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On the Characteristic Operators and Projections and on the Solutions of Weyl Type of Dissipative and Accumulative Operator Systems. III. Separated Boundary Conditions

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For the systems as in the title, boundary-value problems with separated boundary conditions are considered. We prove that the characteristic operator of such problem admits a special expression in terms of the projection (characteristic projection). This allows one to introduce for the above systems the analogues of the Weyl functions and solutions, to establish for them the Weyl type inequalities which turn out to be well known in a number of special cases.

Key words: operator differential equation, characteristic operator, characteristic projection, separated boundary conditions, Weyl function, Weyl type solution, maximal semi-definite subspace.
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This work constitutes Part III of [36]. Notation, definitions, numeration of the sections, statements, formulas etc., as well as the list of references extend those of [36].

3. Separated Condition (1.10). Characteristic Projection

Definition 3.1. *Let $M(\lambda)$ be a c.o. for equation (0.1) on $\mathcal{I}$. We say that condition (1.10) is separated with a nonreal $\lambda = \mu_0$ if for any $\mathcal{H}$-valued function $f(t) \in L^2_{w_{\mu_0}}(\mathcal{I})$ with a compact support solution $x_{\mu_0}(t)$ (1.9), for $\lambda = \mu_0$ of (0.1) satisfies*

$$\lim_{t\downarrow a} \Im\mu_0 U[x_{\mu_0}(t)] \geq 0, \qquad \lim_{t\uparrow b} \Im\mu_0 U[x_{\mu_0}(t)] \leq 0.^\star \tag{3.1}$$

*Every limit in (3.1) exists in view of (1.11).

Note that if $M(\lambda)$ is a c.o. and in one of inequalities (3.1) there is an equality, then another one holds automatically due to (1.10).

The following statement admits a proof similar to that of $n^0 2^0$ of Th. 1.1.

Remark 3.1. *The validity of (3.1) with a nonreal $\lambda = \mu_0$ for an operator function $M(\lambda) \in B(\mathcal{H})$ of the form (1.20) and any $\mathcal{H}$-valued vector function $f(t) \in L^2_{w_{\mu_0}}(\mathcal{I})$ with compact support is equivalent to*

$$\forall t \in \bar{\mathcal{I}} \quad \pm \Im\mu_0 \Gamma^{\pm}_{\mu_0}(t) \le 0 \tag{3.2}$$

with $\Gamma^{\pm}_{\lambda}(t)$ being as in (1.67).

Theorem 3.1. *Let $P = I$, $M(\lambda)$ (1.20) be a c.o. of (0.1), $\Im\mu_0 \neq 0$. Then condition (1.10) corresponding to $M(\lambda)$ is separated with $\lambda = \mu_0$ if and only if*

$$\mathcal{P}^2(\mu_0) = \mathcal{P}(\mu_0). \tag{3.3}$$

P r o o f. Suppose that condition (1.10) is separated with $\lambda = \mu_0$. Take in (3.2) $t = c$. Then one has: $\Im\mu_0 \mathcal{P}^*(\mu_0) G \mathcal{P}(\mu_0) \le 0$, $\Im\mu_0 (I - \mathcal{P}^*(\mu_0)) G (I - \mathcal{P}(\mu_0)) \ge 0$, which implies (3.3) in view of (1.69) and Ths. 2.4, 2.7.

Conversely, suppose (3.3) holds. By $n^o 2^o$ of Th. 1.1 one has

$$\begin{aligned} \forall [\alpha, \beta] \subseteq \bar{\mathcal{I}}: \quad & \Im\mu_0 (\mathcal{P}^*(\mu_0) X^*_{\mu_0}(\beta) Q(\beta) X_{\mu_0}(\beta) \mathcal{P}(\mu_0) \\ & - (I - \mathcal{P}^*(\mu_0)) X^*_{\mu_0}(\alpha) Q(\alpha) X_{\mu_0}(\alpha) (I - \mathcal{P}(\mu_0))) \le 0. \end{aligned} \tag{3.4}$$

Multiply (3.4) by $\mathcal{P}^*(\mu_0)$ from the left and by $\mathcal{P}(\mu_0)$ from the right to deduce that $\forall \beta \in \bar{\mathcal{I}} \ \ \Im\mu_0 \Gamma^{+}_{\mu_0}(\beta) \le 0$. In a similar way one can establish that $\forall \alpha \in \bar{\mathcal{I}}$ $\Im\lambda_0 \Gamma^{-}_{\lambda_0}(\alpha) \ge 0$, so the theorem is proved in view of Remark 3.1.

As a consequence of (1.68), formula (9) of [1] and Ths. 3.1, 2.4, 2.7, formula (1.69) we have

Corollary 3.1. *Let $P = I$, $M(\lambda)$ (1.20) be a c.o. of (0.1) on $\mathcal{I}$. Then in order to claim that condition (1.10) is separated with a nonreal $\lambda = \mu_0$, it is necessary to have simultaneously the two inequalities*

$$\begin{aligned} (I - \mathcal{P}^*(\mu_0)) \Delta_{\mu_0}(\alpha, c) (I - \mathcal{P}^*(\mu_0)) &\le \frac{1}{2\Im\mu_0} (I - \mathcal{P}^*(\mu_0)) G (I - \mathcal{P}(\mu_0)), \\ \mathcal{P}^*(\mu_0) \Delta_{\mu_0}(c, \beta) \mathcal{P}(\mu_0) &\le -\frac{1}{2\Im\mu_0} \mathcal{P}^*(\mu_0) G \mathcal{P}(\mu_0), \end{aligned} \tag{3.5}$$

for all finite $\alpha \le c \le \beta$, $[\alpha, \beta[\subseteq \bar{\mathcal{I}}$, and it is sufficient to have simultaneously the two inequalities (3.5) with $\alpha = c = \beta$.

Remark 3.2. *If $M(\lambda), \mathcal{P}(\lambda) \in B(\mathcal{H})$ are related by (1.20), then*

$$M(\lambda) = M^*(\bar{\lambda}) \Longleftrightarrow \mathcal{P}(\lambda) = G^{-1}(I - \mathcal{P}^*(\bar{\lambda}))G$$
$$\Longleftrightarrow (I - \mathcal{P}^*(\bar{\lambda})G(I - \mathcal{P}(\lambda)) = \mathcal{P}^*(\bar{\lambda})G\mathcal{P}(\lambda) \tag{3.6}$$

and hence

$$\left(M(\lambda) = M^*(\bar{\lambda})\right) \wedge \left((\mathcal{P}^2(\lambda) = \mathcal{P}(\lambda)\right)$$
$$\Longleftrightarrow (I - \mathcal{P}^*(\bar{\lambda})G(I - \mathcal{P}(\lambda)) = \mathcal{P}^*(\bar{\lambda})G\mathcal{P}(\lambda) = 0. \tag{3.7}$$

The following Remark 3.3. establishes a relationship between a c.o. with the separated condition (1.10) and the boundary-value problems with separated boundary conditions which depend on the spectral parameter.

Remark 3.3. *Suppose the interval $\mathcal{I} = (a, b)$ is finite and condition (1.3) holds with $F = \mathcal{H}$. Then:*

1^0. If operator functions $\mathcal{M}_\lambda, \mathcal{N}_\lambda$ from $n^0 1^0$ of Remark 1.1 are such that $\mathcal{M}^*_{\bar{\lambda}} Q(a)\mathcal{M}_\lambda = \mathcal{N}^*_{\bar{\lambda}} Q(b)\mathcal{N}_\lambda$, $\Im\lambda \mathcal{M}^*_\lambda Q(a)\mathcal{M}_\lambda \geq 0$, $\Im\lambda \mathcal{N}^*_\lambda Q(b)\mathcal{N}_\lambda \leq 0$, $\Im\lambda \neq 0$ (i.e., boundary condition (1.14), (1.13) is separated), then a solution of boundary-value problem (0.1), (1.14), (1.13) for any $\mathcal{H}$-valued $f(t) \in L^2_{w_\lambda}(\mathcal{I})$ is given by $x_\lambda(t)$ (1.9), where $M(\lambda)$ is a c.o. of (0.1) on $\mathcal{I}$ for which condition (1.10) is separated. Hence $M(\lambda)$ admits representation (1.20), with $\mathcal{P}(\lambda)$ being a projection which, as one can easily see, is just

$$\mathcal{P}(\lambda) = -X_\lambda^{-1}(b)\mathcal{N}_\lambda \left(X_\lambda^{-1}(a)\mathcal{M}(\lambda) - X_\lambda^{-1}(b)\mathcal{N}(\lambda)\right)^{-1} \tag{3.8}$$

where $(...)^{-1} \in B(\mathcal{H})$. In this setting, boundary condition (1.14), (1.13) is separated $\Leftrightarrow \left(M^*_{\bar{\lambda}} Q(a) M_\lambda = 0\right) \vee \left(N^*_{\bar{\lambda}} Q(b) N_\lambda = 0\right)$, $\Im\lambda \neq 0$.

2^0. If $M(\lambda)$ (1.20) is a c.o. of (0.1) on $\mathcal{I}$ in such a way that $\mathcal{P}^2(\lambda) = \mathcal{P}(\lambda)$, then $x_\lambda(t)$ (1.9) is a solution of some boundary-value problem from $n^0 1^0$ of Remark 1.1 with separated boundary condition (1.14, (1.13)).

P r o o f. All the claims of $n^0 1^0$, except the last one, follow from $n^0 1^0$ of Remark 1.1. As for the last claim of $n^0 1^0$, it follows from

$$\mathcal{M}^*_{\bar{\lambda}} Q(a)\mathcal{M}_\lambda = \mathcal{N}^*_{\bar{\lambda}} Q(b)\mathcal{N}_\lambda$$
$$= (X_{\bar{\lambda}}^{-1}(a)\mathcal{M}_{\bar{\lambda}} - X_{\bar{\lambda}}^{-1}(b)\mathcal{N}_{\bar{\lambda}})^* G\mathcal{P}(\lambda)(I - \mathcal{P}(\lambda)) \left(X_\lambda^{-1}(a)\mathcal{M}_\lambda - X_\lambda^{-1}(b)\mathcal{N}_\lambda\right),$$

which is a consequence of (1.1), (1.15), (1.20), (3.6).

The claim 2^0 follows by an argument of proof of $n^0 2^0$ of Remark 1.1, Th. 3.1, Remark 3.1. Remark 3.3 is proved.

Theorem 3.2. *Let $P = I, M(\lambda)$ be a c.o. of (0.1) on $\mathcal{I}$. Then one has:*

1. If condition (1.10) is separated with some nonreal λ, then it is separated with $\bar{\lambda}$.

2. Suppose $\mathcal{I}$ is finite and there exist λ_0, μ_0 such that $\Im\lambda_0\Im\mu_0 > 0$ and the following is true: 1) condition (1.10) with $\lambda = \lambda_0$ becomes an equality; 2) condition (1.10) with $\lambda = \mu_0$ is separated. Then condition (1.10) is separated with any nonreal λ.

3. If in (0.1) one has $H_\lambda(t) = H_0(t) + \lambda H(t), H_0(t) = H_0^*(t)$ and (1.3) holds with $F = \mathcal{H}$, then $n^0 2^0$ is valid without assuming finiteness of $\mathcal{I}$.

