$$\frac{\partial \widetilde{u}}{\partial t} - x^\alpha\frac{\partial^2 \widetilde{u}}{\partial x^2} = \widetilde{v}^{(1)}(x) - x^\alpha \widetilde{v}_{0xx}(x), \quad x > 0, \ t > 0,$$
$$\widetilde{u}_x|_{x=0} = v_{0x}(0),$$
$$\widetilde{u}(x,0) = \widetilde{v}_0(x),$$

where $\widetilde{v}_0(x)$ and $\widetilde{v}^{(1)}(x)$ are the finite extensions of $v_0(x)$ and $v^{(1)}(x)$ on $x \geq 0$. It follows from Theorem 4.1 below that there exists a unique solution of this problem and the estimate of the form (4.20) is valid so that in particular $\widetilde{u}(x,t) \in C_\alpha^{2+\gamma,\gamma/q}(\overline{G}_T)$. Moreover, $\widetilde{u}(x,0) = v_0(x)$ and $\widetilde{u}_t(x,0) = v^{(1)}(x)$ on $[0,1]$ by the construction. Now we set

$$w(x,t) = \widetilde{u}(x,t) - \widetilde{u}(0,t).$$

Since $\widetilde{u}(0,0) = v_0(0) = 0$ and $\widetilde{u}_t(0,0) = v^{(1)}(0) = 0$ (due to consistency conditions), the function $w(x,t)$ has the desired properties.

We denote also

$$\sigma(t) = \rho^{(1)} \cdot t$$

so that $\sigma(0) = 0$, $\sigma_t(0) = \rho^{(1)} = \rho_t(0)$.

To reduce problem (2.2)–(2.7) to a problem in the spaces $\dot{C}_\alpha^{2+\beta,\beta/q}(\overline{G}_T)$ and $\dot{C}^{1+\beta/q}([0,T])$ with zero initial data, we introduce the new unknown functions

$$u(x,t) = v(x,t) - w(x,t), \qquad \delta(t) = \rho(t) - \sigma(t) \tag{3.3}$$

such that $\delta(0) = 0$, $\delta_t(0) = 0$, $u(x,0) \equiv 0$, $u_t(x,0) \equiv 0$.

We rewrite equation (2.2) in the form

$$L(\rho, v) = \frac{\partial v}{\partial t} - a_{11}(\rho, v)\frac{\partial^2 v}{\partial x^2} + a_1(\rho)\frac{d\rho}{dt}\frac{\partial v}{\partial x} = 0, \tag{3.4}$$

where

$$a_{11}(\rho, v) = \frac{v^\alpha}{(l - s_0 - \rho(t))^2}, \quad a_1(\rho) = \frac{x-1}{l - s_0 - \rho(t)}. \tag{3.5}$$

For the new unknown functions $\delta(t)$ and $u(x,t)$ we obtain the equation

$$L(\delta(t) + \sigma(t), u(x,t) + w(x,t)) = 0. \tag{3.6}$$

Next we single out the main linear part from $L(\delta+\sigma, u+w)$ with respect to (δ, u) so that equation (3.6) takes the form

$$\frac{\partial u}{\partial t} - a_{11}(\sigma, w)\frac{\partial^2 u}{\partial x^2} + a_1(\sigma)\delta_t\frac{\partial w}{\partial x}$$

$$= -L(\sigma, w) - [a_{11}(\sigma, w) - a_{11}(\delta + \sigma, u + w)]\frac{\partial^2 u}{\partial x^2}$$

$$-[a_{11}(\sigma, w) - a_{11}(\delta + \sigma, u + w)]\frac{\partial^2 w}{\partial x^2}$$

$$-[a_1(\delta + \sigma)(\delta_t + \sigma_t) - a_1(\sigma)\sigma_t]\frac{\partial u}{\partial x} - a_1(\sigma)\sigma_t\frac{\partial u}{\partial x}$$

$$-[a_1(\delta+\sigma)(\delta_t+\sigma_t)-a_1(\sigma)\sigma_t-a_1(\sigma)\delta_t]\frac{\partial w}{\partial x}$$
$$=-L(\sigma,w)+F(\delta,u)=F_0(\delta,u).$$

One can check that the function $F(\delta,u)$ contains either "quadratic" terms with respect to (δ,u) or minor terms in the sense of smoothness. For instance, the term

$$[a_{11}(\sigma,w)-a_{11}(\delta+\sigma,u+w)]\frac{\partial^2 u}{\partial x^2}$$
$$=\left[\frac{w^\alpha}{(l-s_0-\sigma(t))^2}-\frac{(u+w)^\alpha}{(l-s_0-\sigma(t)-\delta(t))^2}\right]\frac{\partial^2 u}{\partial x^2}$$
$$=\left[\frac{w^\alpha}{(l-s_0-\sigma(t))^2}-\frac{w^\alpha}{(l-s_0-\sigma(t)-\delta(t))^2}\right]\frac{\partial^2 u}{\partial x^2}$$
$$+\frac{w^\alpha-(u+w)^\alpha}{(l-s_0-\sigma(t)-\delta(t))^2}\frac{\partial^2 u}{\partial x^2} \tag{3.7}$$

consists of the terms "quadratic" in (δ,u), and in the expression

$$[a_1(\delta+\sigma)(\delta_t+\sigma_t)-a_1(\sigma)\sigma_t-a_1(\sigma)\delta_t]\frac{\partial w}{\partial x}$$
$$=\frac{d\delta}{dt}\left(\frac{1}{l-s_0-\sigma(t)-\delta(t)}-\frac{1}{l-s_0-\sigma(t)}\right)\frac{\partial w}{\partial x}$$
$$+\frac{d\sigma}{dt}\left(\frac{1}{l-s_0-\sigma(t)-\delta(t)}-\frac{1}{l-s_0-\sigma(t)}\right)\frac{\partial w}{\partial x}$$

the first term is "quadratic" and the second one is minor. Note also that by the construction of $\sigma(t)$, $w(x,t)$ and (3.2)

$$L(\sigma,w)|_{t=0}=0.$$

Thus problem (2.2)–(2.7) is transformed to the nonlinear problem for the functions $u(x,t)$ and $\delta(t)$

$$\frac{\partial u}{\partial t}-a_{11}(\sigma,w)\frac{\partial^2 u}{\partial x^2}+a_1(\sigma)\delta_t\frac{\partial w}{\partial x}=F_0(\delta,u) \quad \text{in } G_T, \tag{3.8}$$

$$u(x,0)=0,\quad \delta(0)=0,\quad x\in[0,1], \tag{3.9}$$

$$u(0,t)=0,\quad u(1,t)=g(t)-w(1,t),\quad t\in[0,T], \tag{3.10}$$

$$\frac{\partial u}{\partial x}(0,t)=F_1(\delta(t)),\quad t\in[0,T], \tag{3.11}$$

where

$$F_1(\delta(t))=-g_0(t)\frac{l-s_0-\sigma(t)-\delta(t)}{s_0+\sigma(t)+\delta(t)}-\frac{\partial w}{\partial x}(0,t), \tag{3.12}$$

and moreover,

$$u_t(x,0) = 0, \quad \delta_t(0) = 0. \tag{3.13}$$

First we study the linear problem

$$\frac{\partial u}{\partial t} - a_{11}(\sigma, w)\frac{\partial^2 u}{\partial x^2} + a_1(\sigma)\delta_t\frac{\partial w}{\partial x} = f_0(x,t) \text{ in } G_T, \tag{3.14}$$

$$u(x,0) = 0, \quad \delta(0) = 0, \quad x \in [0,1], \tag{3.15}$$

$$u(0,t) = 0, \quad u(1,t) = \varphi(t), \quad t \in [0,T], \tag{3.16}$$

$$\frac{\partial u}{\partial x}(0,t) = f_1(t), \quad t \in [0,T], \tag{3.17}$$

where $\varphi(t) = g(t) - w(1,t)$, $f_0(x,t)$ and $f_1(t)$ correspond to the right hand sides of (3.8) and (3.11) for some fixed $u(x,t)$ and $\delta(t)$ from classes $\dot{C}_\alpha^{2+\beta,\beta/q}(\overline{G}_T)$ and $\dot{C}^{1+\beta/q}([0,T])$, respectively. So we assume

$$f_0(x,t) \in \dot{C}^{\beta,\beta/q}(\overline{G}_T), \quad f_1(t) \in \dot{C}^{\frac{\beta+1-\alpha}{q}}([0,T]), \quad \varphi(t) \in \dot{C}^{1+\beta/q}([0,T]). \tag{3.18}$$

From (3.5) we have

$$a_{11}(\sigma, w) = \frac{w^\alpha}{(l - s_0 - \sigma(t))^2}$$

$$= \frac{1}{(l - s_0 - \sigma(t))^2}\left(\frac{w(x,t)}{x}\right)^\alpha x^\alpha \equiv a(x,t)x^\alpha, \tag{3.19}$$

where $\mu \le a(x,t) \le \mu^{-1}$, $\mu = \text{const} > 0$, for small t since $\sigma(0) = 0$ and $w(x,t)$ is the smooth function with $w(0,t) = 0$, $w_x(0,t) \ge \nu > 0$. Similar arguments give

$$a_1(\sigma)\delta_t\frac{\partial w}{\partial x}(x,t) = \frac{x-1}{l - s_0 - \sigma(t)}\frac{\partial w}{\partial x}(x,t)\frac{d\delta(t)}{dt} \equiv -b(x,t)\frac{d\delta(t)}{dt} \tag{3.20}$$

with $b(x,t) \ge \mu > 0$ for $x \in [0,1/2]$, $t \in [0,T]$.

Theorem 3.1. *Under conditions* (3.18) *there exists a unique solution of problem* (3.14)–(3.17) $u(x,t) \in \dot{C}_\alpha^{2+\beta,\beta/q}(\overline{G}_T)$, $\delta(t) \in \dot{C}^{1+\beta/q}([0,T])$ *for some* $T > 0$ *and*

$$\|u\|_{\alpha,G_T}^{(2+\beta)} + |\delta|_{[0,T]}^{(1+\beta/q)} \le C\left(|f_0|_{G_T}^{(\beta,\beta/q)} + |f_1|_{[0,T]}^{(\frac{\beta+1-\alpha}{q})} + |\varphi|_{[0,T]}^{(1+\beta/q)}\right). \tag{3.21}$$

The proof of Theorem 3.1 uses the well-known procedure (see [12], Ch. 4):

i) partition of unity on G,

ii) investigation of model problems in R_T^+ or R_T,

iii) construction of a regularizator.

In the next section we describe the main model problem related to problem (3.14)–(3.17). We will not discuss problems i) and iii). For discussion of these problems see, for example, [1–3].

Problem (3.8)–(3.12) has the form

$$A(u,\delta) = \mathcal{F}(u,\delta), \tag{3.22}$$

where $A(u,\delta)$ is the linear bounded operator, and $\mathcal{F}(u,\delta)$ is the nonlinear operator

$$\mathcal{F}(u,\delta) = (F_0(\delta,u), F_1(\delta), g(t) - w(1,t)). \tag{3.23}$$

Theorem 3.1 means that the operator A has the bounded inverse operator A^{-1}. Hence, equation (3.22) can be written as

$$(u,\delta) = A^{-1}\mathcal{F}(u,\delta),$$

and in Sec. 5 we will show that the operator $A^{-1}\mathcal{F}$ is contractive.

4. The Model Problem

Consider the differential operator defined by the left hand side in (3.14) and freeze the coefficients $a(x,t)$ from (3.19) and $b(x,t)$ from (3.20) at the point $(x,t) = (0,0)$. Let $a(0,0) = a_0$, $b(0,0) = b_0$, $R^+ = \{x \geq 0\}$, $R^+_T = R^+ \times [0,T]$.

