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On Linear Relations Generated by a Differential Expression and by a Nevanlinna Operator Function

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The families of maximal and minimal relations generated by a differential expression with bounded operator coefficients and by a Nevanlinna operator function are defined. These families are proved to be holomorphic. In the case of finite interval, the space of boundary values is constructed. In terms of boundary conditions, a criterion for the restrictions of maximal relations to be continuously invertible and a criterion for the families of these restrictions to be holomorphic are given. The operators inverse to these restrictions are stated to be integral operators. By using the results obtained, the existence of the characteristic operator on the finite interval and the axis is proved.

Key words: Hilbert space, linear relation, differential expression, holomorphic family of relations, resolvent, characteristic operator, Nevanlinna function.

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Introduction

On a finite or infinite interval (a,b), we consider a differential expression $l_\lambda[y] = l[y] - \mathcal{C}_\lambda y$, where l is a formally self-adjoint differential expression with bounded operator coefficients in a Hilbert space H, $\mathcal{C}_\lambda = \mathcal{C}_\lambda(t)$ is a Nevanlinna operator function, i.e., $\mathcal{C}_\lambda(t)$ is a holomorphic function for $\mathrm{Im}\lambda \neq 0$ whose values are the bounded operators in H such that $\mathcal{C}^*_{\bar\lambda}(t) = \mathcal{C}_\lambda(t)$ and $\mathrm{Im}\,\mathcal{C}_\lambda(t)/\mathrm{Im}\lambda \geqslant 0$.

Let $A(t) = \mathrm{Im}\,\mathcal{C}_{\lambda_0}(t)/\mathrm{Im}\lambda_0$. In the space $L_2(H, A(t); a, b)$, we define the families $L_0(\lambda)$, $L(\lambda)$ of minimal and maximal relations, respectively, and prove that these families are holomorphic for $\mathrm{Im}\lambda \neq 0$. In the case of finite interval, the space of boundary values for the family of maximal relations $L(\lambda)$ is constructed. There

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exists a one-to-one correspondence between the relations $\widehat{L}(\lambda)$ with the property $L_0(\lambda) \subset \widehat{L}(\lambda) \subset L(\lambda)$ and the boundary relations $\theta(\lambda)$. This correspondence is determined by the equality $\gamma(\lambda)\widehat{L}(\lambda) = \theta(\lambda)$, where $\gamma(\lambda)$ is the operator which to each pair from $L(\lambda)$ associates the pair of boundary values (in this case, we denote $\widehat{L}(\lambda) = L_{\theta(\lambda)}(\lambda)$). We prove that 1) for fixed λ, the relation $(L_{\theta(\lambda)}(\lambda))^{-1}$ is a bounded everywhere defined operator if and only if the relation $\theta^{-1}(\lambda)$ is the operator with the same property; in this case the operator $(L_{\theta(\lambda)}(\lambda))^{-1}$ is integral; 2) the operator function $(L_{\theta(\lambda)}(\lambda))^{-1}$ is holomorphic if and only if $\theta^{-1}(\lambda)$ is holomorphic. If the function $\mathcal{C}_\lambda$ is linear, these results correspond with the description of the generalized resolvents from [1–3].

We apply the obtained results to prove the existence of the characteristic operator on the finite interval and the axis. In the papers by V.I. Khrabustovsky [4–6], the definition of the characteristic operator is given. In these papers, for a differential operator of first order the existence of the characteristic operator is established and various problems associated with the characteristic operator are studied. Bibliography on the characteristic operator is given in [4–6].

1. Main Assumptions and Auxiliary Statements

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. On a finite or infinite interval (a,b) we consider a differential expression l of order r

$$l[y] = \begin{cases} \displaystyle\sum_{k=1}^{n} (-1)^k \{(p_{n-k}(t)y^{(k)})^{(k)} - i[(q_{n-k}(t)y^{(k)})^{k-1} + (q_{n-k}(t)y^{(k-1)})^{(k)}]\} + \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + p_n(t)y, \\ \displaystyle\sum_{k=0}^{n} (-1)^k \{i[(q_{n-k}(t)y^{(k)})^{(k+1)} + (q_{n-k}(t)y^{(k+1)})^k] + (p_{n-k}(t)y^{(k)})^{(k)}\}, \end{cases}$$