P r o o f. 1^0 follows from Th. 3.1 and Remark 3.2.

2^0 follows from Remarks 1.1, 3.3 and Th. 2.8.

3^0. Without assumption $P = I$, consider in $L^2_H(\mathcal{I}) \oplus L^2_H(\mathcal{I})$ a linear manifold $L_0' = \Big\{ y(t) \oplus g(t) \Big| \; y(t) \overset{L^2_H(\mathcal{I})}{=} x(t),\; x(t) \in AC_{loc}$ is a vector function with compact support; $g(t) \overset{L^2_H(\mathcal{I})}{=} f(t),\; f(t)$ is an $\mathcal{H}$-valued vector function and $l[x] = H(t)f(t),$ with $l[x(t)] = \frac{i}{2}((Q(t)x(t))' + Q(t)x'(t)) - H_0(t)x(t) \Big\}$, which is symmetric [37, p. 75] by the Lagrange formula.

In what follows we replace generic elements of L_0' by the elements of the form $x(t) \oplus f(t)$. Introduce the notation $L_0 = \bar{L}_0'$.

In the case when $M(\lambda)$ is a c.o., we keep the notation $R(\lambda)$ for an extension by a continuity onto $L^2_H(\mathcal{I})$ of the operator $R_\lambda(1.9)$ originally defined on $\mathcal{H}$-valued vector functions $f(t) \in L^2_H(\mathcal{I})$ with compact support, which is bounded by $n^0 1$ of Th. 1.1.

We need the following two Lemmas; the first one does not require $P = I$. In the statements and in the proofs of Lems. 3.1, 3.2 and Remark 3.3 the notation of operator and its graph coincide.

Lemma 3.1. *$R(\lambda)$ is a generalized resolvent of the subspace L_0 (in the sense of formula (6.25) of [37]).*

P r o o f. Note that $R^*(\lambda) = R(\bar{\lambda})$ since $M^*(\lambda) = M(\bar{\lambda})$, that $R(\lambda)$ satisfies (1.65) with $R_\lambda = R(\lambda)$ and that $R(\lambda)$ depends analytically on λ by [30, p. 195]. These observations imply, in view of Th. 6.8 of [37], that in our case to finish proving of the Lem. 3.1. it remains to verify that

$$R(\lambda)(L_0 - \lambda) \subset \mathbf{I}, \tag{3.9}$$

with $\mathbf{I}$ being the graph of an identity operator in $L^2_H(\mathcal{I})$.

If $L_0' = \{x(t) \oplus f(t)\}$, then

$$R(\lambda)(L_0' - \lambda) = x(t) \oplus x_\lambda(t),$$

with $x_\lambda(t) = R_\lambda(f(t) - \lambda x(t))$. Introduce the notation $z_\lambda(t) = x_\lambda(t) - x(t)$.

If $\operatorname{supp} x(t) \subseteq [\alpha, \beta] \subseteq \bar{\mathcal{I}}$, then by (1.10) one has

$$\Im\lambda((Q(\beta)z_\lambda(\beta), z_\lambda(\beta)) - (Q(\alpha)z_\lambda(\alpha), z_\lambda(\alpha))) \le 0. \tag{3.10}$$

On the other hand,

$$(Q(\beta)z_\lambda(\beta), z_\lambda(\beta)) - (Q(\alpha)z_\lambda(\alpha), z_\lambda(\alpha)) = 2\Im\lambda \int_\alpha^\beta (H(t)z_\lambda(t), z_\lambda(t))dt, \tag{3.11}$$

since

$$l[z_\lambda(t)] = \lambda H(t) z_\lambda(t).$$

It follows from (3.10), (3.11) that $z_\lambda \overset{L^2_H(\mathcal{I})}{=} 0$, hence

$$R(\lambda)(L_0' - \lambda) \subset \mathbf{I}. \tag{3.12}$$

Suppose $y(t) \oplus g(t) \in L_0$ and $L_0' \ni y_n(t) \oplus g_n(t) \to y(t) \oplus g(t)$. Then by (3.12) one has

$$\begin{aligned} R(\lambda)\big(y(t) \oplus [g(t) - \lambda y(t)]\big) &= \lim R(\lambda)\big(y_n(t) \oplus [g_n(t) - \lambda y_n(t)]\big) \\ &= \lim(y_n(t) \oplus y_n(t)) = y(t) \oplus y(t), \end{aligned}$$

so that (3.9), and hence Lem. 3.1, are proved.

Lemma 3.2. *Let $\mathcal{H}$ be an arbitrary Hilbert space, $R(\lambda)$ the generalized resolvent of a symmetric subspace $S \subset \mathcal{H}^2$, and*

$$\exists \lambda_0, \quad \Im\lambda_0 \neq 0, \quad \forall f \in \mathcal{H}: \qquad \|R(\lambda_0)f\|^2 = \frac{\Im(R(\lambda_0)f, f)}{\Im\lambda_0}. \tag{3.13}$$

Then:
1^0. (3.13) is valid for all λ with $\Im\lambda\Im\lambda_0 > 0$.
2^0.

$$R(\lambda) = \begin{cases} (\tilde{S} - \lambda)^{-1}, & \Im\lambda\Im\lambda_0 > 0, \\ (\tilde{S}^* - \lambda)^{-1}, & \Im\lambda\Im\lambda_0 < 0, \end{cases} \tag{3.14}$$

with $\tilde{S}$ being a maximal symmetric extension of S.

P r o o f. 1^0. It is known from [37, p. 95] that

$$R(\lambda) = (T(\lambda) - \lambda)^{-1}, \tag{3.15}$$

with the linear subspace $T(\lambda) \subset \mathcal{H}^2$ being such that

$$\Im T(\lambda) \le 0(\max), \qquad \Im\lambda > 0, \tag{3.16}$$

$$T(\bar{\lambda}) = (T(\lambda))^*. \tag{3.17}$$

Its Cayley transform $C_\mu(T(\lambda))$ with fixed μ, $\Im\lambda\Im\mu > 0$, is a contraction in $\mathcal{H}$ which depends analytically on λ.

Since with $\Im\lambda\Im\mu \neq 0$, $f \in \mathcal{H}$ one has

$$\|(I + (\mu - \bar{\mu})R(\lambda))f\|^2 = \|f\|^2 + 4(\Im\mu)^2 \left(\|R(\lambda)f\|^2 - \frac{\Im(R(\lambda)f, f)}{\Im\mu}\right),$$

$C_{\lambda_0}(T(\lambda_0))$ is an isometry in $\mathcal{H}$ in view of formula (4.17) from [37]. Thus $C_{\lambda_0}(T_\lambda) = C_{\lambda_0}(T_{\lambda_0})$ with $\Im\lambda\Im\lambda_0 > 0$ by [35, p. 210], hence 1^0 is proved in view of formula (4.17) from [37].

2^0. Use 1^0 to deduce from Th. 6.7, the formulas (6.20)–(6.24) from [37] and (3.16) that $T(\lambda) = T(\lambda_0) \overset{def}{=} \tilde{S}$ with $\Im\lambda\Im\lambda_0 > 0$, where $\tilde{S}$ is the maximal symmetric extension of S by Th. 6.2 from [37]. Thus 2^0 is proved in view of (3.15), (3.17). Lemma 3.2 is proved.

Turn back to the proof of Th. 3.2. It is clear from the proof of $n^0 6$ of Th. 1.1 that if (1.3) holds with $F = \mathcal{H}$, then the integral

$$x_\lambda(t) = \int_a^b K(t, s, \lambda)H(t)f(t)dt, \tag{3.18}$$

with $K(t, s, \lambda)$ being as in (1.85), converges and $x_\lambda(t) \in L^2_H(\mathcal{I})$ even in the case when an $\mathcal{H}$-valued $f(t) \in L^2_H(\mathcal{I})$ does not have a compact support. Prove that in this case inequalities (3.1) hold for $x_{\mu_0}(t)$ (3.18), if they hold for $x_{\mu_0}(t)$(3.18), ($\lambda = \mu_0$) with $f(t)$ having a compact support. Let $f_n(t) = \chi_n(t)f(t)$ with $\chi_n(t)$ being characteristic functions of the intervals $(\alpha_n, \beta_n) \uparrow (a, b)$. In view of convergence of (3.18) and (1.70), it is possible to choose for any $\varepsilon > 0$ such N_ε that for all $n \ge N_\varepsilon$ one has

$$\|x_{\mu_0}(c) - x^n_{\mu_0}(c)\| < \varepsilon, \tag{3.19}$$

with $x^n_{\mu_0}(t)$ being given by (3.18) in which $\lambda = \mu_0$, $f(t) = f_n(t)$,

$$\|x_{\mu_0}(t) - x^n_{\mu_0}(t)\|_{L^2_H(\mathcal{I})} < \varepsilon, \tag{3.20}$$

$$\begin{aligned}\forall\alpha \in (a, c) \qquad & \frac{1}{2}\left|(U[x_{\mu_0}(c)] - U[x_{\mu_0}(\alpha)]) - (U[x^n_{\mu_0}(c)] - U[x^n_{\mu_0}(\alpha)])\right| \\ & \le |\Im\mu_0| \left(\|x_{\mu_0}(t)\|^2_{L^2_H(\alpha,c)} - \|x^n_{\mu_0}(t)\|^2_{L^2_H(\alpha,c)}\right) \\ & + \left|\Im((x_{\mu_0}(t), f(t))_{L^2_H(\alpha,c)} - (x^n_{\mu_0}(t), f_n(t))_{L^2_H(\alpha,c)})\right| < \epsilon. \end{aligned} \tag{3.21}$$

Thus in view of (3.19)–(3.21)

$$U[x_{\mu_0}(\alpha)] - U[x^n_{\mu_0}(\alpha)] \to 0$$

uniformly in $\alpha \in (a, c)$. Therefore, for the following limit which exists due to (1.70), one has

$$\lim_{t \downarrow a} \Im\mu_0 U[x_{\mu_0}(t)] = \lim_{\alpha_n \downarrow a} \Im\mu_0 U[x_{\mu_0}(\alpha_n)] = \lim_{\alpha_n \downarrow a} \Im\mu_0 U[x^n_{\mu_0}(\alpha_n)] \geq 0.$$

The second inequality in (3.1) for $x_{\mu_0}(t)$ admits a similar verification. After that $n^0 3$ follows from the Hilbert identity for $R(\lambda)$, which is valid if $\Im\lambda\Im\mu_0 > 0$ in view of Lem. 2.4 from [37] and (1.70), Lems. 3.1, 3.2. The Theorem 3.2 is proved.

Note that assumption 1) in $n^0 2$ of Th. 2.2 could not be omitted in general, as it follows from Remarks 1.1, 2.6.

We are about to expand Lem. 3.1 in the case when λ is involved into $H_\lambda(t)$ nonlinearly as follows:

$$H_\lambda(t) = \lambda H(\lambda) + H^1_\lambda(t), \tag{3.22}$$

with $H^1_\lambda(t)$ satisfying the same conditions as $H_\lambda(t)$, $H(t) \geq 0$.