We are looking for a solution $(u(x,t), \delta(t))$ of the problem

$$\frac{\partial u}{\partial t} - a_0 x^\alpha \frac{\partial^2 u}{\partial x^2} - b_0 \frac{d\delta}{dt} = f(x,t) \text{ in } R^+_T, \tag{4.1}$$

$$u(x,0) = 0, \quad \delta(0) = 0, \quad x \in R^+, \tag{4.2}$$

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(0,t) = f_1(t), \quad t \in [0,T] \tag{4.3}$$

with

$$f(x,t) \in C^{\beta,\beta/q}(R^+_T), \quad f_1(t) \in C^{\frac{\beta+1-\alpha}{q}}([0,T]), \tag{4.4}$$

where function $f(x,t)$ has a finite support.

Note that by all norms in Hölder spaces over unbounded domains R^+ and R^+_T we mean supremum in $M > 0$ of the corresponding norms over sets $R^+ \cap \{|x| \leq M\}$ and $R^+_T \cap \{|x| \leq M\}$.

Define the function

$$\theta(x,t) = u(x,t) - b_0\delta(t). \tag{4.5}$$

The function $\theta(x,t)$ can be found from the equations

$$\frac{\partial\theta}{\partial t} - a_0 x^\alpha \frac{\partial^2\theta}{\partial x^2} = f(x,t) \ \text{ in } R_T^+, \tag{4.6}$$

$$\theta(x,0) = 0, \quad x \in R^+, \tag{4.7}$$

$$\frac{\partial\theta}{\partial x}(0,t) = f_1(t), \quad t \in [0,T]. \tag{4.8}$$

To find the function $\delta(t)$ we have the relation

$$\theta(0,t) + b_0\delta(t) = 0. \tag{4.9}$$

Keeping in mind also the problem of constructing of the function $w(x,t)$ from Sec. 3, we consider the next model problem for the unknown function $u(x,t)$

$$\frac{\partial u}{\partial t} - x^\alpha \frac{\partial^2 u}{\partial x^2} = f(x,t) \ \text{ in } R_T^+, \tag{4.10}$$

$$\frac{\partial u}{\partial x}(0,t) = f_1(t), \quad t \in [0,T], \tag{4.11}$$

$$u(x,0) = u_0(x), \quad x \in R^+, \tag{4.12}$$

where the functions $f(x,t)$ and $u_0(x)$ have finite supports. In this case the following conditions are required in addition to (4.4):

$$u_0(x) \in C_\alpha^{2+2\beta/q}(R^+), \quad f(x,0) \in C^{2\beta/q}(R^+). \tag{4.13}$$

To find a general solution of the equation

$$\frac{\partial u}{\partial t} - x^\alpha \frac{\partial^2 u}{\partial x^2} = 0,$$

we apply the Laplace transform in t such that

$$p\widetilde{u} - x^\alpha \widetilde{u}_{xx} = 0, \tag{4.14}$$

where $\widetilde{u}(x,p)$ is the Laplace image of $u(x,t)$. The general solution of (4.14) is (see [9], 8.491(7))

$$\widetilde{u}(x,p) = c_1 x^{1/2} I_{-1/q}(\frac{2}{q}\sqrt{p}x^{q/2}) + c_2 x^{1/2} K_{-1/q}(\frac{2}{q}\sqrt{p}x^{q/2}),$$

where $q = 2-\alpha$, $I_\mu(z)$, $K_\mu(z)$ are the modified Bessel functions, and c_1, c_2 are arbitrary constants. The Green function to the Neumann problem for equation (4.14) is

$$\widetilde{G}(x,\xi,p) = \begin{cases} \frac{2}{q} I_{-1/q}(\frac{2}{q}p^{1/2}x^{q/2}) K_{-1/q}(\frac{2}{q}p^{1/2}\xi^{q/2}) x^{1/2}\xi^{1/2}\xi^{-\alpha}, & x < \xi, \\ \frac{2}{q} K_{-1/q}(\frac{2}{q}p^{1/2}x^{q/2}) I_{-1/q}(\frac{2}{q}p^{1/2}\xi^{q/2}) x^{1/2}\xi^{1/2}\xi^{-\alpha}, & x > \xi, \end{cases}$$

and the inverse Laplace transform gives (see [9], 6.653) the Green function for problem (4.10)–(4.12)

$$G(x,\xi,t)=c(q)t^{-1+1/q}\left(\frac{x^{q/2}\xi^{q/2}}{q^2t}\right)^{1/q}I_{-1/q}\left(2\frac{(x\xi)^{q/2}}{q^2t}\right)e^{-\frac{x^q+\xi^q}{q^2t}}\xi^{-\alpha}. \quad (4.15)$$

Denote

$$u=\frac{\xi^{q/2}}{qt^{1/2}},\quad v=\frac{x^{q/2}}{qt^{1/2}}$$

and rewrite the Green function as

$$G(x,\xi,t)=c(q)t^{-1/q}(uv)^{1/q}I_{-1/q}(2uv)e^{-(u^2+v^2)}u^{-2\alpha/q}. \quad (4.16)$$

We define the constant $c(q)$ in (4.16) by the condition

$$\int_0^\infty G(x,\xi,t)d\xi=1. \quad (4.17)$$

By direct calculations one can show that the function $G(x,\xi,t-\tau)$ satisfies the equations

$$\frac{\partial G}{\partial t}-x^\alpha\frac{\partial^2 G}{\partial x^2}=0\ \text{ and }\ \frac{\partial G}{\partial \tau}+\frac{\partial^2}{\partial\xi^2}(\xi^\alpha G)=0,\quad \tau<t. \quad (4.18)$$

We use equations (4.18) and the Green formula to get the integral representation of the solution to problem (4.10)–(4.12)

$$u(x,t)=\int_0^t d\tau\int_0^\infty G(x,\xi,t-\tau)f(\xi,\tau)d\xi$$

$$+\int_0^\infty G(x,\xi,t)u_0(\xi)d\xi-\int_0^t(\xi^\alpha G(x,\xi,t-\tau))|_{\xi=0}f_1(\tau)d\tau. \quad (4.19)$$

Theorem 4.1. *Let in* (4.10)–(4.13) *and* (4.4) *the consistency condition* $f_1(0)=u_{0x}(0)$ *be fulfilled. Then there exists a unique solution* $u(x,t)\in C_\alpha^{2+\beta,\beta/q}(R_T^+)$, $\alpha+\beta<1$, $q=2-\alpha$, *such that*

$$\langle x^\alpha u_{xx}\rangle_{B_{R,T}}^{(\beta,\beta/q)}+\langle u_t\rangle_{B_{R,T}}^{(\beta,\beta/q)}+\langle u_x\rangle_{t,B_{R,T}}^{((\beta+1-\alpha)/q)}\le C(R,T)(\langle f\rangle_{R_T^+}^{(\beta,\beta/q)}+\langle f_1\rangle_{[0,T]}^{(\frac{\beta+1-\alpha}{q})}$$

$$+|f(x,0)|_{R^+}^{(2\beta/q)}+|x^\alpha u_{0xx}|_{R^+}^{(2\beta/q)}), \quad (4.20)$$

where $B_{R,T}=\{0\le x\le R\}\times[0,T]$, $R>0$, *and the constant* $C(R,T)$ *is bounded for bounded* R *and* T.

The proof of (4.20) is based on the estimates of the potentials on the right hand side of (4.19) and will be done in Sec. 6.

5. Proof of Theorem 2.1

Now we return to problem (3.22) and the equation

$$(u,\delta) = A^{-1}\mathcal{F}(u,\delta), \tag{5.1}$$

where $\mathcal{F}(u,\delta)$ is defined in (3.23). We introduce the space

$$M = \dot{C}_{\alpha}^{2+\beta,\beta/q}(\overline{G}_T) \times \dot{C}^{1+\beta/q}([0,T]) \tag{5.2}$$

with the elements $z = (u(x,t), \delta(t))$ and the norm

$$\|z\|_M = \|u\|_{\alpha,G_T}^{(2+\beta)} + |\delta|_{[0,T]}^{(1+\beta/q)}, \tag{5.3}$$

and the space

$$W = \dot{C}^{\beta,\beta/q}(\overline{G}_T) \times \dot{C}^{\frac{\beta+1-\alpha}{q}}([0,T]) \times \dot{C}^{1+\beta/q}([0,T]) \tag{5.4}$$

with the elements $f = (f(x,t), f_1(t), \varphi(t))$ and the norm

$$\|f\|_W = |f|_{G_T}^{(\beta,\beta/q)} + |f_1|_{[0,T]}^{(\frac{\beta+1-\alpha}{q})} + |\varphi|_{[0,T]}^{(1+\beta/q)}. \tag{5.5}$$

It is straightforward to check that analogously to the inequality for the standard Hölder norms $|u|_{G_T}^{(l)}$ from [12]

$$|u|_{G_T}^{(l')} \le CT^{\frac{l-l'}{2}}|u|_{G_T}^{(l)}, \tag{5.6}$$

which is valid for the functions $u \in \dot{C}^{l,l/2}(\overline{G}_T)$, $l' < l$, l' integer, (see [12], Ch. 4), in our case we have

$$\|u\|_{\alpha,G_T}^{(2+\beta)} \le CT^{\frac{\gamma-\beta}{q}}\|u\|_{\alpha,G_T}^{(2+\gamma)}, \quad |f|_{G_T}^{(\beta,\beta/q)} \le CT^{\frac{\gamma-\beta}{q}}|f|_{G_T}^{(\gamma,\gamma/q)}, \quad \beta < \gamma < 1 \tag{5.7}$$

for the functions $u \in \dot{C}_{\alpha}^{2+\gamma,\gamma/q}(\overline{G}_T)$ and $f \in \dot{C}^{\gamma,\gamma/q}(\overline{G}_T)$, and analogous inequality is valid for the functions defined on $[0,T]$. We will also use a known inequality

$$|fg|_{G_T}^{(\beta,\beta/q)} \le CT^{\beta/q}|f|_{G_T}^{(\beta,\beta/q)}|g|_{G_T}^{(\beta,\beta/q)} \tag{5.8}$$

for the functions $f, g \in \dot{C}^{\beta,\beta/q}(\overline{G}_T)$. We give here the short outline of the proof of inequalities (5.7).

Consider, for example, a weighted Hölder constant $\langle x^\alpha u_{xx}\rangle^{(\beta)}_{x,\overline{G}_T}$ in the definition of the norm $\|u\|^{(2+\beta)}_{\alpha,\overline{G}_T}$ in the space $C^{2+\beta,\beta/q}_\alpha(\overline{G}_T)$. Let function $u(x,t)$ be in fact more smooth, namely, $u \in C^{2+\gamma,\gamma/q}_\alpha(\overline{G}_T)$ with $0<\beta<\gamma<1$.

According to the definition,

$$\langle x^\alpha u_{xx}\rangle^{(\beta)}_{x,\overline{G}_T} = \sup_{(x,t),(\overline{x},t)\in\overline{G}_T} \frac{|v(x,t)-v(\overline{x},t)|}{|x-\overline{x}|^\beta},$$

where $v(x,t)=x^\alpha u_{xx}$. Consider the ratio $\frac{|v(x,t)-v(\overline{x},t)|}{|x-\overline{x}|^\beta}$.