where $r = 2n$ $(n = 1, 2, \dots)$ or $r = 2n+1$ $(n = 0, 1, 2, \dots)$. The coefficients of l are the bounded self-adjoint operators in H. The leading coefficients, $p_0(t)$ in the case $r = 2n$ and $q_0(t)$ in the case $r = 2n+1$, have bounded inverse operators almost everywhere. According to [7], l is considered to be a quasidifferential expression. The quasiderivatives for the expression l are defined in [7]. Suppose that the functions $p_j(t)$, $q_k(t)$ are strongly continuous, the function $q_0(t)$ in the case $r = 2n+1$ is strongly differentiable, and the norms of the functions

$$p_0^{-1}(t),\ p_0^{-1}(t)q_0(t),\ q_0(t)p_0^{-1}(t)q_0(t),\ p_1(t), \dots, p_n(t),\ q_0(t), \dots, q_{n-1}(t), \quad \text{in case } r = 2n,$$

$$q_0'(t),\ q_1(t), \dots, q_n(t),\ p_0(t), \dots, p_n(t), \quad \text{in case } r = 2n+1,$$

are integrable on every compact interval $[\alpha, \beta] \subset (a,b)$.

The boundary a is called a regular boundary if $a > -\infty$, and one can take $\alpha = a$, and the norm $\|\mathcal{C}_\lambda(t)\|$ is integrable on every interval $[a, \beta)$ $(\beta < b)$, where the function $\mathcal{C}_\lambda(t)$ is defined bellow. In the contrary case, the boundary a is called a singular boundary. The regularity and singularity of the boundary b are defined analogously. For brevity, we assume that if the interval (a, b) is finite, then the boundaries a, b are regular. The singular case is studied on the example of the axis $(-\infty, \infty)$. Other cases of singular boundaries are considered analogously.

Let $\mathcal{C}_\lambda(t)$ be an operator function whose values are the bounded operators in H. Suppose $\mathcal{C}_\lambda(t)$ satisfies the following conditions [4]:

(a) there exist sets $\mathbb{C}_0 \supset \mathbb{C} \setminus \mathbb{R}^1$ and $\mathcal{I}_0 \subset (a, b)$ with the following properties: the measure of the set $(a, b) \setminus \mathcal{I}_0$ equals zero and every point belonging to $\mathbb{C}_0$ has a neighborhood independent of $t \in \mathcal{I}_0$ such that the function $\mathcal{C}_\lambda(t)$ is holomorphic with respect to λ in this neighborhood;

(b) for all $\lambda \in \mathbb{C}_0$, the function $\mathcal{C}_\lambda(t)$ is Bochner locally integrable in the uniform operator topology;

(c) $\mathcal{C}^*_\lambda(t) = \mathcal{C}_{\bar\lambda}(t)$ as well as $\mathrm{Im}\, \mathcal{C}_\lambda(t)/\mathrm{Im}\lambda$ is a nonnegative operator for all $t \in \mathcal{I}_0$ and for all λ such that $\mathrm{Im}\lambda \neq 0$.

We denote $\mathcal{B}_\lambda(t) = \mathrm{Re}\, \mathcal{C}_\lambda(t)$, $\mathcal{A}_\lambda(t) = \mathrm{Im}\, \mathcal{C}_\lambda(t)$. Using condition (c), we get $\mathcal{B}_{\bar\lambda} = \mathcal{B}_\lambda$, $\mathcal{A}_{\bar\lambda}(t) = -\mathcal{A}_\lambda(t)$. Let the operator $\mathfrak{a}_\lambda(t)$ be given by $\mathfrak{a}_\lambda(t) = \mathcal{A}_\lambda(t)/\mathrm{Im}\lambda$ for λ such that $\mathrm{Im}\lambda \neq 0$. It follows from condition (c) that for all $\mu \in \mathbb{C}_0 \cap \mathbb{R}$ there exists $\lim_{\lambda \to \mu \pm i0} \mathfrak{a}_\lambda(t) = \mathfrak{a}_\mu(t)$. The function $\mathfrak{a}_\mu(t)$ is locally integrable [4].

Further, we will use the following statements (see [4]).