Let $M(\lambda)$ be a c.o. of (0.1), (3.22). Then, if an $\mathcal{H}$-valued $f(t) \in L^2_H(\mathcal{I})$ has compact support and *supp*$f(t) \subseteq [\alpha, \beta] \subseteq \bar{\mathcal{I}}$, one has in view of $n^0 2$ of Th. 1.1 that $x_\lambda(t)$ (3.18) with $\Im\lambda \neq 0$ satisfies the inequality

$$\Im\lambda(U[x_\lambda(\beta)] - U[x_\lambda(\alpha)]) \leq 0, \tag{3.23}$$

since $\int_a^b X^*_{\bar\lambda}(t)H(t)f(t)dt \in N^\perp$ by (3.22).

Denote by $\Re_\lambda f = x_\lambda(t)$ (3.18), with an $\mathcal{H}$-valued $f(t) \in L^2_H(\mathcal{I})$ having a compact support. Using (3.23), (3.22), one can prove, just as in the case of $n^0 1$ of Th. 1.1, that

$$\|\Re_\lambda f)\|_{L^2_H(\mathcal{I})} \leq \frac{\Im(\Re_\lambda f, f)_{L^2_H(\mathcal{I})}}{\Im\lambda}, \qquad \Im\lambda \neq 0. \tag{3.24}$$

Denote by $\Re(\lambda)$ the extension by a continuity onto $L^2_H(\mathcal{I})$ of $\Re_\lambda f$ which is bounded by (3.24). Note that $\Re^*(\lambda) = \Re(\bar\lambda)$ since $M^*(\lambda) = M(\bar\lambda)$, that $\Re(\lambda)$ satisfies (3.24) with $\Re_\lambda = \Re(\lambda)$ and that $\Re(\lambda)$ depends analytically on λ by [30, p. 195]. Therefore, in view of Ths. 4.5, 3.2 from [37], we come to the following

Remark 3.4. $\Re(\lambda) = (T(\lambda) - \lambda)^{-1}$, *where the linear subspace* $T(\lambda) \subset L^2_H(\mathcal{I}) \oplus L^2_H(\mathcal{I})$ *satisfies (3.16), (3.17), and its Cayley transform* $C_\mu(T_\lambda)$, *(see [37]) under fixed* μ *and* $\Im\lambda\Im\mu > 0$, *is a contraction in* $L^2_H(\mathcal{I})$ *which depends analytically on* λ.

Definition 3.2. *If an operator function $M(\lambda) \in B(\mathcal{H})$ of the form (1.20) is a c.o. of (0.1) on $\mathcal{I}$ in such a way that $\mathcal{P}(\lambda) = \mathcal{P}^2(\lambda)$, then $\mathcal{P}(\lambda)$ is called a characteristic projection (c.p.) of (0.1) on $\mathcal{I}$ (or merely a c.p.).*

Theorem 3.3. *A c.p. of (0.1) on $\mathcal{I}$ exists if one of the ends of $\mathcal{I}$ is finite or if for some $\lambda_0 \in \mathcal{A} \cap R^1$ the norm $\|X^*_{\lambda_0}(t) w_{\lambda_0}(t) X_{\lambda_0}(t)\|$ is summable at one of the ends of $\mathcal{I}$. Also, a c. p. exists if (1.3) holds with $F = \mathcal{H}$.*

If $Q(t)$ is definite, then a c.p. of (0.1) exists without any additional conditions.

P r o o f. By Lemma 1.2 and the proof of Th. 1.2, it is sufficient to prove Th. 3.3 for (0.1) with an indefinite constant $Q(t) = G$.

If one of the ends of $\mathcal{I}$ is finite or the norm $\|X^*_{\lambda_0}(t) w_{\lambda_0}(t) X_{\lambda_0}(t)\|$ is summable at one of its ends, then, in view of Remark 1.2, the Th. 3.3 has already been proved while proving Th. 1.2.

Assume $\mathcal{I} = R^1$ and (1.3) holds with $F = \mathcal{H}$.

First, suppose that $c = \alpha$. Consider the projection $\mathcal{P}_+(\lambda)$ associated via (1.20) to the c.o. of (0.1) on (c, ∞), which is constructed within the scheme of the proof of Th. 2.1, Case I (see Remark 1.2).

It follows from Th. 3.1, Cor. 3.1 and Lem. 1.1 that

$$\exists \delta(\lambda) > 0 \forall f \in \mathcal{H} : \quad \frac{1}{2\Im\lambda}(\mathcal{P}^*_+(\lambda) G \mathcal{P}_+(\lambda) f, f) \leq -\delta(\lambda) \|\mathcal{P}_+(\lambda) f\|^2 \; (\Im\lambda \neq 0)$$

and by Remark 1.2 and Lem. 1.9 one has

$$\begin{aligned} \exists c_+ > 0 \; \forall f \in \mathcal{H} : \; & (Sgn\Im\lambda)(I - \mathcal{P}^*_+(\lambda)) G((I - \mathcal{P}_+(\lambda)) f) \\ & \geq c_+ \|(I - \mathcal{P}_+(\lambda)) f\|^2 \; (\Im\lambda \neq 0). \end{aligned}$$

On the other hand, replace $\Pi(\lambda)$ by $I - \Pi(\lambda)$ and then use the scheme described in the proof of Case I, Th. 2.1 to produce a c.o. on $(-\infty, c)$ for (0.1). Consider the projection $\mathcal{P}_-(\lambda)$ associated with this c.o. by (1.20). By Remark 1.2 and Lem. 1.9 for $\Im\lambda \neq 0$

$$\begin{aligned} \exists c_- > 0 \; \forall f \in \mathcal{H} : \; & (Sqn\Im\lambda)(\mathcal{P}^*_-(\lambda) G \mathcal{P}_-(\lambda) f, f) \leq -c_- \|\mathcal{P}_-(\lambda) f\|^2, \\ & \Im\lambda(I - \mathcal{P}^*_-(\lambda)) G (I - \mathcal{P}_-(\lambda)) \geq 0. \end{aligned}$$

Therefore by Lem. 2.4, Th. 2.4, and [25, p. 76] one has

$$\mathcal{H} = (I - \mathcal{P}_-(\lambda))\mathcal{H} \dotplus \mathcal{P}_+(\lambda)\mathcal{H}.$$

Denote by $\mathcal{P}(\lambda)$ the projection onto $\mathcal{P}_+(\lambda)\mathcal{H}$ parallel to $(I - \mathcal{P}_-(\lambda))\mathcal{H}$.

By Lemma 1.3 and [1] (see also Cor. 2.3) one has

$$\mathcal{P}(\lambda) = \mathcal{P}_+(\lambda)(\mathcal{P}_+(\lambda) + I - \mathcal{P}_-(\lambda))^{-1},$$

with $(...)^{-1} \in B(\mathcal{H})$. It is easy to see that $\mathcal{P}(\lambda)$ is a desired c.p.

If $c \neq \alpha$, produce in a similar way a c.p. $\mathcal{P}(\lambda)$ for the case when the Cauchi operator of (0.1) is normalized by I at α. Then the desired c.p. is just $X_\lambda^{-1}(\alpha)\mathcal{P}(\lambda)X_\lambda(\alpha)$, so the Th. 3.3 is proved.

Note that with $P \neq I$ (3.1) does not imply (3.3) and (3.3) does not imply (3.1) even in the finite dimensional case, as one can see from

Example 3.1. Let $\mathcal{I} = (0,1)$, $c = 0$, in (0.1):

$$Q(t) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad H_\lambda(t) = \begin{pmatrix} 0 & 0 \\ 0 & i/4 \end{pmatrix}, \quad \Im\lambda > 0.$$

Then:

I. $M(\lambda)$ (1.20) is a c.o. with

$$\mathcal{P}(\lambda) = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \neq \mathcal{P}^2(\lambda), \qquad \Im\lambda > 0.$$

However condition (1.10) is separated with $\Im\lambda > 0$.

II. $M(\lambda)$ (1.20) is a c.o. with

$$\mathcal{P}(\lambda) = \begin{pmatrix} i & 1 \\ 1+i & 1-i \end{pmatrix} = \mathcal{P}^2(\lambda), \qquad \Im\lambda > 0.$$

However condition (1.10) is not separated with $\Im\lambda > 0$.

With $P \neq I$ one has the following analogue of Th. 3.1.

Theorem 3.4. *Suppose $P \neq I$ and $\exists \lambda_0 \in C\backslash R^1$, $\gamma \in \bar{\mathcal{I}}$:*

$$N = \bigcap_{\alpha \leq \gamma} Ker\Delta_{\lambda_0}(\alpha, \gamma) \quad or \quad N = \bigcap_{\beta \geq \gamma} Ker\Delta_{\lambda_0}(\gamma, \beta).$$

Let $M(\lambda)$ be a c.o. of (0.1). If condition (1.10) associated to $M(\lambda)$ is separated with nonreal $\lambda = \mu_0$, then

$$\exists M_0 \in B(\mathcal{H}): \quad PM_0P = 0, \quad M(\mu_0) + M_0 = \left(\mathcal{P} - \frac{1}{2}I\right)(iG)^{-1}.$$

Here the operator $\mathcal{P}$ is extendable from $(GN)^\perp$ to, possibly unbounded, densely defined in $\mathcal{H}$, idempotent which, in the case when either $dimP_-\mathcal{H} < \infty$ or $dimP_+\mathcal{H} < \infty$ with $P_\pm$ being as in (2.1), (2.2), is a bounded projection in $\mathcal{H}$.

4. The Weyl Type Functions and Solutions

In this section $a < c < b$, unless a different assumption is stated. Condition (1.3) with $F = \mathcal{H}$ is valid for $\mathcal{I} = \mathcal{I}_+ = (a, c)$ if $\mathcal{I}_+ \neq \emptyset$ and $\mathcal{I} = \mathcal{I}_- = (c, b)$ if $\mathcal{I}_- \neq \emptyset$.

The following theorem describes a relationship between the c.p. on $\mathcal{I}$ and on $\mathcal{I}_\pm$.

Theorem 4.1. *1^0. Let $\mathcal{P}(\lambda)$ be a c.p. of (0.1) on $\mathcal{I}$, $P_\pm = P_\pm(\lambda) \in B(\mathcal{H})$ be some operator-valued functions that depend analytically on nonreal λ and such that*

$$\pm\Im\lambda P_\pm^* G P_\pm \geq 0,\ \pm\Im\lambda(I - P_\pm^*)G(I - P_\pm) \leq 0,\ \Im\lambda > 0 \ \ or\ \ \Im\lambda < 0, \tag{4.1}$$

$$P_\pm(\lambda) = G^{-1}(I - P_\pm^*(\bar{\lambda}))G, \qquad \Im\lambda \neq 0.^\star \tag{4.2}$$

Then

$$\mathcal{H} = P_+(\lambda)\mathcal{H} \dot{+} \mathcal{P}(\lambda)\mathcal{H} = P_-(\lambda)\mathcal{H} \dot{+} (I - \mathcal{P}(\lambda))\mathcal{H}. \tag{4.3}$$

Denote also by $\mathcal{P}_+(\lambda)$ and $\mathcal{P}_-(\lambda)$ the projections onto $\mathcal{P}(\lambda)\mathcal{H}$ and $P_-(\lambda)\mathcal{H}$, respectively, parallel to $P_+(\lambda)\mathcal{H}$ and $(I - \mathcal{P}(\lambda))\mathcal{H}$, respectively.