Consider two cases. If $|x-\overline{x}|<T^{1/q}$, then

$$\frac{|v(x,t)-v(\overline{x},t)|}{|x-\overline{x}|^\beta} = \frac{|v(x,t)-v(\overline{x},t)|}{|x-\overline{x}|^\gamma}|x-\overline{x}|^{\gamma-\beta} \le \langle v\rangle^\gamma_{x,\overline{G}_T} T^{\frac{\gamma-\beta}{q}}.$$

If now $|x-\overline{x}|\ge T^{1/q}$, then, as $v(x,0)\equiv 0$,

$$\frac{|v(x,t)-v(\overline{x},t)|}{|x-\overline{x}|^\beta} \le \frac{|v(x,t)|+|v(\overline{x},t)|}{T^{1/q}} \le 2\langle v(x,t)\rangle^{\gamma/q}_{t,\overline{G}_T}\frac{t^{\gamma/q}}{T^{\beta/q}} \le 2\langle v(x,t)\rangle^{\gamma/q}_{t,\overline{G}_T} T^{\frac{\gamma-\beta}{q}}.$$

Consequently, in both cases,

$$\langle x^\alpha u_{xx}\rangle^{(\beta)}_{x,\overline{G}_T} \le CT^{\frac{\gamma-\beta}{q}}\|u\|^{(2+\gamma)}_{\alpha,\overline{G}_T}.$$

Other terms in the definition of the norm $\|u\|^{(2+\beta)}_{\alpha,\overline{G}_T}$ can be treated in a quite similar way which leads to (5.7).

Due to our assumptions on the smoothness of the initial data $u_0(x)$, the boundary functions $g_0(t)$ and $g(t)$, and consistency conditions (2.9) with (5.7), we obtain

$$\|\mathcal{F}(0,0)\|_W \le PT^{\frac{\gamma-\beta}{q}}, \tag{5.9}$$

where P is a constant independent of T.

We set

$$M_d = \{z\in M:\ \|z\|_M \le d\},$$

where $d>0$ is sufficiently small and will be given below.

Lemma 5.1. *There holds an inequality*

$$\|\mathcal{F}(z_2)-\mathcal{F}(z_1)\|_W \le C(d+T^\varkappa)\|z_2-z_1\|_M, \tag{5.10}$$

where $\varkappa>0$ is a certain positive number, and the constant C is bounded for bounded d and T.

It follows from this lemma and from (5.9) that if T and d are sufficiently small, then the mapping $z \to A^{-1}\mathcal{F}(z)$ maps M_d into itself and is a contraction in M_d. Hence this mapping has a unique fixed point. This completes the proof of Theorem 2.1.

P r o o f of Lemma 5.1. The origin of the estimate (5.10) is that the expression for $\mathcal{F}(z) = \mathcal{F}(u, \delta)$ contains, as it was mentioned, either "quadratic" terms or minor terms. Thus in the factor $(d + T^{\varkappa})$ in (5.10) the value d appears as we estimate "quadratic" terms and $T^{\varkappa}$ appears for minor terms in view of (5.7).

As a typical example of calculations in the proof of Lemma 5.1, we show the estimate of the term

$$E(z) \equiv \frac{w^{\alpha} - (u+w)^{\alpha}}{(l - s_0 - \sigma(t) - \delta(t))^2} \frac{\partial^2 u}{\partial x^2}$$

in (3.7). By the mean value theorem one can write

$$w^{\alpha} - (u+w)^{\alpha} = -\alpha u \int\limits_0^1 [\varepsilon w + (1-\varepsilon)(u+w)]^{\alpha-1} d\varepsilon$$

and hence

$$E(z) = (x^{\alpha} \frac{\partial^2 u}{\partial x^2}) \left(\frac{u}{x}\right) \frac{\alpha}{(l - s_0 - \sigma(t) - \delta(t))^2} \times \int\limits_0^1 \left[\frac{\varepsilon w + (1-\varepsilon)(u+w)}{x}\right]^{\alpha-1} d\varepsilon. \tag{5.11}$$

Since $\alpha - 1 < 0$, some additional arguments are required to show the smoothness of $E(z)$ in z.

Taking into account that $u(0,t) = w(0,t) = 0$, we use the following representation with some sufficiently small $a > 0$:

$$\frac{\varepsilon w(x,t) + (1-\varepsilon)(u(x,t) + w(x,t))}{x}$$

$$= \begin{cases} \int_0^1 [\varepsilon w_x(\omega x, t) + (1-\varepsilon)(u_x(\omega x, t) + w_x(\omega x, t))] d\omega, & 0 \le x \le a, \\ \frac{\varepsilon w(x,t) + (1-\varepsilon)(u(x,t) + w(x,t))}{x}, & x > a, \end{cases} \tag{5.12}$$

where the mean value theorem is applied.

By the properties of the initial function $u_0(x)$ and $w(x,t)$, $w_x(0,0) \ge \nu > 0$, $u_{0x}(0) \ge \nu > 0$, and because $u(x,t) \in M_d$, for sufficiently small T and d we get

$$w_x(x,t) \ge \nu/2, \quad (w_x + u_x)(x,t) \ge \nu/2 \quad \text{for } 0 \le x \le a, \tag{5.13}$$

and for $x \geq a$

$$\frac{\varepsilon w(x,t) + (1-\varepsilon)(u(x,t)+w(x,t))}{x} \geq \nu_1, \quad \nu_1 = \text{const} > 0 \tag{5.14}$$

as far as $w(x,t) \geq \nu_2$ for $x \geq a$. Moreover, the integrand in (5.12) is bounded from above and the same is true for the left hand side of (5.14). Thus from (5.12), inequalities (5.13), (5.14) and $(u, \delta) \in M_d$ it follows

$$\nu_2 \leq \frac{\varepsilon w(x,t) + (1-\varepsilon)(u(x,t)+w(x,t))}{x} \leq \nu_2^{-1}, \quad \nu_2 = \text{const} > 0. \tag{5.15}$$

We observe also that for small d and T and $t \leq T$

$$\nu_3 \leq l - s_0 - \sigma(t) - \delta(t) \leq \nu_3^{-1}, \quad \nu_3 = \text{const} > 0. \tag{5.16}$$

So we obtain the representation

$$E(z) = \left(x^\alpha \frac{\partial^2 u}{\partial x^2} \right) \left(\frac{u}{x} \right) \Phi(z), \tag{5.17}$$

$$\Phi(z) = \frac{\alpha}{(l - s_0 - \sigma(t) - \delta(t))^2} \int\limits_0^1 \left[\frac{\varepsilon w + (1-\varepsilon)(u+w)}{x} \right]^{\alpha-1} d\varepsilon,$$

and from (5.15), (5.16) it follows that $\Phi(z)$ is the smooth function in z for $z \in M_d$ and small $t > 0$.

The difference $E(z_2) - E(z_1)$, $z_1, z_2 \in M_d$, is evaluated as follows:

$$|E(z_2) - E(z_1)|_{G_T}^{(\beta,\beta/q)}$$

$$\leq \left| \left(x^\alpha \frac{\partial^2 u_2}{\partial x^2} - x^\alpha \frac{\partial^2 u_1}{\partial x^2} \right) \left(\frac{u_2}{x} \right) \Phi(z_2) \right|_{G_T}^{(\beta,\beta/q)}$$

$$+ \left| \left(x^\alpha \frac{\partial^2 u_1}{\partial x^2} \right) \left(\frac{u_2}{x} - \frac{u_1}{x} \right) \Phi(z_2) \right|_{G_T}^{(\beta,\beta/q)}$$

$$+ \left| \left(x^\alpha \frac{\partial^2 u_1}{\partial x^2} \right) \left(\frac{u_1}{x} \right) (\Phi(z_2) - \Phi(z_1)) \right|_{G_T}^{(\beta,\beta/q)}.$$

To continue this estimate we note that

$$\frac{u}{x} = \int\limits_0^1 u_x(\varepsilon x) d\varepsilon, \quad |\Phi(z)|_{G_T}^{(\beta,\beta/q)} \leq \text{const}, \quad z \in M_d,$$

$$|\Phi(z_2)-\Phi(z_1)|^{(\beta,\beta/q)}_{G_T} \le \mathrm{const}\cdot\|z_2-z_1\|_M, \quad z_1,z_2\in M_d.$$

This leads to

$$|E(z_2)-E(z_1)|^{(\beta,\beta/q)}_{G_T} \le \mathrm{const}\cdot(\|z_2\|_M+\|z_1\|_M)\|z_2-z_1\|_M$$

$$\le \mathrm{const}\cdot d\,\|z_2-z_1\|_M, \quad z_1,z_2\in M_d.$$

Similar calculations of the rest of the terms in the difference $\mathcal{F}(z_2)-\mathcal{F}(z_1)$ prove Lemma 5.1. ■

6. Proof of Theorem 4.1

In this section we use representation (4.19) to obtain the estimates from Theorem 4.1. First we deduce some properties of the Green function $G(x,\xi,t)$ from (4.16).

The Bessel function $I_\mu(z)$ has the series expansion

$$I_\mu(z)=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\mu+k+1)}\left(\frac{z}{2}\right)^{\mu+2k}, \tag{6.1}$$

where $\Gamma(x)$ is the Gamma function, and the asymptotic expansion for large z

$$I_\mu(z)\sim\frac{e^z}{\sqrt{2\pi z}}\left(1+\frac{C}{z}+\ldots\right). \tag{6.2}$$

Here and below we will denote by C various positive constants. From these representations it follows that

$$I_{-1/q}\sim C\begin{cases} z^{-1/q} & \text{for } z\le 1,\\ e^z z^{-1/2} & \text{for } z>1.\end{cases} \tag{6.3}$$

Using (6.1)–(6.3), we can get the following estimates:

$$|G(x,\xi,t)|\le Ct^{-1/q}u^{-2\alpha/q}\begin{cases} e^{-\left(u^2+v^2\right)}, & uv\le 1,\\ e^{-\gamma(u-v)^2}(uv)^{1/q-1/2}, & uv>1,\end{cases} \tag{6.4}$$

$$|G_t(x,\xi,t)|\le Ct^{-1-1/q}u^{-2\alpha/q}\begin{cases} e^{-\left(u^2+v^2\right)}, & uv\le 1,\\ e^{-\gamma(u-v)^2}(uv)^{1/q-1/2}, & uv>1,\end{cases} \tag{6.5}$$

$$|G_x(x,\xi,t)|\le Ct^{-1/q}\frac{u^{-2\alpha/q}}{x}\begin{cases} e^{-\left(u^2+v^2\right)}\left(v^2+u^2v^2\right), & uv\le 1,\\ e^{-\gamma(u-v)^2}(1+v)(uv)^{1/q-1/2}, & uv>1,\end{cases} \tag{6.6}$$

$$|G_{xt}(x,\xi,t)| \le Ct^{-1-1/q}\frac{u^{-2\alpha/q}}{x}\begin{cases} e^{-\left(u^2+v^2\right)}\left(v^2+u^2v^2\right), & uv \le 1, \\ e^{-\gamma(u-v)^2}(1+v)(uv)^{1/q-1/2}, & uv > 1, \end{cases} \tag{6.7}$$

$$|G_{tt}(x,\xi,t)| \le Ct^{-2-1/q}u^{-2\alpha/q}\begin{cases} e^{-\left(u^2+v^2\right)}, & uv \le 1, \\ e^{-\gamma(u-v)^2}(uv)^{1/q-1/2}, & uv > 1, \end{cases} \tag{6.8}$$

where γ is some positive constant.