Statement 1. *If for some element $g \in H$ there exist numbers $t_0 \in \mathcal{I}_0$ and $\lambda_0 \in \mathbb{C}_0$ such that $\mathfrak{a}_{\lambda_0}(t_0)g = 0$, then for all $\lambda \in \mathbb{C}_0$ the equalities $\mathfrak{a}_\lambda(t_0)g = 0$ and $(\mathcal{B}_\lambda(t_0) - \mathcal{B}_{\lambda_0}(t_0))g = 0$ hold.*

Statement 2. *Suppose $t \in \mathcal{I}_0$, $\lambda, \mu \in \mathbb{C}_0$ and a sequence of the elements $g_n \in H$ satisfy the condition $(\mathfrak{a}_\lambda(t)g_n, g_n) \to 0$ as $n \to \infty$. Then $(\lambda - \mu)^{-1}(\mathcal{C}_\lambda(t) - \mathcal{C}_\mu(t))g_n \to 0$.*

Statement 3. *For any compact $\mathrm{K} \subset \mathbb{C}_0$ there exist constants $k_1, k_2 > 0$ such that for all $h \in H$, $t \in \mathcal{I}_0$, $\lambda_1, \lambda_2 \in \mathrm{K}$ the inequality $k_1(\mathfrak{a}_{\lambda_1}(t)h, h)^{1/2} \leqslant (\mathfrak{a}_{\lambda_2}(t)h, h)^{1/2} \leqslant k_2(\mathfrak{a}_{\lambda_1}(t)h, h)^{1/2}$ holds (thus the constants k_1, k_2 are independent of $t \in \mathcal{I}_0$, $\lambda \in \mathrm{K}$).*

We denote $l_\lambda[y] = l[y] - (\mathcal{B}_\lambda(t) + i\mathcal{A}_\lambda(t))y = l[y] - \mathcal{C}_\lambda(t)y$. Let $W_{j,\lambda}(t)$ be the operator solution of the equation $l_\lambda[y] = 0$ satisfying the initial conditions $W^{[k-1]}_{j,\lambda}(t_0) = \delta_{j,k}E$, where $t_0 \in (a, b)$, $\delta_{j,k}$ is the Kronecker symbol, E is the identity operator, $j, k = 1, \ldots, r$. By $W_\lambda(t)$ we denote the operator one-row matrix $W_\lambda(t) = (W_{1,\lambda}(t), \ldots, W_{r,\lambda}(t))$. The operator $W_\lambda(t)$ is a continuous mapping of H^r into H. The adjoint operator $W^*_\lambda(t)$ maps continuously H into H^r. For fixed t, the function $W_\lambda(t)$ is holomorphic on $\mathbb{C}_0$.

If l_λ is defined for the function y, then we denote $\widehat{y} = (y^{[0]}, y^{[1]}, \ldots, y^{[r-1]})^T$ (T in the upper index denotes transposition). Let $u = (u_1, \ldots, u_n)$ be a system of functions such that $l_\lambda[u_j]$ exists for $j = 1, \ldots, n$. By $\widehat{u}$ we denote the matrix with the jth column $\widehat{u}_j$ ($j = 1, \ldots, n$). Similar notations are used for the operator functions. Note that all quasiderivatives up to order $r-1$ inclusive coincide for the expressions l and l_λ.

We consider the operator matrices of orders $2n$ and $2n+1$ for the expression l in the cases $r = 2n$ and $r = 2n+1$, respectively:

$$J_{2n}(t) = \begin{pmatrix} & & & & & -E \\ & & & & \cdots & \\ & & & -E & & \\ & & E & & & \\ & \cdots & & & & \\ E & & & & & \end{pmatrix},$$

$$J_{2n+1}(t) = \begin{pmatrix} & & & & & & -E \\ & & & & & \cdots & \\ & & & & -E & & \\ & & & 2iq_0^{-1}(t) & & & \\ & & E & & & & \\ & \cdots & & & & & \\ E & & & & & & \end{pmatrix},$$

where all non indicated elements are equal to zero. (In matrix J_{2n+1} the element $2iq_0^{-1}(t)$ stands on the intersection of the row $n+1$ and the column $n+1$.)