Then $\mathcal{P}_\pm(\lambda)$ are c.p. of (0.1) on $\mathcal{I} = \mathcal{I}_\pm$ in such a way that

$$\mathcal{P}_+(\lambda) = \mathcal{P}(\lambda)(\mathcal{P}(\lambda) + P_+(\lambda))^{-1}, \mathcal{P}_-(\lambda) = P_-(\lambda)(P_-(\lambda) + I - \mathcal{P}(\lambda))^{-1}, \tag{4.4}$$

with

$$(\mathcal{P}(\lambda) + P_+(\lambda))^{-1},\ (P_-(\lambda) + I - \mathcal{P}(\lambda))^{-1} \in B(\mathcal{H}). \tag{4.5}$$

2^0. Let $\mathcal{P}_\pm(\lambda)$ be a pair of c.p. of (0.1) with

$$Q(t) = Q^\pm(t),\ Q^\pm(c) = G,\ H_\lambda(t) = H_\lambda^\pm(t),\ t \in \mathcal{I}_\pm, \tag{4.6}$$

on $\mathcal{I} = \mathcal{I}_\pm$, then $\mathcal{H} = \mathcal{P}_+(\lambda)\mathcal{H} \dot{+} (I - \mathcal{P}_-(\lambda))\mathcal{H}$.

Suppose $\mathcal{P}(\lambda)$ projects onto $\mathcal{P}_+(\lambda)\mathcal{H}$ parallel to $(I - \mathcal{P}_-(\lambda))\mathcal{H}$. Then $\mathcal{P}(\lambda)$ is a c.p. of (0.1),(4.6) on $\mathcal{I} = (a, b)$ in such a way that

$$\mathcal{P}(\lambda) = \mathcal{P}_+(\lambda)S_-(\lambda)(\mathcal{P}_+(\lambda)S_-(\lambda) + (I - \mathcal{P}_-(\lambda))S_+(\lambda))^{-1}, \tag{4.7}$$

where $S_+(\lambda)$ and $S_-(\lambda)$ are the Riesz projections for the operator $(sgn\Im\lambda)G$ that correspond to positive and negative parts of its spectrum, respectively; $(\mathcal{P}_+(\lambda)S_-(\lambda)$ $+(I - \mathcal{P}_-(\lambda))S_+(\lambda))^{-1} \in B(\mathcal{H})$.

If the c.p. $\mathcal{P}_\pm(\lambda)$ is generated by the c.p. $\mathcal{P}(\lambda)$ according to (4.4) in $n^0 1^0$ of the theorem, then $n^0 2^0$ results exactly this $\mathcal{P}(\lambda)$.

*(4.1) implies by Lem. 2.4, Ths. 2.4, 2.7, that $P_\pm^2(\lambda) = P_\pm(\lambda)$ for $\Im\lambda > 0$ or $\Im\lambda < 0$, so for all nonreal λ by (4.2).

P r o o f. In view of (4.1), (4.2), Th. 2.4, Lem. 2.4, (3.6), (3.7), [25, p. 73], $P_\pm(\lambda)\mathcal{H}$ and $(I - P_\pm(\lambda))\mathcal{H}$ are respectively maximal $\pm\Im\lambda G$-nonnegative and maximal $\pm\Im G$-nonpositive subspaces for nonreal λ. In view of Corollary 3.1, condition (3.1) with $F = \mathcal{H}$ for $\mathcal{I} = \mathcal{I}_\pm$ and [25, p. 71] (or Th. 2.4, (1.69) (or Lem. 2.4)), $\mathcal{P}(\lambda)\mathcal{H}$ and $(I - \mathcal{P}(\lambda))\mathcal{H}$ are respectively maximal uniformly $\mp\Im\lambda G$-positive and maximal uniformly $\pm\Im\lambda G$-positive subspaces for nonreal λ. Hence we have (4.3) by [25, p. 76] and (4.5) by Lem. 1.3. Thus we have (4.4) by [1] (or Cor. 2.3) and hence $\mathcal{P}_\pm(\lambda)$ depends analytically on nonreal λ. Thus $\mathcal{P}_\pm(\lambda)$ are c.p. of (0.1) on $\mathcal{I}_\pm$ since $\mathcal{P}(\lambda)$ is a c.p. and by (3.6), (3.7), (4.2).

2^0 is proved similarly to 1^0. The Theorem is proved.

The following remark allows, in particular, to transform a c.p. so that the corresponding to it boundary condition at one end of interval is not changed, but boundary condition at another end coincides with any given. This Remark is proved in the same way as Th. 4.1.

Remark 4.1. *1^0. a) Let $\tilde{\mathcal{P}}(\lambda)$ be a c. p. of (0.1) on $\mathcal{I}$. Then, if one sets $P_+(\lambda) = I - \tilde{\mathcal{P}}(\lambda)$, $P_-(\lambda) = \tilde{\mathcal{P}}(\lambda)$ in 1^0 of Th. 4.1, then $\mathcal{P}_\pm(\lambda)$ (4.4) becomes a c.p. of (0.1) not only on $\mathcal{I}_\pm$, but on $\mathcal{I}$ as well. b) Let $\tilde{\mathcal{P}}_\pm(\lambda)$ be a c.p. of (0.1),(4.6) on $\mathcal{I} = \mathcal{I}_\pm$. Then, if one sets $P_+(\lambda) = I - \tilde{\mathcal{P}}_-(\lambda)$, $P_-(\lambda) = \tilde{\mathcal{P}}_+(\lambda)$ in 1^0 of Th. 4.1, then (4.4) becomes a c.p. of (0.1) not only on $\mathcal{I}_\pm$, but also on $\mathcal{I}$ as well with*

$$Q(t) = \begin{cases} Q(t), & t \in \mathcal{I}_+ \\ Q^-(t), & t \in \mathcal{I}_- \end{cases}, \quad H_\lambda(t) = \begin{cases} H_\lambda(t), & t \in \mathcal{I}_+ \\ H_\lambda^-(t), & t \in \mathcal{I}_- \end{cases}$$

in the case of $\mathcal{P}_+(\lambda)$ and with

$$Q(t) = \begin{cases} Q^+(t), & t \in \mathcal{I}_+ \\ Q(t), & t \in \mathcal{I}_- \end{cases}, \quad H_\lambda(t) = \begin{cases} H_\lambda^+(t), & t \in \mathcal{I}_+ \\ H_\lambda(t), & t \in \mathcal{I}_- \end{cases}$$

in the case of $\mathcal{P}_-(\lambda)$.

2^0. If one replaces in (4.7) $I - \mathcal{P}_-(\lambda)(\mathcal{P}_+(\lambda))$ by $P_+(\lambda)(P_-(\lambda))$ as in 1^0 of Th. 4.1, then one still has in (4.7) $(...)^{-1} \in B(\mathcal{H})$, however $\mathcal{P}(\lambda)$ (4.7) in general is no longer a c.p. of (0.1) on $\mathcal{I}$, but a c.p. on $\mathcal{I}_+(\mathcal{I}_-)$, that projects onto $\mathcal{P}_+(\lambda)(P_-(\lambda))$ parallel to $P_+(\lambda)\mathcal{H}((I - \mathcal{P}_-(\lambda))\mathcal{H})$.[⋆]

We are about to demonstrate a procedure of producing the operator Weyl type functions and solutions of (0.1) that uses projections from Ths. 3.1, 4.1.

In view of (1.22), (2.1), (2.2) it is easy to see that

$$\exists\Gamma(t) \in B(\mathcal{H}): \quad \Gamma^{-1}(t) \in B(\mathcal{H}), \quad \Gamma(t) \in AC_{loc}, \quad \Gamma^*(t)Q(t)\Gamma(t) = P_+ - P_-, \tag{4.8}$$

[⋆]The transformations of c.p. on $\mathcal{I}_\pm$ such that they don't change boundary condition at the point c construct in the similar way.

with $P_\pm$ being complementary orthogonal projections that do not depend on t.

Theorem 4.2. *Let $\mathcal{P}(\lambda)$ be a c.p. of (0.1) on (a,b). Then there exist unique strict contractions $K_\pm(\lambda) = K_\pm^*(\bar\lambda)$ that depend analytically on nonreal λ such that*

$$K_\pm(\lambda) \in B(P_\mp\mathcal{H}, P_\pm\mathcal{H}), \Im\lambda > 0, K_\pm(\lambda) \in B(P_\pm\mathcal{H}, P_\mp\mathcal{H}), \Im\lambda < 0, \tag{4.9}$$

$$\mathcal{P}(\lambda) = \begin{cases} \Gamma(c)(P_- + K_+(\lambda)P_-)(I_- - K_-(\lambda)K_+(\lambda))^{-1}(P_- - K_-(\lambda)P_+)\Gamma^{-1}(c), \\ \Im\lambda > 0, \\ \Gamma(c)(P_+ + K_+(\lambda)P_+)(I_+ - K_-(\lambda)K_+(\lambda))^{-1}(P_+ - K_-(\lambda)P_-)\Gamma^{-1}(c), \\ \Im\lambda < 0, \end{cases} \tag{4.10}$$

(here $I_\pm$ are the identity operators in $P_\pm\mathcal{H}$), and for the operator solutions

$$\Psi_\pm(t,\lambda) = \begin{cases} X_\lambda(t)\Gamma(c)(P_\mp + K_\pm(\lambda)P_\mp), & \Im\lambda > 0, \\ X_\lambda(t)\Gamma(c)(P_\pm + K_\pm(\lambda)P_\pm), & \Im\lambda < 0, \end{cases} \tag{4.11}$$

of the homogeneous equation (0.1), (4.8) one has

$$\int\limits_{J_\pm} \Psi_\pm^*(t,\lambda)w_\lambda(t)\Psi_\pm(t,\lambda)dt \le \begin{cases} \frac{1}{2\Im\lambda}P_\mp(I_\mp - K_\pm^*(\lambda)K_\pm(\lambda))P_\mp, & \Im\lambda > 0, \\ \frac{-1}{2\Im\lambda}P_\pm(I_\pm - K_\pm^*(\lambda)K_\pm(\lambda))P_\pm, & \Im\lambda < 0, \end{cases} \tag{4.12}$$

with $J_\pm$ being such finite intervals that $J_- \subseteq (a,c)$, $J_+ \subseteq (c,b)$.

Conversely, suppose that for the operator functions $K_\pm(\lambda) = K_\pm^*(\bar\lambda)$ that depend analytically on nonreal λ the relations (4.9), (4.11), (4.12) hold. Then $K_\pm(\lambda)$ are strict contractions and $\mathcal{P}(\lambda)$ (4.10) is a c.p. of (0.1) (4.8) on (a,b).