To check, for example, the estimate of $G_x(x,\xi,t)$ for $uv \le 1$ in (6.6) we note that for small z

$$z^{1/q}I_{-1/q}(z) \sim C\left(1+\frac{2-\alpha}{1-\alpha}\frac{z^2}{4}+\dots\right),$$

and $v_x = Cv/x,\ z_x = Cz/x$, where $z = 2uv$, u, v — from (4.16). Therefore

$$G_x(x,\xi,t) \sim -Ct^{-1/q}u^{-2\alpha/q}2ve^{-\left(u^2+v^2\right)}\frac{v}{x}z^{1/q}I_{-1/q}(z)$$

$$+Ct^{-1/q}u^{-2\alpha/q}e^{-\left(u^2+v^2\right)}\frac{z}{x}\frac{d}{dz}\left(z^{1/q}I_{-1/q}(z)\right)$$

$$\sim Ct^{-1/q}u^{-2\alpha/q}\frac{1}{x}e^{-\left(u^2+v^2\right)}\left(v^2+u^2v^2\right).$$

Denote

$$J_1(v) = \int_0^\infty u^a e^{-\gamma(u-v)^2}du,$$

$$J_2(v) = \int_{1/2v}^\infty u^a e^{-\gamma(u-v)^2}du,$$

$$J_3(v) = \int_0^{1/2v} u^a e^{-\gamma(u-v)^2}du.$$

Lemma 6.1. *Let* $a > -1,\ \gamma > 0,\ v \ge 0$. *The next inequalities are valid:*

$$J_1(v) \le C\begin{cases} v^a & for \quad v \ge 1, \\ 1, & for \quad v < 1, \end{cases} \tag{6.9}$$

$$J_2(v) \le C\begin{cases} v^a & for \quad v \ge 1, \\ e^{-\gamma_1/v^2}, & for \quad v < 1, \end{cases} \tag{6.10}$$

$$J_3(v) \le C\begin{cases} e^{-\gamma_1 v^2}, & for \quad v \ge 1, \\ 1, & for \quad v < 1, \end{cases} \tag{6.11}$$

where $0 < \gamma_1 \le \gamma$.

P r o o f. To prove (6.9) we split the integral $J_1(v)$

$$J_1(v) = \left(\int_0^{v/2} + \int_{v/2}^{3v/2} + \int_{3v/2}^{\infty} \right) \left(u^a e^{-\gamma(u-v)^2} du \right) \equiv A_1(v) + A_2(v) + A_3(v). \tag{6.12}$$

We have $|u-v|^2 \geq v^2/4$ for the integrand in A_1 and then

$$A_1(v) \leq \int_0^{v/2} u^a e^{-\gamma v^2/4} du \leq C v^{a+1} e^{-\gamma v^2/4} \leq C e^{-\gamma v^2/8}.$$

Since $u \in (v/2, 3v/2)$ in $A_2(v)$,

$$A_2(v) \leq C v^a \int_{v/2}^{3v/2} e^{-\gamma(u-v)^2} du \leq C v^a \int_{-\infty}^{\infty} e^{-\gamma(u-v)^2} du \leq C v^a.$$

For $a \geq 0$

$$A_3(v) = \int_{v/2}^{\infty} (v+z)^a e^{-\gamma z^2} dz \leq C \int_{-\infty}^{\infty} (v^a + z^a) e^{-\gamma z^2} dz \leq C(v^a + 1),$$

and for $a \in (-1, 0)$

$$A_3(v) = \int_{v/2}^{\infty} (v+z)^a e^{-\gamma z^2} dz \leq C v^a \int_{-\infty}^{\infty} e^{-\gamma z^2} dz \leq C v^a.$$

The estimates of $A_i(v)$, $i = 1, 2, 3$, give (6.9).

We first consider the integral $J_2(v)$ for $v \leq 1/2$. Then in $J_2(v)$ $u - v \geq \frac{1}{2v} - v = \frac{1-2v^2}{2v} \geq 1/4v$ and

$$J_2(v) = \int_{1/2v}^{\infty} u^a e^{-\gamma(u-v)^2} du \leq \int_{1/2v}^{\infty} u^a e^{-\frac{\gamma}{2}(u-v)^2} e^{-\frac{\gamma}{32}\frac{1}{v^2}} du$$

$$\leq e^{-\frac{\gamma}{32v^2}} \int_0^{\infty} u^a e^{-\frac{\gamma}{2}(u-v)^2} du \leq C e^{-\frac{\gamma}{32v^2}} v^a \leq C e^{-\gamma_1/v^2},$$

where we used estimate (6.9). The estimates for $J_2(v)$ for $v > 1$ and $1/2 < v < 1$ follow from (6.9).

One can check that for $v > 1$ and $u > 1/2v$ there is $v - u \geq v/2$, then

$$J_3(v) \leq \int_0^{1/2v} u^a e^{-\gamma v^2/4} du \leq e^{-\gamma v^2/4} \int_0^{1/2v} u^a du \leq C e^{-\gamma v^2/4}.$$

For $v \leq 1$ we can estimate $J_3(v)$ by (6.9). This completes the proof of (6.11) and Lemma 6.1. ∎

Lemma 6.2. *Let* $x, \xi > 0$, $q = 2 - \alpha$, $\alpha \in (0,1)$.
i) For $\xi > x$

$$\xi^{q/2} - x^{q/2} \geq C \begin{cases} (\xi - x) x^{q/2-1}, & \xi - x \leq x, \\ (\xi - x)^{q/2}, & \xi - x > x. \end{cases} \tag{6.13}$$

ii) For $\xi < x$

$$x^{q/2} - \xi^{q/2} \geq C (x - \xi) x^{q/2-1}. \tag{6.14}$$

iii) For $\delta > 0$, $2\delta/q < 1$, *and* $f(x) \in C^{2\delta/q}(R^+)$

$$|f(x) - f(\xi)| \leq C(x) |f|_{R^+}^{(2\delta/q)} \left| x^{q/2} - \xi^{q/2} \right|^{2\delta/q}, \tag{6.15}$$

where $C(x) = \text{const} \cdot \max\left(1, x^{\alpha\delta/q}\right)$.

P r o o f. In the case i) let $A_1 = \xi^{q/2} - x^{q/2}$. Successively using the change of variables $\eta = x + z$ and $y = z/(\xi - x)$, we get

$$A_1 = \frac{2}{q} \int_x^{\xi} \eta^{\frac{2}{q}-1} d\eta = \frac{2}{q} \int_0^{\xi - x} (x + z)^{\frac{2}{q}-1} dz = \frac{2}{q} (\xi - x) \int_0^1 \frac{dy}{(x + (\xi - x) y)^{1-q/2}}.$$

For $\xi - x > x$

$$A_1 = \frac{2}{q} (\xi - x)^{q/2} \int_0^1 \frac{dy}{\left(\frac{x}{\xi - x} + y\right)^{\alpha/2}} \geq \frac{2}{q} (\xi - x)^{q/2} \int_0^1 \frac{dy}{(1 + y)^{\alpha/2}} \geq C (\xi - x)^{q/2}.$$

For $\xi - x \leq x$

$$A_1 = \frac{2}{q} (\xi - x) \int_0^1 \frac{dy}{x^{\alpha/2} \left(1 + \frac{\xi - x}{x} y\right)^{\alpha/2}} \geq C (\xi - x) x^{-\alpha/2}.$$

These estimates prove (6.13).

In the case ii) let $B(x,\xi) = x^{q/2} - \xi^{q/2}$. Similarly to the case i), we have

$$B(x,\xi) = \frac{2}{q}(x-\xi)\int_0^1 \frac{dy}{x^{\alpha/2}\left(1-\frac{x-\xi}{x}y\right)^{\alpha/2}}$$

$$\geq \frac{2}{q}(x-\xi)\,x^{q/2-1}\int_0^1 dy = C(x-\xi)\,x^{q/2-1}$$

and, hence, (6.14).

Let

$$F(x,\xi) = \frac{|f(x)-f(\xi)|}{\left|x^{q/2}-\xi^{q/2}\right|^{2\delta/q}}, \quad \Delta f = f(x) - f(\xi).$$

We prove that $F(x,\xi)$ is bounded if x is bounded. For $\xi > x$ and $\xi - x \leq x$ by (6.13)

$$F(x,\xi) \leq C\frac{|\Delta f|}{\left((\xi-x)\,x^{-\alpha/2}\right)^{2\delta/q}} \leq Cx^{\alpha\delta/q}\langle f\rangle_x^{(2\delta/q)}.$$

For $\xi > x$ and $\xi - x > x$ again by (6.13)

$$F(x,\xi) \leq C\frac{|\Delta f|}{(\xi-x)^{\delta}} \leq C\langle f\rangle_x^{(2\delta/q)}\frac{|x-\xi|^{2\delta/q}}{|x-\xi|^{\delta}} = C\langle f\rangle_x^{(2\delta/q)}|x-\xi|^{\delta\alpha/q},$$

and we obtain (6.15) under $|x-\xi| \leq 1$. For $|x-\xi| > 1$

$$F(x,\xi) \leq C\max|f|.$$

Finally, for $\xi < x$

$$F(x,\xi) \leq C\frac{|\Delta f|}{\left((x-\xi)\,x^{-\alpha/2}\right)^{2\delta/q}} \leq Cx^{\alpha\delta/q}\langle f\rangle_x^{(2\delta/q)}.$$

Lemma 6.2 is proved. ■

Lemma 6.3. *Let $f(x) \in C^{2\delta/q}(R^+)$, $2\delta/q < 1$. Then the following integral exists and*

$$\lim_{t\to 0}\int_0^\infty G(x,\xi,t)\,(f(x)-f(\xi))\,d\xi = 0. \tag{6.16}$$

P r o o f. In the integral

$$K(x,t) = \int_0^\infty G(x,\xi,t)\,(f(x)-f(\xi))\,d\xi$$

we change the variable according to $\xi = \left(qt^{1/2}u\right)^{2/q} = C\left(q\right)t^{1/q}u^{2/q}$. Below we will frequently use this change $\xi \to u$. Using (6.15) and (6.4), we get

$$\left|f\left(x\right)-f\left(\xi\right)\right| \leq C\left(x\right)\left|f\right|_{R^+}^{(2\delta/q)}\left|v-u\right|^{2\delta/q}t^{\delta/q}$$

and

$$K\left(x,t\right) \leq C\left(x\right)\left|f\right|_{R^+}^{(2\delta/q)}t^{\delta/q}\left(K_1\left(x,t\right)+K_2\left(x,t\right)\right), \tag{6.17}$$

$$K_1\left(x,t\right) = \int\limits_0^{1/v} e^{-\left(u^2+v^2\right)}u^{-\frac{2\alpha}{q}+\frac{2}{q}-1}\left|v-u\right|^{2\delta/q}du,$$

$$K_2\left(x,t\right) = \int\limits_{1/v}^{\infty} e^{-\gamma(u-v)^2}\left(uv\right)^{1/q-1/2}u^{-\frac{2\alpha}{q}+\frac{2}{q}-1}\left|v-u\right|^{2\delta/q}du.$$

To estimate $K_1\left(x,t\right)$, we use the inequality $\left|v-u\right|^{2\delta/q} \leq C\left(v^{2\delta/q}+u^{2\delta/q}\right)$ and (6.9), then

$$\left|K_1\left(x,t\right)\right| \leq C\left\{e^{-v^2}\int\limits_0^{1/v}e^{-u^2}u^{-\frac{\alpha}{q}+\frac{2\delta}{q}}du + e^{-v^2}v^{2\delta/q}\int\limits_0^{1/v}e^{-u^2}u^{-\alpha/q}du\right\} \leq \text{const.}$$

In the case of $K_2\left(x,t\right)$, we use (6.10) to obtain

$$\begin{aligned}\left|K_2\left(x,t\right)\right| &\leq C\int\limits_{1/v}^{\infty}e^{-\gamma_1(u-v)^2}\left(uv\right)^{1/q-1/2}u^{-\frac{2\alpha}{q}+\frac{2}{q}-1}du \\ &= Cv^{\alpha/2q}\int\limits_{1/v}^{\infty}e^{-\gamma_1(u-v)^2}u^{-\alpha/2q}du \leq \text{const.}\end{aligned}$$