Suppose the expression l is defined for the functions y, z, and $l[y]$, $l[z]$ are locally integrable on (a, b). Then Lagrange's formula has the form

$$\int\limits_\alpha^\beta (l[y], z)dt - \int\limits_\alpha^\beta (y, l[z])dt = (J_r(t)\widehat{y}(t), \widehat{z}(t))|_\alpha^\beta, \quad a < \alpha \leqslant \beta < b. \tag{1}$$

In (1), we take $y(t) = W_\lambda(t)c$, $z(t) = W_{\bar\lambda}(t)d$ ($c, d \in H^r$). Since $l[y] = (\mathcal{B}_\lambda + i\mathcal{A}_\lambda)y$, $l[z] = (\mathcal{B}_\lambda - i\mathcal{A}_\lambda)z$, we obtain

$$\widehat{W}^*_{\bar\lambda}(t)J_r(t)\widehat{W}_\lambda(t) = J_r(t_0). \tag{2}$$

It follows from (2) and from the "method of the variation of arbitrary constants" that the general solution of the equation $l_\lambda[y] = f(t)$ is represented in the form

$$y_\lambda(t, f) = W_\lambda(t)\left(c + J_r^{-1}(t_0)\int\limits_{t_0}^t W^*_{\bar\lambda}(s)f(s)ds\right), \tag{3}$$

where $c \in H^r$, the function f is locally integrable on (a,b). Consequently,

$$\widehat{y}_\lambda(t,f) = \widehat{W}_\lambda(t)\left(c + J_r^{-1}(t_0)\int\limits_{t_0}^{t} W^*_{\bar\lambda}(s)f(s)ds\right). \tag{4}$$

We fix some $\lambda_0 \in \mathbb{C}_0$. For brevity, we denote $A(t) = \mathfrak{a}_{\lambda_0}(t)$. On the set of functions continuous and finite on the interval (a,b) and ranging in H, we introduce the quasi-scalar product

$$(y,z)_A = \int\limits_a^b (A(t)y(t), z(t))dt. \tag{5}$$

We identify the functions such that $(y,y)_A = 0$ with zero and perform the completion. Then we obtain a Hilbert space denoted by $\mathfrak{H} = L_2(H, A(t); a, b)$. The elements of $\mathfrak{H}$ are the classes of functions identified with each order in the norm

$$\|y\|_A = \left(\int\limits_a^b \left\|A^{1/2}(t)y(t)\right\|^2 dt\right)^{1/2}. \tag{6}$$

In order not to complicate terminology, we denote the class of functions with representative y by the same symbol. We will also say that the function y belongs to $\mathfrak{H}$. We treat the equalities between functions belonging to $\mathfrak{H}$ as the equalities between corresponding equivalence classes.

It follows from Statements 1, 3 that the space $\mathfrak{H}$ does not depend on the choice of point $\lambda \in \mathbb{C}_0$ in the sense below. If we change $A(t) = \mathfrak{a}_{\lambda_0}(t)$ either to $\mathfrak{a}_\lambda(t)$ $(\lambda \in \mathbb{C}_0)$ or to $\mathcal{A}_\lambda(t)$ $(\mathrm{Im}\lambda > 0)$ in (5), we obtain the same set $\mathfrak{H}$ supplied with the equivalent norm.

According to Statement 1, the set $G(t) = \ker \mathfrak{a}_\lambda(t)$ $(t \in \mathcal{I}_0)$ is independent of $\lambda \in \mathbb{C}_0$. Let $H(t)$ be the orthogonal complement of $G(t)$ in H, i.e, $H(t) = H \ominus G(t)$; and $A_0(t)$ be the restriction of $A(t)$ to $H(t)$. By $H_{-1/2}(t)$ we denote the completion of $H(t)$ with respect to the norm $(A_0(t)x, x)^{1/2}$. The space $H_{-1/2}(t)$ can be treated as a space with negative norm with regard to $H(t)$ [8, Ch. 2]. Let $H_{1/2}(t)$ denote the corresponding space with a positive norm. It follows from Statements 1, 3 that the spaces $H_{1/2}(t)$, $H_{-1/2}(t)$ do not depend on the replacement of $A(t) = \mathfrak{a}_{\lambda_0}(t)$ by $\mathfrak{a}_\lambda(t)$ $(\lambda \in \mathbb{C}_0)$ or by $\mathcal{A}_\lambda(t)$ $(\mathrm{Im}\lambda > 0)$ in the sense below. After replacing we obtain the same sets $H_{1/2}(t)$, $H_{-1/2}(t)$, but the norms on $H_{1/2}(t)$, $H_{-1/2}(t)$ are changed to the equivalent norms. Moreover, the constants in the inequalities between the norms do not depend on $t \in \mathcal{I}_0$ and $\lambda \in \mathrm{K}$, where the compact $\mathrm{K} \subset \mathbb{C}_0$.