P r o o f. Let $\mathcal{P}(\lambda)$ be a c.p. of (0.1), (4.8) on (a,b). Then, in view of Cor. 3.1, condition (3.1) with $F = \mathcal{H}$ for $\mathcal{I} = \mathcal{I}_\pm$ and [25, p. 71] (or (1.69) (or Lem. 2.4), Th. 2.4), the subspaces

$$\mathcal{H}_-(\lambda) = (I - \mathcal{P}(\lambda))\mathcal{H}, \qquad \mathcal{H}_+(\lambda) = \mathcal{P}(\lambda)\mathcal{H} \tag{4.13}$$

are respectively maximal uniformly $\pm\Im\lambda G$-positive and maximal uniformly $\mp\Im\lambda G$-positive for nonreal λ. Therefore [24, p. 100], [25, Ch. I, § 8] there exist unique strict contractions $K_\pm(\lambda)$ (4.9) such that

$$\mathcal{H}_\pm(\lambda) = \begin{cases} \Gamma(c)\{P_\mp f \oplus K_\pm(\lambda)P_\mp f | f \in \mathcal{H}\}, & \Im\lambda > 0, \\ \Gamma(c)\{P_\pm f \oplus K_\pm(\lambda)P_\pm f | f \in \mathcal{H}\}, & \Im\lambda < 0. \end{cases} \tag{4.14}$$

Thus (4.10) is valid for $\mathcal{P}(\lambda)$. Substitute (4.10), (4.11) into (3.5) (with μ_0 being replaced by λ) to get (4.12).

Prove that $K_\pm(\lambda) = K_\pm^*(\bar\lambda)$ and depends on a nonreal λ analytically. Introduce the notation $\mathcal{H}_c(\lambda) = P_c(\lambda)\mathcal{H}$, with

$$P_c(\lambda) = \begin{cases} \Gamma(c)P_+\Gamma^{-1}(c), & \Im\lambda > 0 \\ \Gamma(c)P_-\Gamma^{-1}(c), & \Im\lambda < 0. \end{cases} \tag{4.15}$$

By [25, p. 76] one has

$$\mathcal{H} = \mathcal{H}_c(\lambda) \dot{+} \mathcal{H}_+(\lambda).$$

Denote by $\mathcal{P}_+(\lambda)$ the projection onto $\mathcal{H}_+(\lambda)$ parallel to $\mathcal{H}_c$.

By (3.6)

$$\mathcal{P}_+(\lambda) = G^{-1}(I - \mathcal{P}_+^*(\bar\lambda)))G, \tag{4.16}$$

and, obviously

$$\mathcal{P}_+(\lambda) = \begin{cases} \Gamma(c)(P_- + K_+(\lambda)P_-)\Gamma^{-1}(c), & \Im\lambda > 0, \\ \Gamma(c)(P_+ + K_+(\lambda)P_+)\Gamma^{-1}(c), & \Im\lambda < 0. \end{cases} \tag{4.17}$$

Compare (4.8), (4.16), (4.17) to observe that $K_+(\lambda) = K_+^*(\bar\lambda)$. By Th. 4.1 the operator function $\mathcal{P}_+(\lambda)$ is a c.p. on (c, b), hence it depends analytically on a nonreal λ. Thus $K_+(\lambda)$ depends analytically on nonreal λ in view of (4.17).

The same properties for $K_-(\lambda)$ can be proved in a similar way.

Conversely, suppose that (4.9), (4.11), (4.12) are valid for the operator functions $K_\pm(\lambda) = K_\pm^*(\bar\lambda)$ that depend analytically on a nonreal λ. In view of condition (1.3) with $F = \mathcal{H}$ for $\mathcal{I} = \mathcal{I}_\pm$, one can easily deduce from (4.12) that

$$\exists\delta_1 = \delta_1(\lambda) > 0: \quad \begin{cases} \forall f \in P_\mp\mathcal{H} : ((I_\mp - K_\pm^*K_\pm)f, f) \geq \delta_1\|f\|^2, & \Im\lambda > 0, \\ \forall f \in P_\pm\mathcal{H} : ((I_\pm - K_\pm^*K_\pm)f, f) \geq \delta_1\|f\|^2, & \Im\lambda < 0. \end{cases}$$

It follows that (4.10) determines an operator $\mathcal{P}(\lambda) \in B(\mathcal{H})$ which depends analytically on a nonreal λ. Thus (4.12), (4.11) imply (3.5), ($\mu_0 = \lambda$) and so (1.68) (by (9) of [1]) with $\mathcal{P}(\lambda)$ as in (4.10). Consider the operator $\mathcal{P}_+(\lambda)$ (4.17) together with its analogue $\mathcal{P}_-(\lambda)$ for (b, c), which are obviously projections. Since $\mathcal{P}_\pm(\lambda)$ satisfies relations similarly to (4.16), it follows from (3.6), (3.7) that such a relation is valid for $\mathcal{P}(\lambda)$. Hence $\mathcal{P}(\lambda)$ is a c.p. in view of Th. 1.1, Cor. 3.1 and Th. 3.1. The Theorem 4.2 is proved.

We call the operator functions $K_-(\lambda) = K_-^*(\bar\lambda)$, $K_+(\lambda) = K_+^*(\bar\lambda)$ (4.9), which depend analytically on a nonreal λ, and which satisfy (4.12), (4.11), the Weyl functions of (0.1), (4.8) on (a,c) and on (c,b) respectively (by a similarity to [13]–[17])*. We call the corresponding solutions $\psi_-(t,\lambda) = \Psi_-(t,\lambda)\big|_{P_\pm\mathcal{H}}$ and $\psi_+(t,\lambda) = \Psi_+(t,\lambda)\big|_{P_\mp\mathcal{H}}$ $(\pm\Im\lambda > 0)$ the Weyl solutions of (0.1), (4.8) on (a,c) and on (c,b) respectively.

Theorem 4.3. *Suppose that the interval (a,b) is finite and the operator functions $K_-(\lambda)$ and $K_+(\lambda)$ are the Weyl functions of (0.1), (4.8) on (a,c) and on (c,b), respectively. Then there exist unique contractions $U_\pm(\lambda) = U_\pm^*(\bar\lambda)$ that depend analytically on a nonreal λ and such that:*

$$U_\pm(\lambda) \in B(P_\mp\mathcal{H}, P_\pm\mathcal{H}),\ \Im\lambda > 0,\ U_\pm(\lambda) \in B(P_\pm\mathcal{H}, P_\mp\mathcal{H}),\ \Im\lambda < 0, \tag{4.18}$$

$$\begin{aligned} P_-K_-(\lambda)P_+ &= P_-\Gamma^{-1}(c)(I - \Pi_-(\lambda))\Gamma(c)P_+ \quad (\Im\lambda > 0),\\ P_+K_-(\lambda)P_- &= P_+\Gamma^{-1}(c)(I - \Pi_-(\lambda))\Gamma(c)P_- \quad (\Im\lambda < 0); \end{aligned} \tag{4.19}$$

$$\begin{aligned} P_+K_+(\lambda)P_- &= P_+\Gamma^{-1}(c)\Pi_+(\lambda)\Gamma(c)P_- \quad (\Im\lambda > 0),\\ P_-K_+(\lambda)P_+ &= P_-\Gamma^{-1}(c)\Pi_+(\lambda)\Gamma(c)P_+ \quad (\Im\lambda < 0); \end{aligned} \tag{4.20}$$

with

$$I - \Pi_-(\lambda) =$$
$$= \begin{cases} X_\lambda^{-1}(a)\Gamma(a)(P_+ + U_-(\lambda)P_+)(X_\lambda^{-1}(a)\Gamma(a)(P_+ + U_-(\lambda)P_+) - P_-\Gamma(c))^{-1},\\ \Im\lambda > 0,\\ X_\lambda^{-1}(a)\Gamma(a)(P_- + U_-(\lambda)P_-)(X_\lambda^{-1}(a)\Gamma(a)(P_- + U_-(\lambda)P_-) - P_+\Gamma(c))^{-1},\\ \Im\lambda < 0, \end{cases} \tag{4.21}$$

$$\Pi_+(\lambda) =$$
$$= \begin{cases} X_\lambda^{-1}(b)\Gamma(b)(P_- + U_+(\lambda)P_-)(X_\lambda^{-1}(b)\Gamma(b)(P_- + U_+(\lambda)P_-) - P_+\Gamma(c))^{-1},\\ \Im\lambda > 0\\ X_\lambda^{-1}(b)\Gamma(b)(P_+ + U_+(\lambda)P_+)(X_\lambda^{-1}(b)\Gamma(b)(P_+ + U_+(\lambda)P_+) - P_-\Gamma(c))^{-1},\\ \Im\lambda < 0. \end{cases}. \tag{4.22}$$

(In (4.22), (4.21) one has $(...)^{-1} \in B(\mathcal{H})$). The operators $\Pi_-(\lambda)$ and $\Pi_+(\lambda)$ are c.p. of (0.1) on (a,c) and on (c,b), respectively, and $x_\lambda(t)$ (1.9), (1.20), (4.10)

*Note that in view of n^{o}5^o of Th. 1.1 the validity of inequalities (4.11), (4.12) for arbitrary $K_\pm(\lambda)$ (4.9) in one of the complex half-planes implies validity of their analogs in another half-plane if one sets up $K_\pm(\bar\lambda) = K_\pm^*(\lambda)$.

is a solution of the boundary-value problem (0.1), (1.14) with

$$\mathcal{M}_\lambda = \begin{cases} \Gamma(a)(P_+ + U_-(\lambda)P_+), \ \Im\lambda > 0, \\ \Gamma(a)(P_- + U_-(\lambda)P_-), \ \Im\lambda < 0, \end{cases}$$

$$\mathcal{N}_\lambda = \begin{cases} \Gamma(b)(P_- + U_+(\lambda)P_-), \ \Im\lambda > 0, \\ \Gamma(b)(P_+ + U_+(\lambda)P_+), \ \Im\lambda < 0. \end{cases} \tag{4.23}$$

Conversely, suppose that (4.18) holds for the contractions $U_\pm(\lambda) = U_\pm^*(\bar\lambda)$ *depending analytically on a nonreal* λ. *Then the operators* $(...)^{-1} \in B(\mathcal{H})$ *in (4.22), (4.21) and operators (4.19), (4.20), associated to (4.22), (4.21), are the Weyl functions of (0.1) on* (a, c) *and* (c, b), *respectively.*

P r o o f. Let $K_+(\lambda)$ be a Weyl function of (0.1), (4.8) on (c, b). Then, as one can observe from the proof of Th. 4.2, $\mathcal{P}_+(\lambda)$ (4.17) is a c.p. of (0.1), (4.8) on (c, b). Therefore the subspace $X_\lambda(b)\mathcal{P}_+(\lambda)\mathcal{H}$ is maximal $\pm\Im\lambda Q(b)$ nonpositive by Cor. 3.1, [25, p. 71] (or (1.69) (or Lem. 2.4), Th. 2.4), Lem. 2.6. Hence [24, p. 100], [25, Ch. I, § 8] there exists such a unique contraction $U_+(\lambda)$ (4.18) that

$$X_\lambda(b)\mathcal{P}_+(\lambda)\mathcal{H} = \begin{cases} \Gamma(b)(P_- + U_+(\lambda)P_-)\mathcal{H}, \quad \Im\lambda > 0, \\ \Gamma(b)(P_+ + U_+(\lambda)P_+)\mathcal{H}, \quad \Im\lambda < 0. \end{cases} \tag{4.24}$$

Since by (1.1) and Remark 3.2

$$\mathcal{P}_+^*(\bar\lambda)X_{\bar\lambda}^*(b)Q(b)X_\lambda(b)\mathcal{P}_+(\lambda) = 0,$$

and hence one has $U(\lambda) = U^*(\bar\lambda)$.