Now Lemma 6.3 follows from (6.17). ∎

Lemma 6.4. *Let*

$$w\left(x,t\right) = \int\limits_0^{\infty}G\left(x,\xi,t\right)f\left(\xi\right)d\xi, \quad f\left(x\right) \in C^{2\beta/q}\left(R^+\right), \quad 2\beta/q < 1, \quad \beta \in \left(0,1\right).$$

Then

$$\langle w\rangle_{x,B_{R,T}}^{(\beta)} + \langle w\rangle_{t,B_{R,T}}^{(\beta/q)} \leq C\left(R\right)\left|f\right|_{R^+}^{(2\beta/q)}. \tag{6.18}$$

P r o o f. Recall the notation: $\widetilde{\Omega}_T = ([0,\infty) \cap B_R)_T$. From (4.17) it follows

$$w(x,t) = \int_0^\infty G(x,\xi,t)\left[f(\xi) - f(x)\right] d\xi + f(x) = z(x,t) + f(x),$$

so that we need to estimate the function $z(x,t)$. First we evaluate the Hölder constant of $z(x,t)$ in t. Let $0 < t < \bar{t}$. We have

$$\left|z(x,t) - z\left(x,\bar{t}\right)\right| \le \left|\int_t^{\bar{t}} z_\tau(x,\tau)\, d\tau\right| \tag{6.19}$$

and from the representation of $z(x,t)$ and Lemma 6.2

$$|z_t| \le C(R)\,|f|_{R^+}^{(2\beta/q)} \int_0^\infty |G_t(x,\xi,t)| \left|x^{q/2} - \xi^{q/2}\right|^{\frac{2\beta}{q}} d\xi.$$

The change of variable $\xi \to u$ leads to

$$|z_t| \le C(R)\,|f|_{R^+}^{(2\beta/q)}\, t^{-1+\beta/q}\left(K_1(x,t) + K_2(x,t)\right) \le C(R)\,|f|_{R^+}^{(2\beta/q)}\, t^{-1+\beta/q},$$

where $K_1(x,t)$, $K_2(x,t)$ were introduced in the proof of Lemma 6.3. Hence,

$$\left|z(x,t) - z\left(x,\bar{t}\right)\right| \le C(R)\,|f|_{R^+}^{(2\beta/q)} \int_t^{\bar{t}} \tau^{-1+\beta/q} d\tau \le C(R)\,|f|_{R^+}^{(2\beta/q)} \left|\bar{t} - t\right|^{\beta/q}. \tag{6.20}$$

To estimate the Hölder constant of $z(x,t)$ with respect to x we consider two cases. Let $\Delta x = \overline{x} - x > 0$ and $\Delta x \ge t^{1/q}$. By Lemma 6.3 $z(x,0) = 0$ and by (6.20), we get

$$\frac{|z(\overline{x},t) - z(x,t)|}{|\Delta x|^\beta} \le \frac{|z(\overline{x},t) - z(x,0)|}{t^{\beta/q}} + \frac{|z(x,t) - z(x,0)|}{t^{\beta/q}} \le C(R)\,|f|_{R^+}^{(2\beta/q)}, \tag{6.21}$$

i.e., we have the required estimate.

In the case $\Delta x < t^{1/q}$ we consider two possibilities. The first one is $\Delta x < x/2$. Then

$$z(\overline{x},t) - z(x,t) = \int_0^\infty \left(G(\overline{x},\xi,t) - G(x,\xi,t)\right)\left(f(\xi) - f(x)\right) d\xi$$

$$+ (f(x) - f(\overline{x})) \int_0^\infty G(\overline{x}, \xi, t)\, d\xi = i_1(x, \overline{x}, t) + i_2(x, \overline{x}, t).$$

Using (4.17), we can evaluate $i_2(x, \overline{x}, t)$

$$|i_2(x, \overline{x}, t)| \le |f|^{(2\beta/q)}_{R^+} |\overline{x} - x|^{2\beta/q} = |f|^{(2\beta/q)}_{R^+} |\overline{x} - x|^{\beta} |\overline{x} - x|^{\alpha\beta/q}$$

$$\le |f|^{(2\beta/q)}_{R^+} |\overline{x} - x|^{\beta} |\Delta x|^{\alpha\beta/q} \le C(R) |f|^{(2\beta/q)}_{R^+} |\overline{x} - x|^{\beta}. \qquad (6.22)$$

To estimate $i_1(x, \overline{x}, t)$ we apply the mean value theorem. Let $\theta \in [x, \overline{x}]$, then

$$|i_1(x, \overline{x}, t)| \le C(R) |f|^{(2\beta/q)}_{R^+} \int_0^\infty G_\theta(\theta, \xi, t) |\Delta x| \left| x^{q/2} - \xi^{q/2} \right|^{2\beta/q} d\xi.$$

Due to $\Delta x < x/2$ the values θ, $\overline{x}, x$ are equivalent. Therefore we change θ by x below. We change ξ to u in the integral and use estimate (6.6) that gives

$$|i_1(x, \overline{x}, t)| \le C(R) |f|^{(2\beta/q)}_{R^+} \frac{\Delta x}{x} t^{\beta/q} (i_{11}(x, \overline{x}, t) + i_{12}(x, \overline{x}, t)),$$

$$i_{11}(x, \overline{x}, t) = \int_0^{1/v} e^{-\left(u^2+v^2\right)} v^2 \left(1 + u^2\right) u^{-\frac{\alpha}{q}} |v - u|^{2\beta/q} du,$$

$$i_{12}(x, \overline{x}, t) = \int_{1/v}^\infty e^{-\gamma(u-v)^2} (1 + v)(uv)^{1/q - 1/2} u^{-\frac{\alpha}{q}} |v - u|^{2\beta/q} du.$$

Since $e^{-\left(u^2+v^2\right)} |v - u|^{2\beta/q} \le C e^{-\gamma\left(u^2+v^2\right)}$, $0 < \gamma < 1$, we get

$$|i_{11}(x, \overline{x}, t)| \le C v^2 e^{-\gamma v^2} \int_0^{1/v} \left(1 + u^2\right) u^{-\frac{\alpha}{q}} e^{-\gamma u^2} du,$$

and one can see that either the estimate

$$|i_{11}(x, \overline{x}, t)| \le C v \qquad (6.23)$$

or

$$|i_{11}(x, \overline{x}, t)| \le C v^2 \qquad (6.24)$$

is valid.

Similarly, with Lemma 6.1

$$\left|i_{12}\left(x,\overline{x},t\right)\right| \le C\left(1+v\right)v^{1/q-1/2}\int\limits_{1/v}^{\infty} e^{-\gamma(u-v)^2}u^{-\frac{\alpha}{q}}du$$

$$\le C\left(1+v\right)v^{\alpha/2q}\begin{cases} v^{-\alpha/2q}, & v\ge 1,\\ e^{-\delta_1/v^2}, & v<1,\end{cases}$$

and again

$$\left|i_{12}\left(x,\overline{x},t\right)\right| \le Cv, \tag{6.25}$$

or

$$\left|i_{12}\left(x,\overline{x},t\right)\right| \le Cv^2. \tag{6.26}$$

Now we use estimates (6.23) and (6.25) and can write

$$\left|i_1\left(x,\overline{x},t\right)\right| \le C\left(R\right)\left|f\right|_{R^+}^{(2\beta/q)}\frac{\Delta x}{x}t^{\beta/q}v.$$

Note that in the case under consideration $v=x^{q/2}/qt^{1/2}$ and

$$\frac{|\Delta x|}{|\Delta x|^{\beta}}\frac{x^{q/2}}{xt^{1/2-\beta/q}} = \left|\frac{\Delta x}{x}\right|^{1-q/2}\left(\frac{\Delta x}{t^{1/q}}\right)^{q/2-\beta} \le \text{const}.$$

It means

$$\left|i_1\left(x,\overline{x},t\right)\right| \le C\left(R\right)\left|f\right|_{R^+}^{(2\beta/q)}\left|\Delta x\right|^{\beta}. \tag{6.27}$$

Thus for $\Delta x\le x/2$ and $\Delta x\le t^{1/q}$

$$\left|z\left(\overline{x},t\right)-z\left(x,t\right)\right| \le C\left(R\right)\left|f\right|_{R^+}^{(2\beta/q)}\left|\Delta x\right|^{\beta}. \tag{6.28}$$

Finally, we consider the case $\Delta x\ge x/2$ and $\Delta x\le t^{1/q}$. We have

$$\frac{\left|z\left(\overline{x},t\right)-z\left(x,t\right)\right|}{|\Delta x|^{\beta}} \le C\left(\frac{\left|z\left(\overline{x},t\right)-z\left(0,t\right)\right|}{\overline{x}^{\beta}}+\frac{\left|z\left(x,t\right)-z\left(0,t\right)\right|}{x^{\beta}}\right)$$

since $2\Delta x\ge x$ and $3\Delta x\ge \overline{x}$. We evaluate, as an example, the second term on the right hand side. To this end we use the inequality

$$\left|z\left(x,t\right)-z\left(0,t\right)\right| \le \int\limits_0^{\infty}\left|G\left(x,\xi,t\right)-G\left(0,\xi,t\right)\right|\left|f\left(\xi\right)-f\left(x\right)\right|d\xi$$

$$+\left|\int\limits_0^{\infty}G\left(0,\xi,t\right)\left(f\left(0\right)-f\left(x\right)\right)d\xi\right| = i_3\left(x,\overline{x},t\right)+i_4\left(x,\overline{x},t\right).$$

The term $i_4(x,\overline{x},t)$ is evaluated by (4.17) and $2\Delta x \geq x$ as follows:

$$|i_4(x,\overline{x},t)| \leq |f(0)-f(x)| \leq \langle f\rangle_{x,R^+}^{(2\beta/q)} x^{2\beta/q} = \langle f\rangle_{x,R^+}^{(2\beta/q)} x^{\beta} x^{2\beta/q-\beta}$$
$$\leq C(R)\langle f\rangle_{x,R^+}^{(2\beta/q)} |\Delta x|^{\beta}. \tag{6.29}$$

The term $i_3(x,\overline{x},t)$ is similar to $i_1(x,\overline{x},t)$ and to estimate it we use the mean value theorem. Let $\theta \in [0,x]$ and $\overline{v} = \theta^{q/2}/qt^{1/2}$. In this case we apply inequalities (6.23) and (6.25) and get

$$|i_3(x,\overline{x},t)| \leq C(R)|f|_{R^+}^{(2\beta/q)} \frac{x}{\theta} t^{\beta/q}\overline{v}^2 \leq C(R)|f|_{R^+}^{(2\beta/q)} x^{\beta}$$
$$\leq C(R)|f|_{R^+}^{(2\beta/q)} |\Delta x|^{\beta}, \tag{6.30}$$

since

$$\frac{x^{1-\beta}}{\theta} t^{\beta/q}\left(\frac{\theta^{q/2}}{t^{1/2}}\right)^2 = x^{1-\beta}t^{-1+\beta/q}\theta^{q-1} \leq t^{\frac{1-\beta}{q}-1+\frac{\beta}{q}+\frac{q-1}{q}} = 1.$$

Inequalities (6.20), (6.21), (6.28), (6.29), and (6.30) lead to inequality (6.18). Lemma 6.4 is proved. ■

R e m a r k 6.1. The reader will see that in our subsequent evaluations in Lemmas 6.5–6.8 we will apply the approach which is analogous to the approach used in the proof of Lemma 6.4. We will use the change of variable $\xi \to u$, split the integration domain according to the small and large values of $z = 2uv$, and apply the estimates of the Green function.