Suppose Im$\lambda > 0$. Let $\mathcal{A}^{1/2}_{0,\lambda}(t)$, $\mathcal{A}_{0,\lambda}(t)$ be the restrictions respectively of $\mathcal{A}^{1/2}_{\lambda}(t)$, $\mathcal{A}_{\lambda}(t)$ to $H(t)$. It follows from the above argument and the definition of positive and negative spaces that the operators $\mathcal{A}^{1/2}_{0,\lambda}(t)$, $\mathcal{A}_{0,\lambda}(t)$ possess the continuous extensions $\tilde{\mathcal{A}}^{1/2}_{0,\lambda}(t)$, $\tilde{\mathcal{A}}_{0,\lambda}(t)$ to $H_{-1/2}(t)$. The operator $\tilde{\mathcal{A}}^{1/2}_{0,\lambda}(t)$ ($\tilde{\mathcal{A}}_{0,\lambda}(t)$) is the continuous and one-to-one mapping of $H_{-1/2}(t)$ onto $H(t)$ (onto $H_{1/2}(t)$, respectively). Moreover, the equality $\tilde{\mathcal{A}}_{0,\lambda}(t) = \mathcal{A}^{1/2}_{0,\lambda}(t)\tilde{\mathcal{A}}^{1/2}_{0,\lambda}(t)$ holds. The operator $\mathcal{A}^{1/2}_{0,\lambda}(t)$ is a bijection of $H(t)$ onto $H_{1/2}(t)$. For Im$\lambda<0$, we set $\tilde{\mathcal{A}}_{0,\lambda}(t)=-\tilde{\mathcal{A}}_{0,\bar{\lambda}}(t)$.

Let $\tilde{\mathcal{A}}_{\lambda}(t)$ (Im$\lambda\neq 0$) denote the operator defined on $H_{-1/2}(t)\oplus G(t)$ which is equal to $\tilde{\mathcal{A}}_{0,\lambda}(t)$ on $H_{-1/2}(t)$ and to zero on $G(t)$. Obviously, $\tilde{\mathcal{A}}_{\lambda}(t)$ is an extension of $\mathcal{A}_{\lambda}(t)$. We apply the similar notations for the operators $\mathfrak{a}_{\lambda}(t)$ and $A(t)$. In particular, $\tilde{A}(t)$ is an extension of $A(t)$ to $H_{-1/2}(t)\oplus G(t)$ obtained in the same way as the extension $\tilde{\mathcal{A}}_{\lambda}(t)$. The above argument and Statement 3 imply

$$\left\|\tilde{\mathcal{A}}_{\lambda}(t)\tilde{\mathcal{A}}^{-1}_{0,\mu}(t)x\right\|_{H_{1/2}(t)} \leqslant k\left\|x\right\|_{H_{1/2}(t)}, \quad x\in H_{1/2}(t),$$
$$\left\|\tilde{\mathcal{A}}_{\lambda}(t)x_1\right\|_{H_{1/2}(t)} \leqslant k_1\left\|x_1\right\|_{H_{-1/2}(t)}, \quad x_1\in H_{-1/2}(t), \tag{7}$$

where the constants k, k_1 are independent of $t\in\mathcal{I}_0$. (Here and further, the symbols k, $k_1,...$ denote positive constants that are different in various inequalities, in general.)

In [1], it was shown that the spaces $H_{-1/2}(t)$ are measurable with respect to the parameter t [9, Ch. 1] whenever for the family of measurable functions one takes the family of functions of the form $\tilde{A}^{-1}_0(t)A^{1/2}(t)h(t)$, where $h(t)$ is a measurable function with the values in H. The space $\mathfrak{H}$ is a measurable sum of the spaces $H_{-1/2}(t)$, and $\mathfrak{H}$ consists of the elements (i.e., classes of functions) with representatives of the form $\tilde{A}^{-1}_0(t)A^{1/2}(t)h(t)$, where $h(t)\in L_2(H;a,b)$ [1].