Consider the boundary-value problem (0.1), (4.8), (1.14) on $(a, b) = (c, b)$ with $\mathcal{M}_\lambda = P_c(\lambda)\Gamma(c)$ with $P_c(\lambda)$ (4.15), $\mathcal{N}(\lambda)$ (4.23).

It satisfies all the assumptions of Remarks 1.1, 3.3 by Cor. 2.1 (except analyticity for $\mathcal{N}(\lambda)$ so far). Thus projection (3.8) associated with the above problem is just $\Pi_+(\lambda)$ (4.22). On the other hand,

$$\Pi_+(\lambda)\mathcal{H} = \begin{cases} X_\lambda^{-1}\Gamma(b)(P_- + U_+(\lambda)P_-)\mathcal{H}, \ \Im\lambda > 0 \\ X_\lambda^{-1}\Gamma(b)(P_+ + U_+(\lambda)P_+)\mathcal{H}, \ \Im\lambda < 0 \end{cases} = \mathcal{P}_+(\lambda)\mathcal{H},$$

$$(I - \Pi_+(\lambda))\mathcal{H} = (I - \mathcal{P}_+(\lambda))\mathcal{H}$$

by (4.24), (4.17). Hence $\mathcal{P}_+(\lambda) = \Pi_+(\lambda)$, which, together with (4.17), implies (4.20).

Analyticity for $U_+(\lambda) = U_+^*(\bar\lambda)$ is deducible from the fact that with $\Im\lambda > 0$,

$$\Gamma(c)(P_- + U_+(\lambda)P_-)\Gamma^{-1}(c) = Y_\lambda(b)\mathcal{P}_+(\lambda)(Y_\lambda(b)\mathcal{P}_+(\lambda) + P_c(\lambda))^{-1},$$

where $(...)^{-1} \in B(\mathcal{H})$, $Y_\lambda(t)$ see Lem. 1.2. The claims related to $K_-(\lambda)$, $U_-(\lambda)$ can be proved in a similar way.

Use (4.24) and its analogue for the endpoint a to deduce that $x_\lambda(t)$ (1.9), (1.20), (4.10) is a solution of the boundary-value problem (0.1), (1.14), (4.23).

Conversely, suppose that a contraction $U_+(\lambda) = U_+^*(\bar{\lambda})$ which depends analytically on a nonreal λ, satisfies (4.18). By Remarks 1.1, 3.3, a c.p. of problem (0.1), (4.8), (1.14) on $(a,b) = (c,b)$ with $\mathcal{M}_\lambda = P_c(\lambda)\Gamma(c), \mathcal{N}_\lambda$ (4.23) is just $\Pi_+(\lambda)$ (4.22). Hence (see the proof of Th. 4.2) there exists the Weyl function $K_+(\lambda)$ of (0.1), (4.8) on (c,b) such that $\Pi_+(\lambda) = \mathcal{P}_+(\lambda)$ (4.17) $\Rightarrow$ (4.20), (4.22) for $K_+(\lambda)$. The statements related to $U_-(\lambda)$ can be proved in a similar way. The Theorem 4.3 is proved.

Theorem 4.4. *Let $\mathcal{P}(\lambda)$ be a c.p. of (0.1), (4.8) on $(a,b) = (c,b)$. Then there exist the unique Weyl function $K_+(\lambda)$ of this equation on (c,b) and the unique contraction $U_-(\lambda) = U_-^*(\bar{\lambda})$ (4.18) which depends analytically on a nonreal λ, such that*

$$\mathcal{P}(\lambda) = \begin{cases} \Gamma(c)(P_- + K_+(\lambda)P_-)(I_- - U_-(\lambda)K_+(\lambda))^{-1}(P_- - U_-(\lambda)P_+)\Gamma^{-1}(c), \\ \Im\lambda > 0, \\ \Gamma(c)(P_+ + K_+(\lambda)P_+)(I_+ - U_-(\lambda)K_+(\lambda))^{-1}(P_+ - U_-(\lambda)P_-)\Gamma^{-1}(c), \\ \Im\lambda < 0, \end{cases} \tag{4.25}$$

and for any $\mathcal{H}$-valued vector function $f(t) \in L^2_{w_\lambda}(c,b)$ with compact support the solution of (0.1), (4.8) $x_\lambda(t)$ (1.9), (1.20), (4.25) satisfies at c the following boundary condition:

$$\exists h = h(f,\lambda): \quad y(c) = \mathcal{M}_\lambda h, \tag{4.26}$$

with $\mathcal{M}_\lambda$ as in (4.23).

Conversely, let $K_+(\lambda)$ be an arbitrary Weyl function of (0.1), (4.8) on (c,b) and $U_-(\lambda) = U_-^*(\bar{\lambda})$ (4.18)be an arbitrary contraction that depends analytically on nonreal λ. Then (4.25) is a c.p. of (0.1), (4.8) on (c,b).

P r o o f. Let $\mathcal{P}(\lambda)$ be a c.p. of (0.1), (4.8) on (c,b). Thus by Th. 3.1 and Cor. 3.1, (3.5) is valid with $a = c$ and μ_0 being replaced by any nonreal λ. Therefore the subspaces $\mathcal{H}_-(\lambda)$ (4.13) and $\mathcal{H}_+(\lambda)$ (4.13) are maximal and $\Im\lambda G$-nonnegative and uniformly negative, respectively. Hence [24, p. 100], [25, Ch. I, § 8] there exist the unique contraction $U_-(\lambda)$ (4.18) and the unique strict contraction $K_+(\lambda)$ (4.9) such that (4.14) holds with $K_-(\lambda)$ being replaced by $U_-(\lambda)$. Thus $\mathcal{P}(\lambda)$ satisfies (4.25), whence (4.26). Substitute (4.25) into (3.5) (with μ_0 being replaced by λ) to deduce that $K_+(\lambda)$ satisfies (4.12). The proof of relations

$K_+(\lambda) = K_+^*(\lambda)$, $U_-(\lambda) = U_-^*(\bar{\lambda})$ goes through like that of the similar relations for $K_\pm(\lambda)$ in Th. 4.2.

Analyticity for $K_+(\lambda)$ and $U_-(\lambda)$ follows by Lem. 3.1 from

$$\mathcal{P}_+(\lambda) = \mathcal{P}(\lambda)(\mathcal{P}(\lambda) + \mathcal{P}_c(\lambda))^{-1}, \tag{4.27}$$
$$\Gamma(c)(P_+ + U_-(\lambda)P_+)\Gamma^{-1}(c) = (I - \mathcal{P}(\lambda))(I - \mathcal{P}(\lambda) + I - P_c(\lambda))^{-1}, \ \Im\lambda > 0,$$

respectively, with $\mathcal{P}_+(\lambda)$ as in (4.17), $P_c(\lambda)$ as in (4.15), and $(...)^{-1} \in B(\mathcal{H})$.

Conversely, let $K_+(\lambda)$ be the Weyl function of (0.1), (4.8) on (c, b) and $U_-(\lambda) = U_-^*(\bar{\lambda})$ (4.18) be a contraction that depends analytically on a nonreal λ. Then (4.25) determines the operator $\mathcal{P}(\lambda) \in B(\mathcal{H})$ which is a c.p. in view of proof of Th. 4.2 and $\mathcal{M}_{\bar{\lambda}}^* G \mathcal{M}_\lambda = 0$. The Theorem 4.4 is proved.

Lemma 1.3 and the proof of Th. 4.2 imply

Remark 4.2. $K_+(\lambda)$ *(4.9) is a Weyl function of (0.1), (4.8) on* (c, b) *if and only if* $\mathcal{P}_+(\lambda)$ *(4.17) is a c.p. of this equation on* (c, b)*. This c.p., hence also* $K_+(\lambda)$*, can be derived from the c.p.* $\mathcal{P}(\lambda)$ *(4.25) using (4.27), (4.15).*

Remark 4.3. *Let* $\mathcal{P}(\lambda)$ *be a c.p. of equation (0.1), (4.8) on* (c, b) *(hence* $\mathcal{P}(\lambda)$ *admits representation (4.25)), and with a nonreal* $\lambda = \mu_0$ *and any* $\mathcal{H}$*-valued vector functions* $f(t) \in L^2_{w_{\mu_0}}(c, b)$ *with the compact support for solutions* $x_{\mu_0}(t)$ *(1.9),* $(\lambda = \mu_0)$ *of this equation corresponding to* $M(\mu_0)$ *(1.20), (4.25),* $(\lambda = \mu_0)$ *one has*

$$\lim_{\beta \uparrow b} U[x_{\mu_0}(\beta)] = 0. \tag{4.28}$$

Then inequality (4.12) for the solution $\Psi_+(t, \mu_0)$ *(4.11) becomes an equality with* $\lambda = \mu_0$ *and* J_+ *being replaced by* $(c, b)^\star$*.*

Conversely, let $K_+(\lambda)$ *be the Weyl function of (0.1), (4.8), and suppose that for some nonreal* $\lambda = \mu_0$ *and the associated solution* $\Psi_+(t, \mu_0)$ *(4.11) of (0.1), (4.8) on* (c, b)*, inequality (4.12) becomes the equality with* J_+ *being replaced by* $(c, b)^\star$*. Let the contraction* $U_-(\lambda)$ *satisfies (4.18),* $(\lambda = \mu_0)$*. Then with* $\lambda = \mu_0$ *and any* $\mathcal{H}$*-valued vector functions* $f(t) \in L^2_{w_{\mu_0}}(c, b)$ *with compact support, (4.28) holds for solutions* $x_{\mu_0}(t)$ *(1.9), (1.20), (4.25),* $(\lambda = \mu_0)$*.*

P r o o f. Assume for certainty $\Im\mu_0 > 0$. It follows that one can use vector functions of the form (1.75) with compact support to deduce from (4.28) that

$$s - \lim_{\beta \uparrow b} U[X_{\mu_0}(\beta)\mathcal{P}(\mu_0)] = 0. \tag{4.29}$$

*Where $\int\limits_c^b = s - \lim\limits_{J_+ \uparrow (c,b)} \int\limits_{J_+}$.

Therefore it is easy to conclude from (4.27) that

$$s - \lim_{\beta \uparrow b} U[X_{\mu_0}(\beta)\mathcal{P}_+(\mu_0)] = 0, \tag{4.30}$$

with $\mathcal{P}_+(\lambda)$ being as in (4.17). On the other hand, by (1.77) and formula (9) from [1] one has

$$\mathcal{P}_+^*(\mu_0)\Delta_{\mu_0}(c,\beta)\mathcal{P}_+(\mu_0) = \frac{1}{2\Im\mu_0}(U[X_{\mu_0}(\beta)\mathcal{P}_+(\mu_0)] - U[X_{\mu_0}(c)\mathcal{P}_+(\mu_0)]) \tag{4.31}$$

whence in view of (4.17),

$$\int_c^\beta \Psi_+^*(t,\mu_0)w_{\mu_0}(t)\Psi_+(t,\mu_0)dt$$

$$= \frac{1}{2\Im\mu_0}\left(P_-(I_- - K_+^*(\mu_0)K_+(\mu_0))P_- + U[X_{\mu_0}(\beta)\mathcal{P}_+(\mu_0)\Gamma(c)]\right), \tag{4.32}$$

so that the equality in (4.10) with $\lambda = \mu_0$ and J_+ being replaced by (c,b), is proved by a virtue of (4.30).