Lemma 6.5. *Let*

$$w(x,t) = \int_0^{\infty} G(x,\xi,t)\, f(\xi,t)\, d\xi, \quad f(x,t) \in C^{\beta,\beta/q}\left(R_T^+\right),\ f(x,0) = 0. \tag{6.31}$$

Then

$$\langle w\rangle_{x,R_T^+}^{(\beta)} + \langle w\rangle_{t,R_T^+}^{(\beta/q)} \leq C\langle f\rangle_{t,R_T^+}^{(\beta/q)}. \tag{6.32}$$

P r o o f. First we estimate the Hölder constant of $w(x,t)$ in t. Let $0 < t < \overline{t}$, $\Delta t = \overline{t} - t$. In the case of $\Delta t > t/2$ it follows that $\Delta t = \overline{t} - t > \overline{t} - 2\Delta t$. Then $\Delta t > \overline{t}/3$ and hence

$$\frac{\left|w(x,\overline{t}) - w(x,t)\right|}{|\Delta t|^{\beta/q}} \leq C\left(\frac{\left|w(x,\overline{t})\right|}{\overline{t}^{\beta/q}} + \frac{|w(x,t)|}{t^{\beta/q}}\right).$$

Taking into account $f(x,0)=0$, we can estimate the terms on the right hand side. For example, by (4.17)

$$\frac{|w(x,t)|}{t^{\beta/q}} \le \int_0^\infty G(x,\xi,t)\,\frac{|f(\xi,t)|}{t^{\beta/q}}d\xi \le \langle f\rangle_{t,R_T^+}^{(\beta/q)} \int_0^\infty G(x,\xi,t)\,d\xi \le \langle f\rangle_{t,R_T^+}^{(\beta/q)}.$$

Therefore in this case

$$\langle w\rangle_{t,R_T^+}^{(\beta/q)} \le C\,\langle f\rangle_{t,R_T^+}^{(\beta/q)}. \tag{6.33}$$

Let now $\Delta t < t/2 < \overline{t}/2$. We use the representation

$$w\left(x,\overline{t}\right) - w(x,t) = \int_0^\infty G\left(x,\xi,\overline{t}\right)\left[f\left(\xi,\overline{t}\right) - f(\xi,t)\right]d\xi$$

$$+\int_0^\infty \left[G\left(x,\xi,\overline{t}\right) - G(x,\xi,t)\right] f(\xi,t)\,d\xi = w_1(x,t) + w_2(x,t).$$

For the function $w_1(x,t)$ we obtain

$$|w_1(x,t)| \le \langle f\rangle_{t,R_T^+}^{(\beta/q)} |\Delta t|^{\beta/q} \int_0^\infty \left|G\left(x,\xi,\overline{t}\right)\right| d\xi = \langle f\rangle_{t,R_T^+}^{(\beta/q)} |\Delta t|^{\beta/q}. \tag{6.34}$$

We apply the mean value theorem to estimate $w_2(x,t)$ and note that estimate (6.5) means $|G_t(x,\xi,t)| \le Ct^{-1}|G(x,\xi,t)|$. Moreover, the inequality $\Delta t < t/2$ implies that for any $\theta \in \left[t,\overline{t}\right]$ the values θ, t, and $\overline{t}$ are equivalent, $\theta \sim t \sim \overline{t}$. Then

$$|w_2(x,t)| \le C\frac{\Delta t}{\theta}\langle f\rangle_{t,R_T^+}^{(\beta/q)} t^{\beta/q} \int_0^\infty |G(x,\xi,\theta)|\,d\xi \le C\frac{\Delta t}{t}\langle f\rangle_{t,R_T^+}^{(\beta/q)} t^{\beta/q}$$

$$= C\langle f\rangle_{t,R_T^+}^{(\beta/q)} |\Delta t|^{\beta/q}\frac{|\Delta t|^{1-\beta/q}}{t^{1-\beta/q}} \le C\langle f\rangle_{t,R_T^+}^{(\beta/q)} |\Delta t|^{\beta/q}. \tag{6.35}$$

From (6.33)–(6.35) it follows that

$$\langle w\rangle_{t,R_T^+}^{(\beta/q)} \le C\,\langle f\rangle_{t,R_T^+}^{(\beta/q)} \tag{6.36}$$

in all cases.

Our next step is to evaluate the Hölder constant of $w(x,t)$ with respect to x. Let $0 < \Delta x = \overline{x} - x$. For $\Delta x > t^{1/q}$ due to $w(x,0) = 0$ we obtain

$$\frac{|w(\overline{x},t) - w(x,t)|}{|\Delta x|^{\beta}} \le \frac{|w(\overline{x},t) - w(\overline{x},0)|}{t^{\beta/q}} + \frac{|w(x,t) - w(x,0)|}{t^{\beta/q}}$$
$$\le 2\langle w\rangle_{t,R_T^+}^{(\beta/q)} \le C\langle f\rangle_{t,R_T^+}^{(\beta/q)}. \tag{6.37}$$

Next we suppose $\Delta x \le t^{1/q}$ and $\Delta x \le x/2$. Consider the difference

$$w(\overline{x},t) - w(x,t) = \int_0^{\infty} [G(\overline{x},\xi,t) - G(x,\xi,t)]\, f(\xi,t)\, d\xi. \tag{6.38}$$

The mean value theorem gives

$$G(\overline{x},\xi,t) - G(x,\xi,t) = G_x(\theta,\xi,t)(\overline{x} - x), \quad \theta \in [x,\overline{x}],$$

and since $\theta \sim x \sim \overline{x}$ for $\Delta x \le x/2$

$$G(\overline{x},\xi,t) - G(x,\xi,t) \le C\,|G_x(x,\xi,t)|\,|\Delta x|.$$

We use this inequality in (6.38) and arrive at the situation which we already encountered in the proof of Lemma 6.4 (see inequalities (6.23) and (6.25)). Therefore

$$|w(\overline{x},t) - w(x,t)| \le C\langle f\rangle_{t,R_T^+}^{(\beta/q)} \frac{\Delta x}{x} t^{\beta/q} \frac{x^{q/2}}{t^{1/2}}$$
$$= C\langle f\rangle_{t,R_T^+}^{(\beta/q)} |\Delta x|^{\beta} \frac{|\Delta x|^{1-\beta}}{x} x^{q/2} t^{\beta/q-1/2}$$
$$= C\langle f\rangle_{t,R_T^+}^{(\beta/q)} |\Delta x|^{\beta} \left(\frac{\Delta x}{x}\right)^{1-q/2} \left(\frac{\Delta x}{t^{1/q}}\right)^{q/2-\beta} \le C\langle f\rangle_{t,R_T^+}^{(\beta/q)} |\Delta x|^{\beta}. \tag{6.39}$$

In the case $\Delta x > x/2$ and $\Delta x \le t^{1/q}$ we use the inequality

$$\frac{|w(\overline{x},t) - w(x,t)|}{|\Delta x|^{\beta}} \le C\left\{\frac{|w(\overline{x},t) - w(0,t)|}{\overline{x}^{\beta}} + \frac{|w(x,t) - w(0,t)|}{x^{\beta}}\right\},$$

and, for instance, the second term on the right hand side is estimated as follows (see the estimate of i_3 in the proof of Lemma 6.4):

$$\frac{|w(x,t) - w(0,t)|}{x^{\beta}} \le C\langle f\rangle_{t,R_T^+}^{(\beta/q)} \frac{x}{x^{\beta}} t^{\beta/q} \int_0^{\infty} |G_x(\theta,\xi,t)|\, d\xi$$
$$\le C\langle f\rangle_{t,R_T^+}^{(\beta/q)} x^{1-\beta} t^{\beta/q} \frac{1}{\theta} \left(\frac{\theta^{q/2}}{t^{1/2}}\right)^2 \le C\langle f\rangle_{t,R_T^+}^{(\beta/q)}.$$

In this case we obtain

$$|w(\overline{x},t) - w(x,t)| \le C \langle f \rangle_{t,R_T^+}^{(\beta/q)} |\Delta x|^\beta . \tag{6.40}$$

Inequalities (6.36), (6.37), (6.39), and (6.40) give (6.32). Lemma 6.5 is proved. ■

Corollary 6.1. *Let* $f(x,t) \in C^{\beta,\beta/q}\left(R_T^+\right)$, $f(x,0) \in C^{2\beta/q}(R^+)$,

$$w(x,t) = \int_0^\infty G(x,\xi,t)\ f(\xi,t)\, d\xi.$$

The inequality

$$\langle w \rangle_{x,B_{R,T}}^{(\beta)} + \langle w \rangle_{t,B_{R,T}}^{(\beta/q)} \le C(R)\left(\langle f \rangle_{t,R_T^+}^{(\beta/q)} + |f(x,0)|_{R^+}^{(2\beta/q)} \right) \tag{6.41}$$

is valid.

Inequality (6.41) is the consequence of (6.18) and (6.32). ■

Lemma 6.6. (*The potential of the initial data*)
Let

$$w(x,t) = \int_0^\infty G(x,\xi,t)\ u_0(\xi)\, d\xi, \quad x^\alpha u_{0xx} \in C^{2\beta/q}\left(R^+\right). \tag{6.42}$$

Then

$$\langle w_t \rangle_{x,\widetilde{\Omega}_T}^{(\beta)} + \langle w_t \rangle_{t,\widetilde{\Omega}_T}^{(\beta/q)} + \langle w_x \rangle_{t,\widetilde{\Omega}_T}^{(\beta+1-\alpha)/q} \le C(R) \left| x^\alpha u_{0xx} \right|_{R^+}^{2\beta/q}. \tag{6.43}$$

P r o o f. We can consider the case of $u_0(0) = u_{0x}(0) = 0$ since otherwise we can introduce the function $v(x,t) = u(x,t) - u_0(0) - u_{0x}(0)\, x$ in problem (4.10)–(4.12). Using (4.18) and integrating by parts, we get

$$w_t(x,t) = \int_0^\infty G_t(x,\xi,t)\ u_0(\xi)\, d\xi = \int_0^\infty \left(\xi^\alpha G(x,\xi,t)\right)_{\xi\xi} u_0(\xi)\, d\xi$$
$$= \int_0^\infty G(x,\xi,t)\ \xi^\alpha u_{0\xi\xi}(\xi)\, d\xi.$$

Lemma 6.4 gives the estimates of $w_t(x,t)$, and the estimate of $w_x(x,t)$ can be obtained similarly. ■

Lemma 6.7. (*The volume potential*)
Let

$$w(x,t) = \int_0^t d\tau \int_0^\infty G(x,\xi,t-\tau)\ f(\xi,\tau)\,d\xi, \quad f(x,t) \in C^{\beta,\beta/q}\left(R_T^+\right), \tag{6.44}$$

$f(x,0) \in C^{2\beta/q}\left(R^+\right).$

Then

$$\langle w_t\rangle_{x,B_{R,T}}^{(\beta)} + \langle w_t\rangle_{t,B_{R,T}}^{(\beta/q)} + \langle w_x\rangle_{t,\widetilde{\Omega}_T}^{(\beta+1-\alpha)/q} \le C(R)\left(\langle f\rangle_{t,R_T^+}^{(\beta/q)} + |f(x,0)|_{R^+}^{(2\beta/q)}\right). \tag{6.45}$$

P r o o f. First we derive the representation for $w_t(x,t)$. Let

$$w_h(x,t) = \int_0^{t-h} d\tau \int_0^\infty G(x,\xi,t-\tau)\ f(\xi,\tau)\,d\xi,$$

then

$$\frac{\partial w_h}{\partial t}(x,t) = \int_0^{t-h} d\tau \int_0^\infty G_t(x,\xi,t-\tau)\ f(\xi,\tau)\,d\xi + \int_0^\infty G(x,\xi,h)\ f(\xi,t-h)\,d\xi$$

$$= \int_0^{t-h} d\tau \int_0^\infty G_t(x,\xi,t-\tau)\ [f(\xi,\tau) - f(\xi,t)]\,d\xi$$

$$- \int_0^\infty d\xi f(\xi,t) \int_0^{t-h} d\tau G_\tau(x,\xi,t-\tau)$$

$$+ \int_0^\infty G(x,\xi,h)\ f(\xi,t-h)\,d\xi = \int_0^{t-h} d\tau \int_0^\infty G_t(x,\xi,t-\tau)\ [f(\xi,\tau) - f(\xi,t)]\,d\xi$$

$$- \int_0^\infty G(x,\xi,h)\ f(\xi,t)\,d\xi + \int_0^\infty G(x,\xi,t)\ f(\xi,t)\,d\xi + \int_0^\infty G(x,\xi,h)\ f(\xi,t-h)\,d\xi.$$

Now we go to the limit as $h \to 0$ and get

$$\frac{\partial w}{\partial t}(x,t) = \int_0^t d\tau \int_0^\infty G_t(x,\xi,t-\tau)\ [f(\xi,\tau) - f(\xi,t)]\,d\xi + \int_0^\infty G(x,\xi,t)\ f(\xi,t)\,d\xi$$

$$= w_1(x,t) + w_2(x,t). \tag{6.46}$$

The estimate of the function $w_2(x,t)$ is given by (6.41).

Since

$$f(\xi,\tau)-f(\xi,t)=(f(\xi,\tau)-f(\xi,0))-(f(\xi,t)-f(\xi,0)),$$

we can assume that in the representation of $w_1(x,t)$, $f(x,t)$ has the property $f(x,0)=0$.

Consider the smoothness of $w_1(x,t)$ with respect to x. Let $0<\Delta x=\overline{x}-x$ and $\Delta x\le x/2$. We have

$$w_1(\overline{x},t)-w_1(x,t)$$

$$=\int\limits_{t-(\Delta x)^q}^{t}d\tau\int\limits_0^\infty G_t(\overline{x},\xi,t-\tau)\,[f(\xi,\tau)-f(\xi,t)]\,d\xi$$
$$-\int\limits_{t-(\Delta x)^q}^{t}d\tau\int\limits_0^\infty G_t(x,\xi,t-\tau)\,[f(\xi,\tau)-f(\xi,t)]\,d\xi$$

$$+\int\limits_0^{t-(\Delta x)^q}d\tau\int\limits_0^\infty[G_t(\overline{x},\xi,t-\tau)-G_t(x,\xi,t-\tau)]\,[f(\xi,\tau)-f(\xi,t)]\,d\xi$$
$$=i_1(\overline{x},x,t)+i_2(\overline{x},x,t)+i_3(\overline{x},x,t). \qquad (6.47)$$

The comparison of (6.4) and (6.5) gives $|G_t|\le Ct^{-1}|G|$. With this remark and (4.17) we have

$$|i_1(\overline{x},x,t)|\le C\langle f\rangle_{t,R_T^+}^{(\beta/q)}\int\limits_{t-(\Delta x)^q}^{t}d\tau\int\limits_0^\infty(t-\tau)^{-1+\beta/q}\,|G(\overline{x},\xi,t-\tau)|\,d\xi$$

$$\le C\langle f\rangle_{t,R_T^+}^{(\beta/q)}\int\limits_{t-(\Delta x)^q}^{t}d\tau\,(t-\tau)^{-1+\beta/q}\le C\langle f\rangle_t^{(\beta/q)}|\Delta x|^\beta.$$

The integral $i_2(\overline{x},x,t)$ is evaluated in a similar way. To estimate $i_3(\overline{x},x,t)$, we change the integration variable $\xi\to u$, use inequality (6.7), and note that due to $\Delta x\le x/2$ the values $\theta\in[x,\overline{x}]$, x, and $\overline{x}$ are equivalent. As the result, we obtain

$$|i_3(\overline{x},x,t)|\le C\langle f\rangle_{t,R_T^+}^{(\beta/q)}|\Delta x|\int\limits_0^{t-(\Delta x)^q}(t-\tau)^{\beta/q}\,d\tau\int\limits_0^\infty|G_{tx}(\theta,\xi,t-\tau)|\,d\xi$$

$$\leq C \langle f \rangle_{t,R_T^+}^{(\beta/q)} \frac{|\Delta x|}{x} \int\limits_0^{t-(\Delta x)^q} (t-\tau)^{-1+\beta/q} \, d\tau e^{-v^2} v^2 \int\limits_0^{1/v} e^{-u^2} \left(1+u^2\right) u^{-\alpha/q} du$$

$$+ C \langle f \rangle_{t,R_T^+}^{(\beta/q)} \frac{|\Delta x|}{x} \int\limits_0^{t-(\Delta x)^q} (t-\tau)^{-1+\beta/q} \, d\tau v^{1/q-1/2}$$

$$\times (1+v) \int\limits_{1/v}^{\infty} u^{1/q-1/2-\alpha/q} e^{-\gamma(v-u)^2} du.$$

We drop the estimates of the internal integrals (see, for example, the proof of Lemma 6.4) and write down

$$|i_3 (\overline{x}, x, t)| \leq C \langle f \rangle_{t,R_T^+}^{(\beta/q)} \frac{|\Delta x|}{x} \int\limits_0^{t-(\Delta x)^q} (t-\tau)^{-1+\beta/q} \frac{x^{q/2}}{(t-\tau)^{1/2}} d\tau$$

$$\leq C \langle f \rangle_{t,R_T^+}^{(\beta/q)} \frac{|\Delta x|}{x} x^{q/2} \int\limits_{|\Delta x|^q}^{\infty} y^{-1-1/2+\beta/q} dy.$$

The last integral exists since, under our assumptions, $\frac{1}{2} - \frac{\beta}{q} > 0$. Finally we get

$$|i_3 (\overline{x}, x, t)| \leq C \langle f \rangle_{t,R_T^+}^{(\beta/q)} \frac{|\Delta x|}{x} x^{q/2} |\Delta x|^{q(-1/2+\beta/q)}$$

$$= C \langle f \rangle_{t,R_T^+}^{(\beta/q)} |\Delta x|^{\beta} \left(\frac{\Delta x}{x}\right)^{1-q/2} \leq C \langle f \rangle_t^{(\beta/q)} |\Delta x|^{\beta}.$$

Now let $\Delta x \geq x/2$. In this case we use the equality

$$w_1 (\overline{x}, t) - w_1 (x, t) = [w_1 (\overline{x}, t) - w_1 (0, t)] + [w_1 (x, t) - w_1 (0, t)].$$

To estimate the second term on the right hand side, we represent it analogously to the way it was done in (6.47)

$$w_1 (x, t) - w_1 (0, t) = a_1 (x, t) + a_2 (x, t) + a_3 (x, t).$$

The values of $a_1 (x, t), a_2 (x, t)$ are estimated similarly to $i_1 (\overline{x}, x, t)$, and $a_3 (x, t)$ is estimated similarly to $i_3 (\overline{x}, x, t)$, but in the case of $a_3 (x, t)$ we use the inequalities in (6.23) and (6.25). We arrive at

$$|w_1 (\overline{x}, t) - w_1 (x, t)| \leq C \langle f \rangle_{t,R_T^+}^{(\beta/q)} |\Delta x|^{\beta}.$$

So, we have proved that

$$\langle w_1(x,t)\rangle^{(\beta)}_{x,R_T^+} \le C\,\langle f\rangle^{(\beta/q)}_{t,R_T^+}. \tag{6.48}$$

To evaluate the Hölder constant of the function $w_1(x,t)$ with respect to t we use the representation $(0<\Delta t=\bar t-t)$

$$w_1(x,\bar t)-w_1(x,t)$$

$$= \int\limits_{\bar t-2\Delta t}^{\bar t} d\tau \int\limits_0^\infty G_t(\overline{x},\xi,\bar t-\tau)\,\left[f(\xi,\tau)-f(\xi,\bar t)\right]d\xi$$
$$-\int\limits_{t-\Delta t}^{t} d\tau \int\limits_0^\infty G_t(x,\xi,t-\tau)\,\left[f(\xi,\tau)-f(\xi,t)\right]d\xi$$

$$+\int\limits_0^{t-\Delta t} d\tau \int\limits_0^\infty \left[G_t(x,\xi,\bar t-\tau)-G_t(x,\xi,t-\tau)\right]\left[f(\xi,\tau)-f(\xi,t)\right]d\xi$$

$$+\int\limits_0^{t-\Delta t} d\tau \int\limits_0^\infty G_t(x,\xi,\bar t-\tau)\,\left[f(\xi,\tau)-f(\xi,\bar t)\right]d\xi$$
$$= A_1(x,t,\bar t)+A_2(x,t,\bar t)+A_3(x,t,\bar t)+A_4(x,t,\bar t). \tag{6.49}$$

We can estimate $A_4(x,t,\bar t)$ as follows:

$$\left|A_4(x,t,\bar t)\right| \le \left|\int\limits_0^\infty \left[f(\xi,t)-f(\xi,\bar t)\right]d\xi \int\limits_0^{t-\Delta t} d\tau G_\tau(\overline{x},\xi,\bar t-\tau)\right|$$

$$\le C\,\langle f\rangle^{(\beta/q)}_{t,R_T^+}\left|\bar t-t\right|^{\beta/q}\int\limits_0^\infty \left[\left|G(x,\xi,\bar t)\right|+\left|G(x,\xi,\Delta t)\right|\right]d\xi \le C\,\langle f\rangle^{(\beta/q)}_{t,R_T^+}\left|\Delta t\right|^{\beta/q}.$$

The integrals $A_1(x,t,\bar t)$, $A_2(x,t,\bar t)$ are estimated similarly

$$\left|A_1(x,t,\bar t)\right| \le C\,\langle f\rangle^{(\beta/q)}_{t,R_T^+}\int\limits_{\bar t-2\Delta t}^{\bar t}\left(\bar t-t\right)^{-1+\beta/q}d\tau\int\limits_0^\infty \left|G(x,\xi,\bar t-\tau)\right|d\xi$$

$$\leq C\left\langle f\right\rangle_{t,R_T^+}^{(\beta/q)}\int\limits_{\bar t-2\Delta t}^{\bar t}\left(\bar t-t\right)^{-1+\beta/q}d\tau\leq C\left\langle f\right\rangle_{t,R_T^+}^{(\beta/q)}\left|\Delta t\right|^{\beta/q}.$$

In the integral $A_3\left(x,t,\bar t\right)$ it is natural to assume $\Delta t\leq t$. With this condition for any $\theta\in\left[t,\bar t\right]$ the values θ, t, $\bar t$ are equivalent, so that if we apply the mean value theorem, we obtain

$$G_t\left(x,\xi,\bar t-\tau\right)-G_t\left(x,\xi,t-\tau\right)=G_{tt}\left(x,\xi,\theta-\tau\right)\Delta t\sim G_{tt}\left(x,\xi,t-\tau\right)\Delta t.$$

We observe that from (6.8) and (6.4) it follows that

$$\left|G_{tt}\left(x,\xi,t-\tau\right)\right|\leq C\left|t-\tau\right|^{-2}\left|G\left(x,\xi,t-\tau\right)\right|.$$

Therefore

$$\left|A_3\left(x,t,\bar t\right)\right|\leq C\left\langle f\right\rangle_{t,R_T^+}^{(\beta/q)}\Delta t\int\limits_0^{t-\Delta t}d\tau\left(t-\tau\right)^{-2+\beta/q}$$
$$\leq C\left\langle f\right\rangle_{t,R_T^+}^{(\beta/q)}\Delta t\int\limits_{-\infty}^{t-\Delta t}d\tau\left(t-\tau\right)^{-2+\beta/q}\leq C\left\langle f\right\rangle_{t,R_T^+}^{(\beta/q)}\left(\Delta t\right)^{\beta/q}.$$