Conversely, suppose that the relation just proved is true. Then (4.32), (4.31) imply (4.30), hence in view of (4.25) one has also (4.29), which implies (4.28). The Remark 4.3 is proved.

As the consequence of Th. 4.3 proof, [35, p. 210], Remark 4.1, Lemmas 3.1, 3.2 we have

Remark 4.4. *Let $\mathcal{P}(\lambda)$ be a c.p. of equation (0.1), (4.8) on (c,b). Then condition (4.28) implies the similar condition for λ such that $\Im\lambda\Im\mu_0 > 0$, if $b < \infty$ or if $H_\lambda(t) = H_0(t) + \lambda H(t)$, $H_0(t) = H_0^*(t)$.*

The statements, which are similar to Th. 4.4, Remarks 4.2–4.4 hold for the interval (a,c).

Classes of equations (0.1) such that with $dim\mathcal{H} < \infty$ (4.28) holds for any c.p., are described in [38, 39].

We illustrate below Ths. 4.2–4.4 in three basic cases. To simplify notation assume that $\Im\lambda > 0$.

I. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $Q(t)\begin{pmatrix} I_1 & 0 \\ 0 & -I_2 \end{pmatrix}$, with $I_j, j = 1,2$ being the identity operators in $\mathcal{H}_j$.

In this case one has

$$\Gamma = I, \qquad P_+ = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix},$$

and the Weyl functions $K_+(\lambda) \in B(\mathcal{H}_2, \mathcal{H}_1)$, $K_-(\lambda) \in B(\mathcal{H}_1, \mathcal{H}_2)$ (contractions $U_\pm(\lambda)$ act in a similar way). Therefore

$$P_- + K_+(\lambda)P_- = \begin{pmatrix} 0 & K_+(\lambda) \\ 0 & I_2 \end{pmatrix},$$
$$P_- - K_-(\lambda)P_+ = \begin{pmatrix} 0 & 0 \\ -K_-(\lambda) & I_2 \end{pmatrix},$$

hence the projection $\mathcal{P}(\lambda)$ (4.10) in Th. 4.2 is just

$$\mathcal{P}(\lambda) = \begin{pmatrix} K_+(\lambda) \\ I_2 \end{pmatrix} (I_2 - K_-(\lambda)K_+(\lambda))^{-1}(-K_-(\lambda), I_2),$$

and the projection $\mathcal{P}(\lambda)$ (4.25) in Th. 4.4 is given by

$$\mathcal{P}(\lambda) = \begin{pmatrix} K_+(\lambda) \\ I_2 \end{pmatrix} (I_2 - U_-(\lambda)K_+(\lambda))^{-1}(-U_-(\lambda), I_2).$$

The Weyl solutions are given by

$$\psi_-(t,\lambda) = \begin{pmatrix} x_1(t,\lambda) + x_2(t,\lambda)K_-(\lambda) \\ x_3(t,\lambda) + x_4(t,\lambda)K_-(\lambda) \end{pmatrix}, \ \psi_+(t,\lambda) = \begin{pmatrix} x_1(t,\lambda)K_+(\lambda) + x_2(t,\lambda) \\ x_3(t,\lambda)K_+(\lambda) + x_4(t,\lambda) \end{pmatrix}$$

with

$$x_1(t,\lambda) \in B(\mathcal{H}_1), \quad x_2(t,\lambda) \in B(\mathcal{H}_2, \mathcal{H}_1),$$
$$x_3(t,\lambda) \in B(\mathcal{H}_1, \mathcal{H}_2), x_4(t,\lambda) \in B(\mathcal{H}_2),$$
$$\begin{pmatrix} x_1(t,\lambda) & x_2(t,\lambda) \\ x_3(t,\lambda) & x_4(t,\lambda) \end{pmatrix} = X_\lambda(t). \tag{4.33}$$

The inequalities (4.12) are equivalent to

$$\int\limits_{J_\pm} \psi_\pm^*(t,\lambda)w_\lambda(t)\psi_\pm(t,\lambda)dt \le \frac{1}{2\Im\lambda}(I_\mp - K_\pm^*(\lambda)K_\pm(\lambda)), \ I_+ = I_1, \ I_- = I_2.$$

The operators (4.19), (4.20) from Th. 4.3 are given by

$$P_-K_-(\lambda)P_+ = \begin{pmatrix} 0 & 0 \\ K_-(\lambda) & 0 \end{pmatrix}, \quad P_+K_+(\lambda)P_- = \begin{pmatrix} 0 & K_+(\lambda) \\ 0 & 0 \end{pmatrix},$$

with the Weyl functions $K_\pm(\lambda)$, by a virtue of (4.22), (4.21), being (in the case $dim \mathcal{H} < \infty$ cf. [8])

$$K_-(\lambda) = -(x_2^*(a,\bar\lambda) - x_4^*(a,\bar\lambda)U_-(\lambda))(x_1^*(b,\bar\lambda) - x_3^*(b,\bar\lambda)U_-(\lambda))^{-1}, \tag{4.34}$$
$$K_+(\lambda) = -(x_1^*(b,\bar\lambda)U_+(\lambda) - x_3^*(b,\bar\lambda))(x_2^*(b,\bar\lambda)U_+(\lambda) - x_4^*(b,\bar\lambda))^{-1}, \tag{4.35}$$

(in (4.34) $(...)^{-1} \in B(\mathcal{H}_1)$; in (4.35) one has $(...)^{-1} \in B(\mathcal{H}_2)$).

II. Let

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1 = \mathcal{H}_1^2, \qquad Q(t) = \begin{pmatrix} 0 & iI_1 \\ -iI_1 & 0 \end{pmatrix}. \tag{4.36}$$

In this case one has

$$\Gamma = I, \qquad P_\pm = \frac{1}{2i} \begin{pmatrix} I_1 \\ \mp iI_1 \end{pmatrix} (iI_1, \mp I_1).$$

It is easy to see that the following representations are valid

$$P_\pm K_\pm(\lambda) P_\mp = \frac{1}{2i} \begin{pmatrix} \pm ik_\pm(\lambda) \\ k_\pm(\lambda) \end{pmatrix} \left(iI_1, \pm I_1\right),$$

with $k_\pm(\lambda)$ being strict contractions in $\mathcal{H}_1$, so the projection $\mathcal{P}$ (4.10) in Th. 4.2 is given by

$$\mathcal{P}(\lambda) = \frac{1}{2i} \begin{pmatrix} I_1 + ik_+(\lambda) \\ iI_1 + k_+(\lambda) \end{pmatrix} (I_1 - k_-(\lambda)k_+(\lambda))^{-1}(iI_1 - k_-(\lambda), I_1 - ik_-(\lambda)), \tag{4.37}$$

and inequalities (4.12) in Th. 4.2 are equivalent to

$$\int\limits_{J_\pm} \begin{pmatrix} I_1 \pm ik_\pm(\lambda) \\ \pm iI_1 + k_\pm(\lambda) \end{pmatrix}^* X_\lambda^*(t) w_\lambda(t) X_\lambda(t) \begin{pmatrix} I_1 \pm ik_\pm(\lambda) \\ \pm iI + k_\pm(\lambda) \end{pmatrix} dt$$

$$\leq \frac{1}{\Im\lambda}(I_1 - k_\pm^*(\lambda)k_\pm(\lambda)). \tag{4.38}$$

If one passes in (4.37), (4.38) from strict contractions $k_\pm(\lambda)$ to the Nevanlinna operator functions

$$\pm m_\pm(\lambda) = \pm(\pm iI_1 + k_\pm(\lambda))(I_1 \pm ik_\pm(\lambda))^{-1}, \; \pm\Im m_\pm(\lambda) >> 0,$$

and takes into account that

$$m_-(\lambda) - m_+(\lambda) = 2i(-I_1 + ik_-(\lambda))^{-1}(I_1 - k_-(\lambda)k_+(\lambda))(I_1 + ik_+(\lambda))^{-1},$$

then projection (4.37) acquires the following form known from [1]:

$$\mathcal{P}(\lambda) = \begin{pmatrix} I_1 \\ m_+(\lambda) \end{pmatrix} (m_-(\lambda) - m_+(\lambda))^{-1}(m_-(\lambda), -I_1), \tag{4.39}$$

and inequalities (4.38) become

$$\int\limits_{J_\pm} \psi_\pm^*(t,\lambda) w_\lambda(t) \psi_\pm(t,\lambda) dt \le \pm \frac{\Im m_\pm(\lambda)}{\Im \lambda}, \tag{4.40}$$

with

$$\psi_\pm(t,\lambda) = \begin{pmatrix} x_1(t,\lambda) + x_2(t,\lambda) m_\pm(\lambda) \\ x_3(t,\lambda) + x_4(t,\lambda) m_\pm(\lambda) \end{pmatrix}. \tag{4.41}$$

In a similar way, the projection $\mathcal{P}$ (4.25) in Th. 4.4 is given by

$$\mathcal{P}(\lambda) = \frac{1}{2i} \begin{pmatrix} I_1 \\ m_+(\lambda) \end{pmatrix}$$

$$\times (I_1 + ik_+(\lambda))(I_1 - u_-(\lambda)k_+(\lambda))^{-1}(iI_1 - u_-(\lambda), I_1 - iu_-(\lambda)), \tag{4.42}$$

with $u_-(\lambda)$ being the contraction in $\mathcal{H}_1$ that depends analytically on λ.

Take into account that

$$2i(I_1 - u_-(\lambda)k_+(\lambda))(I_1 + ik_+(\lambda))^{-1} = iI_1 - u_-(\lambda) + (I_1 - iu_-(\lambda))m_+(\lambda)$$

and denote $u(\lambda) = -iu_-(\lambda)$

$$a_1(\lambda) = -i(u(\lambda) + I_1), \qquad a_2 = u(\lambda) - I_1, \tag{4.43}$$

to observe that the projection $\mathcal{P}(\lambda)$ (4.25), (4.42) acquires the form

$$\mathcal{P}(\lambda) = \begin{pmatrix} I_1 \\ m_+(\lambda) \end{pmatrix} (a_2(\lambda) - a_1(\lambda) m_+(\lambda))^{-1} (a_2(\lambda), -a_1(\lambda)). \tag{4.44}$$

Thus in boundary condition (4.26) we can set

$$\mathcal{M}_\lambda = \begin{pmatrix} a_1(\lambda) & 0 \\ a_2(\lambda) & 0 \end{pmatrix}.$$

We follow terminology of [13]–[17] by calling the operators $m_-(\lambda) \in B(\mathcal{H}_1)$ and $m_+(\lambda) \in B(\mathcal{H}_1)$ that depend analytically on λ and satisfy (4.40), (4.41), the Weyl functions of (0.1), (4.36) on (a,c) and (c,b), respectively.

Formula (4.39) indicates that definition of the Weyl functions in this work is equivalent to that of [1]*.

*In [1] $m_\pm(\lambda)$ are denoted by $n_\pm(\lambda)$.