Thus it is proved that

$$\left\langle w_1\left(x,t\right)\right\rangle_{t,R_T^+}^{(\beta/q)}\leq C\left\langle f\right\rangle_{t,R_T^+}^{(\beta/q)}. \tag{6.50}$$

The inequalities (6.48), (6.50) together with the estimate of the function $w_2\left(x,t\right)$ give (6.45) for $w_t(x,t)$. The estimate of $w_x(x,t)$ is obtained in a similar way. This completes the proof. ■

R e m a r k 6.2. Since the function $w\left(x,t\right)$ from (6.44) is a solution of the equation $w_t-x^\alpha w_{xx}=f\left(x,t\right)$, we get

$$\left\langle x^\alpha w_{xx}\right\rangle_{x,B_{R,T}}^{(\beta)}+\left\langle x^\alpha w_{xx}\right\rangle_{t,B_{R,T}}^{(\beta/q)}\leq C\left(R\right)\left(\left\langle f\right\rangle_{t,R_T^+}^{(\beta/q)}+\left|f\left(x,0\right)\right|_{R^+}^{(2\beta/q)}\right) \tag{6.51}$$

■

The kernel of the simple layer potential in (4.19) is

$$g\left(x,t\right)=G\left(x,\xi,t\right)\xi^\alpha\mid_{\xi=0}=Ct^{-1+1/q}e^{-\frac{x^q}{q^2t}}. \tag{6.52}$$

Lemma 6.8. (*The simple layer potential*)
Let

$$w(x,t)=\int\limits_0^t g(x,t-\tau)\,f_1(\tau)\,d\tau,\quad f_1(t)\in C^{\frac{1+\beta-\alpha}{q}}([0,T]),\ f_1(0)=0. \quad (6.53)$$

Then

$$\langle w_t\rangle_x^{(\beta)}+\langle w_t\rangle_t^{(\beta/q)}+\langle w_x\rangle_t^{(\beta+1-\alpha)/q}\le C\langle f_1\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}. \quad (6.54)$$

P r o o f. To prove Theorem 4.1 it is sufficient to consider the case of $f_1(0)=0$ since otherwise we can introduce the other unknown function as it was proposed in the proof of Lemma 6.7. We represent the derivative $w_t(x,t)$ in the form

$$w_t(x,t)=\int\limits_0^t g_t(x,t-\tau)\,f_1(\tau)\,d\tau$$

$$=\int\limits_0^t g_t(x,t-\tau)\,[f_1(\tau)-f_1(t)]\,d\tau-f_1(t)\int\limits_0^t g_\tau(x,t-\tau)\,d\tau$$

$$=\int\limits_0^t g_t(x,t-\tau)\,[f_1(\tau)-f_1(t)]\,d\tau-f_1(t)\,g(x,t)=w_1(x,t)+w_2(x,t). \quad (6.55)$$

For the function $w_2(x,t)$ the Hölder constant in t is obtained as follows. We represent $w_2(x,t)$ as

$$w_2(x,t)=g_1(t)\,g_2(x,t),\quad g_1(t)=f_1(t)/t^{(1-\alpha)/q},\quad g_2(x,t)=e^{-\frac{x^q}{q^2t}}$$

and use the inequality

$$\langle w_2(x,t)\rangle_t^{(\beta/q)}\le\langle g_1(t)\rangle_t^{(\beta/q)}\max|g_2(x,t)|+\max|g_1(t)|\,\langle g_2(x,t)\rangle_t^{(\beta/q)}. \quad (6.56)$$

The value $\langle g_2(x,t)\rangle_t^{(\beta/q)}$ can be evaluated as follows:

$$\langle g_2(x,t)\rangle_t^{(\beta/q)}\le t^{1-\beta/q}\max|g_{2t}|\le Ct^{1-\beta/q}t^{-1}=Ct^{-\frac{\beta}{q}}.$$

Since $g_1(t)\le\langle f_1\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}t^{\beta/q}$,

$$\max|g_1(t)|\,\langle g_2(x,t)\rangle_t^{(\beta/q)}\le C\langle f_1\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}.$$

The term $\langle g_1(t)\rangle_t^{(\beta/q)} \max|g_2(x,t)|$ is estimated similarly, thus

$$\langle w_2(x,t)\rangle_t^{(\beta/q)} \le C\langle f_1\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}. \tag{6.57}$$

For the function $w_1(x,t)$ we consider the difference

$$\begin{aligned} w_1(x,\bar t) - w_1(x,t) = &\int\limits_{\bar t-2\Delta t}^{\bar t} g_t(x,\bar t-\tau)\left[f_1(\tau)-f_1(\bar t)\right]d\tau \\ &- \int\limits_{t-\Delta t}^{t} g_t(x,t-\tau)\left[f_1(\tau)-f_1(t)\right]d\tau \end{aligned}$$

$$\begin{aligned} &+ \int\limits_{0}^{t-\Delta t}\left[g_t(x,\bar t-\tau)-g_t(x,t-\tau)\right]\left[f_1(\tau)-f_1(t)\right]d\tau \\ &+ \left[f_1(t)-f_1(\bar t)\right]\int\limits_{0}^{t-\Delta t} g_t(x,\bar t-\tau)\,d\tau = \sum_{k=1}^{4} b_k(x,t,\bar t). \end{aligned} \tag{6.58}$$

The estimate of $b_1(x,t,\bar t)$ has the form

$$\left|b_1(x,t,\bar t)\right| \le C\langle f_1\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)} \int\limits_{\bar t-2\Delta t}^{\bar t} (\bar t-\tau)^{-2+\frac{1}{q}+\frac{1+\beta-\alpha}{q}} e^{-\frac{x^q}{q^2(\bar t-\tau)}}d\tau$$

$$\le C\langle f_1\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)} \int\limits_{0}^{2\Delta t} y^{-2+\frac{2}{q}+\frac{\beta-\alpha}{q}}dy \le C\langle f_1\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)} |\Delta t|^{\beta/q}.$$

The integral $b_2(x,t,\bar t)$ is estimated in the same way. It is natural to assume that $\Delta t \le t$ in $b_3(x,t,\bar t)$, otherwise this integral is absent. The application of the mean value theorem leads to ($\gamma = \text{const} > 0$)

$$\left|b_3(x,t,\bar t)\right| \le C\langle f_1\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)} |\Delta t| \int\limits_{0}^{t-\Delta t} (t-\tau)^{-3+\frac{1}{q}+\frac{1+\beta-\alpha}{q}} e^{-\gamma\frac{x^q}{q^2(\bar t-\tau)}}d\tau$$

$$\le C\langle f_1\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)} |\Delta t| \int\limits_{\Delta t}^{t} y^{-3+\frac{1}{q}+\frac{1+\beta-\alpha}{q}}dy \le C\langle f_1\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)} |\Delta t|^{\beta/q}.$$

Finally,

$$\left|b_4\left(x,t,\overline{t}\right)\right| \le C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)} |\Delta t|^{\frac{1+\beta-\alpha}{q}} \int\limits_0^{t-\Delta t} (t-\tau)^{-2+1/q}\, d\tau$$

$$\le C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta}{q}\right)} |\Delta t|^{\frac{1+\beta-\alpha}{q}} \int\limits_{\Delta t}^{\infty} y^{-2+\frac{1}{q}} dy \le C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)} |\Delta t|^{\beta/q}.$$

As the result, we have shown that

$$\langle w_t\rangle_t^{(\beta/q)} \le C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}. \tag{6.59}$$

To estimate the Hölder constant for $w_t(x,t)$ in x we set $0<\Delta x=\overline{x}-x$ and first consider the case of $\Delta x \ge t^{1/q}$. Since $w_t(x,0)=0$, we can write

$$\frac{\left|w_t\left(\overline{x},t\right)-w_t\left(x,t\right)\right|}{|\Delta x|^{\beta}} \le \frac{\left|w_t\left(\overline{x},t\right)\right|}{t^{\beta/q}} + \frac{\left|w_t\left(x,t\right)\right|}{t^{\beta/q}} \le C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}$$

by using (6.59). Now we consider the case of $\Delta x < t^{1/q}$, first for the function $w_2\left(\overline{x},t\right)$. By the mean value theorem

$$g\left(\overline{x},t\right)-g\left(x,t\right) = Ct^{-1+1/q}\left[e^{-\frac{\overline{x}^q}{q^2 t}} - e^{-\frac{x^q}{q^2 t}}\right]$$

$$= Ct^{-1+1/q}\Delta x\frac{\theta^{q-1}}{t}e^{-\frac{\theta^q}{q^2 t}}, \quad \theta\in\left[x,\overline{x}\right],$$

so that

$$\left|w_2\left(\overline{x},t\right)-w_2\left(x,t\right)\right| \le C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)} \Delta x t^{-1+\frac{1+\beta-\alpha}{q}}$$

$$\le C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)} |\Delta x|^{\beta}\, t^{-1+\frac{1+\beta-\alpha}{q}+\frac{1-\beta}{q}}$$

$$\le C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)} |\Delta x|^{\beta}.$$

For the function $w_1(x,t)$ we study the difference

$$w_1\left(\overline{x},t\right)-w_1\left(x,t\right) = \int\limits_{t-(\Delta x)^q}^{t} g_t\left(\overline{x},t-\tau\right)\left(f_1(\tau)-f_1(t)\right)d\tau$$

$$- \int\limits_{t-(\Delta x)^q}^{t} g_t\left(x,t-\tau\right)\left(f_1(\tau)-f_1(t)\right)d\tau$$

$$+\int\limits_0^{t-(\Delta x)^q}\left[g_t\left(\overline{x},t-\tau\right)-g_t\left(x,t-\tau\right)\right]\left(f_1\left(\tau\right)-f_1\left(t\right)\right)d\tau=\sum_{k=1}^{3}a_k\left(x,\overline{x},t\right).$$

It is easy to estimate $a_1\left(x,\overline{x},t\right)$ and $a_2\left(x,\overline{x},t\right)$, for example, taking into account $\Delta x<t^{1/q}$,

$$\left|a_1\left(x,\overline{x},t\right)\right|\leq\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}\int\limits_{t-(\Delta x)^q}^{t}\left(t-\tau\right)^{\frac{1+\beta-\alpha}{q}-2+\frac{1}{q}}e^{-\frac{\overline{x}^q}{q^2(t-\tau)}}d\tau$$

$$\leq C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}\int\limits_0^{(\Delta x)^q}y^{\frac{1+\beta-\alpha}{q}-2+\frac{1}{q}}dy\leq C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}\left|\Delta x\right|^{\beta}.$$

Finally we estimate $a_3\left(x,\overline{x},t\right)$ by the mean value theorem

$$\left|a_3\left(x,\overline{x},t\right)\right|\leq C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}\Delta x\int\limits_0^{t-(\Delta x)^q}\left(t-\tau\right)^{-2+\frac{1+\beta-\alpha}{q}}e^{-\gamma\frac{\theta^q}{q^2(t-\tau)}}d\tau$$

$$\leq C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}\Delta x\int\limits_{(\Delta x)^q}^{\infty}y^{-2+\frac{1+\beta-\alpha}{q}}dy$$

$$\leq C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}\Delta x^{1+q\left(-1+\frac{1+\beta-\alpha}{q}\right)}\leq C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}\left|\Delta x\right|^{\beta}.$$

Thus

$$\left\langle w_t\right\rangle_x^{(\beta)}\leq C\left\langle f_1\right\rangle_t^{\left(\frac{1+\beta-\alpha}{q}\right)}. \tag{6.60}$$

The estimates of the function $w_t(x,t)$ are proved by inequalities (6.59), (6.60). The estimate of $w_x(x,t)$ is obtained in a similar way. ∎

Theorem 4.1 is the consequence of Lemmas 6.6–6.8.

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