Remark 4.5. *(In the case* $dim\mathcal{H} < \infty$ *cf. [8]). Suppose the interval* (a,b) *is finite. Then the operators* $m_-(\lambda) \in B(\mathcal{H}_1)$ *and* $m_+(\lambda) \in B(\mathcal{H}_1)$ *are the Weyl functions of (0.1), (4.36) on* (a,c) *and on* (c,b)*, respectively if and only if they admit the representation*

$$m_-(\lambda) = (x_1^*(a,\bar{\lambda})a_2(\lambda) - x_3^*(a,\bar{\lambda})a_1(\lambda))(x_4^*(a,\bar{\lambda})a_1(\lambda) - x_2^*(a,\bar{\lambda})a_2(\lambda))^{-1}, \tag{4.45}$$

$$m_+(\lambda) = (x_1^*(b,\bar{\lambda})b_2(\lambda) - x_3^*(b,\bar{\lambda})b_1(\lambda))(x_4^*(b,\bar{\lambda})b_1(\lambda) - x_2^*(b,\bar{\lambda})b_2(\lambda))^{-1}, \tag{4.46}$$

with $x_j(t,\lambda) \in B(\mathcal{H}_1)$ *see (4.33),*

$$a_1(\lambda) = -i(U(\lambda) + I_1), \qquad a_2 = U(\lambda) - I_1, \tag{4.47}$$

$$b_1(\lambda) = V(\lambda) - I_1, \qquad b_2 = -i(V(\lambda) + I_1), \tag{4.48}$$

$U(\lambda)$, $V(\lambda)$ *being arbitrary contractions in* $\mathcal{H}_1$ *(the latter assumption guaranties that in (4.45), (4.46) one has* $(...)^{-1} \in B(\mathcal{H}_1)$*) that depend analytically on* λ*.*

In this setting solution of (0.1), (4.36) $x_\lambda(t)$ *(1.9), (1.20), (4.39), (4.45), (4.46) is a solution of the boundary-value problem (0.1), (1.14) with*

$$\mathcal{M}_\lambda = \begin{pmatrix} a_1(\lambda) & 0 \\ a_2(\lambda) & 0 \end{pmatrix}, \qquad \mathcal{N}_\lambda = \begin{pmatrix} 0 & b_1(\lambda) \\ 0 & b_2(\lambda) \end{pmatrix},$$

$a_j(\lambda)$, $b_j(\lambda) \in B(\mathcal{H}_1)$, *see (4.47), (4.48).*

Remark 4.6. 1^0 *(cf. Remark 4.1). One can choose the contraction* $u(\lambda)$ *in (4.43) so that in (4.43) one has* $a_1^{-1} \in B(\mathcal{H}_1)$*, and* $a_2(\lambda)a_1^{-1}(\lambda) = m_-(\lambda)$ *is an arbitrary Weyl function of (0.1), (4.36) on* (a,c)*. Under this choice of* $u(\lambda)$ *the projections (4.44), (4.39) coincide, hence* $M(\lambda)$ *(1.20), (4.44) is also a characteristic operator of (0.1), (4.36) on* (a,b)*.*

2^0*. If in (4.43)* $u(\lambda)$ *is unitary (hence by [35, p. 210]* $u(\lambda) = u$ *does not depend on* λ*), then the operators* a_1*,* a_2 *admit the representation (cf. [33])*

$$a_1 = \cos\alpha \cdot K, \qquad a_2 = \sin\alpha \cdot K,$$

with $\alpha = \alpha^* \in B(\mathcal{H}_1), K = -2ie^{-i\alpha}$*, and the projection (4.44) can be written in the form*

$$\mathcal{P}(\lambda) = \begin{pmatrix} I_1 \\ m_+(\lambda) \end{pmatrix} (\sin\alpha - \cos\alpha m_+(\lambda))^{-1} (\sin\alpha, -\cos\alpha)$$

with $(...) \in B(\mathcal{H}_1)$*.*

In definition (1.9), (1.10) of c.o. (0.1), (4.36) the operator solution $X_\lambda(t)$ is often replaced by

$$X_\beta(t,\lambda) = X_\lambda(t) \begin{pmatrix} \sin\beta & \cos\beta \\ -\cos\beta & \sin\beta \end{pmatrix},$$

with $\beta = \beta^* \in B(\mathcal{H}_1)$.

Remark 4.7. *(In the case dim$\mathcal{H}$ = 2 cf. [17]). Suppose that the operators α, β commute. Then, if one replaces in the definition of c.o. of (0.1), (4.36) the operator solution $X_\lambda(t)$ by $X_\beta(t,\lambda)$, the characteristic projection $\mathcal{P}(\lambda)$ from $n^0 2^0$ of Remark 4.6 turns into the characteristic projection*

$$\mathcal{P}_\beta(\lambda) = \begin{pmatrix} I_1 \\ m_+(\lambda,\beta) \end{pmatrix}$$

$$\times(\sin(\alpha-\beta)+\cos(\alpha-\beta)m_+(\lambda,\beta))^{-1}(\cos(\alpha-\beta),\sin(\alpha-\beta)), \qquad (4.49)$$

with a new Weyl function $m_+(\lambda,\beta)$ being related to the Weyl function $m_+(\lambda)$ as follows:

$$m_+(\lambda,\beta) = (\cos\beta+\sin\beta m_+(\lambda))(\sin\beta - cos\beta m_+(\lambda))^{-1} \qquad (4.50)$$

(in (4.49), (4.50) one has $(...)^{-1} \in B(\mathcal{H})$).

We are about to demonstrate that the Weyl functions $m_\pm(\lambda)$ are analogues of the Weyl functions of Dirac and Sturm–Liouville equations, as well as analogues for characteristic matrix of a scalar symmetric differential operator of even order on the semiaxis.

In the case of the Dirac type homogeneous equation (0.1), (4.36) when the weight $\Im w_\lambda(t) = I$, i.e., for the equation

$$\begin{pmatrix} 0 & -I_1 \\ I_1 & 0 \end{pmatrix} x'(t) + H_0(t)x(t) = \lambda x(t), \qquad (4.51)$$

with $H_0(t) = H_0^*(t) \in B(\mathcal{H})$, the inequalities (4.40) are equivalent to the inequalities as follows:

$$\int\limits_{J_\pm} \big[(x_1(t,\lambda)+x_2(t,\lambda)m_\pm(\lambda))^*(x_1(t,\lambda)+x_2(t,\lambda)m_\pm(\lambda))$$

$$+(x_3(t,\lambda)+x_4(t,\lambda)m_\pm(\lambda))^*(x_3(t,\lambda)+x_4(t,\lambda)m_\pm(\lambda))\big]\,dt \le \pm\frac{\Im m_\pm(\lambda)}{\Im\lambda}, \qquad (4.52)$$

i.e., $m_\pm(\lambda)$ (or $m_\pm^{-1}(\lambda)$) are operator analogues of the scalar Weyl functions for the Dirac equation (see, e.g., [13], [17]).

One has a different but equivalent to (4.51) form of the Dirac equation, which is commonly used:

$$i\begin{pmatrix} I_1 & 0 \\ 0 & -I_1 \end{pmatrix} y'(t) + \tilde{H}_0(t)y(t) = \lambda y(t), \qquad (4.53)$$

with $\tilde{H}_0(t) = 2S^{*-1}H_0(t)S^{-1}$, $S = \begin{pmatrix} I_1 & iI_1 \\ iI_1 & I_1 \end{pmatrix}$.

In this case inequalities (4.52) are equivalent to the claim that for the operator solutions

$$\psi_\pm(t,\lambda) = F(t) + G(t)m_\pm(\lambda) \in B(\mathcal{H}_1, \mathcal{H}_1^2)$$

of (4.53), where $F(t), G(t) \in B(\mathcal{H}_1, \mathcal{H}_1^2)$ are such operator solutions of (4.53) that $\{F(0), G(0)\} = S$, i.e., $F(0) = \begin{pmatrix} I_1 \\ iI_1 \end{pmatrix}, G(0) = \begin{pmatrix} iI_1 \\ I_1 \end{pmatrix} \in B(\mathcal{H}_1, \mathcal{H}_1^2)$, the following inequalities hold

$$\int\limits_{J_\pm} \psi_\pm^*(t,\lambda)\psi_\pm(t,\lambda)dt \leq \pm\frac{2\Im m_\pm(\lambda)}{\Im\lambda}.$$

Consequently, $m_\pm(\lambda)$ are also operator analogues of the Weyl functions for the Dirac equation of the form (4.53), considered for the case $dim\mathcal{H} = 2$ in the work by V.A. Marchenko [15].

In the case when $\mathcal{H}_1 = \mathcal{H}_2^n$ and the weight given by $w_\lambda(t) = diag(I_2, O_2, .., O_2)$ (in particular, a symmetric equation of an arbitrary even order $2n$ in $\mathcal{H}_2$ (e.g., the Sturm–Liouville equation) reduces to equation (0.1), (4.36) with such weight), inequalities (4.40) are equivalent to

$$\int\limits_{J_\pm}[(x_1(t,\lambda),\ldots,x_n(t,\lambda)) + (x_{n+1}(t,\lambda),\ldots,x_{2n}(t,\lambda))m_\pm(\lambda)]^*$$

$$\times[(x_1(t,\lambda),\ldots,x_n(t,\lambda)) + (x_{n+1}(t,\lambda),\ldots,x_{2n}(t,\lambda))m_\pm(\lambda)]dt \leq \pm\frac{\Im m_\pm(\lambda)}{\Im\lambda},$$

with $x_j(t,\lambda) = x_{1j}(t,\lambda)$ being the first line operator elements of the operator matrix $X_\lambda(t) = (x_{ij}(t,\lambda))_{i,j=1}^{2n}, x_{ij}(t,\lambda) \in B(\mathcal{H}_2)$. That is, the operators $m_\pm(\lambda)$ (or $m_\pm^{-1}(\lambda)$) are analogues of the Weyl functions of the Sturm–Liouville equation [13]–[15], [17], as well as an operator analogue of characteristic matrix [40] for the scalar symmetric equation of even order $2n$ on the semiaxis.

III. Let

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1, \qquad Q(t) = \begin{pmatrix} 0 & I_1 \\ I_1 & 0 \end{pmatrix}. \tag{4.54}$$

This case reduces to the case II since

$$\begin{pmatrix} I_1 & 0 \\ 0 & iI_1 \end{pmatrix}\begin{pmatrix} 0 & iI_1 \\ -iI_1 & 0 \end{pmatrix}\begin{pmatrix} I_1 & 0 \\ 0 & -iI_1 \end{pmatrix} = \begin{pmatrix} 0 & I_1 \\ I_1 & 0 \end{pmatrix}.$$

In particular, (4.40), (4.46) in the case (4.54) turns to the inequalities obtained for the case $dim\mathcal{H} < \infty$ in [16, p. 337] for the Weyl type solutions of equation $x'(t) = i\lambda\begin{pmatrix} 0 & I_1 \\ I_1 & 0 \end{pmatrix}H(t)x(t)$ with $t \in \mathcal{I}_+$, $0 \leq H(t) \in B(\mathcal{H})$, $\int_c^l H(t)dt >> 0$ for some $l \in \bar{\mathcal{I}}_+$.

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