We denote $\mathcal{B}_{\lambda,\mu}(t) = \mathcal{B}_{\lambda}(t) - \mathcal{B}_{\mu}(t)$, where $\lambda,\mu\in\mathbb{C}_0$, $t\in\mathcal{I}_0$. It follows from Statement 1 that $G(t) = \ker\mathfrak{a}_{\lambda}(t)\subset\ker\mathcal{B}_{\lambda,\mu}(t)$. Therefore, $\overline{\mathcal{R}(\mathcal{B}_{\lambda,\mu}(t))}\subset\overline{\mathcal{R}(\mathfrak{a}_{\lambda}(t))} = H(t)$. By $\mathcal{B}^{(0)}_{\lambda,\mu}(t)$ denote the restriction of $\mathcal{B}_{\lambda,\mu}(t)$ to $H(t)$.

Lemma 1. *For all $t\in\mathcal{I}_0$, $\lambda,\mu\in\mathbb{C}_0$, the operator $\mathcal{B}^{(0)}_{\lambda,\mu}(t)$ possesses the continuous extension $\tilde{\mathcal{B}}^{(0)}_{\lambda,\mu}(t)$ onto the space $H_{-1/2}(t)$. The operator $\tilde{\mathcal{B}}^{(0)}_{\lambda,\mu}(t)$ maps $H_{-1/2}(t)$ into $H_{1/2}(t)$ continuously.*

Proof. Following [4], we represent the function $\mathcal{C}_{\lambda}(t)$ in the form

$$\mathcal{C}_{\lambda}(t) = \mathrm{Re}\,\mathcal{C}_i(t) + \lambda\mathcal{U}(t) + \int\limits_{-\infty}^{\infty}\left(\frac{1}{\tau-\lambda} - \frac{\tau}{1+\tau^2}\right)d\sigma_t(\tau), \tag{8}$$

where $\mathcal{U}(t)$ are nonnegative bounded operators in H, $\sigma_t(\tau)$ is a nondecreasing operator function for each fixed $t \in \mathcal{I}_0$ whose values are bounded operators in H and $\int_{-\infty}^{\infty} \frac{d(\sigma_t(\tau)g,g)}{1+\tau^2} < \infty$ for any $g \in H$. Then

$$\mathfrak{a}_\lambda(t) = \mathcal{U}(t) + \int\limits_{-\infty}^{+\infty} \frac{d\sigma_t(\tau)}{|\tau-\lambda|^2}. \tag{9}$$

Using (8), (9), we obtain

$$\begin{aligned}
&\left|(\lambda-\mu)^{-1}((\mathcal{C}_\lambda(t)-\mathcal{C}_\mu(t))g,h)\right| \leqslant \left\|\mathcal{U}^{1/2}(t)g\right\| \left\|\mathcal{U}^{1/2}(t)h\right\| \\
&\qquad + \left(\int\limits_{-\infty}^{\infty} \frac{(d\sigma_t(\tau)g,g)}{|t-\lambda|^2}\right)^{1/2} \left(\int\limits_{-\infty}^{\infty} \frac{(d\sigma_t(\tau)h,h)}{|t-\mu|^2}\right)^{1/2} \\
&\leqslant \left(\left\|\mathcal{U}^{1/2}(t)g\right\|^2 + \int\limits_{-\infty}^{\infty} \frac{(d\sigma_t(\tau)g,g)}{|t-\lambda|^2}\right)^{1/2} \left(\left\|\mathcal{U}^{1/2}(t)h\right\|^2 + \int\limits_{-\infty}^{\infty} \frac{(d\sigma_t(\tau)h,h)}{|t-\mu|^2}\right)^{1/2} \\
&\qquad = (\mathfrak{a}_\lambda(t)g,g)^{1/2}(\mathfrak{a}_\mu(t)h,h)^{1/2} = \left\|\mathfrak{a}_\lambda^{1/2}(t)g\right\| \left\|\mathfrak{a}_\mu^{1/2}(t)h\right\|
\end{aligned}$$

for all $g,h \in H$. Hence, $|(\mathcal{B}_{\lambda,\mu}(t)g,h)| \leqslant |\lambda-\mu| \left\|\mathfrak{a}_\lambda^{1/2}(t)g\right\| \left\|\mathfrak{a}_\mu^{1/2}(t)h\right\|$. According to Statement 3, there exists a constant $k > 0$ such that k is independent of $t \in \mathcal{I}_0$, and

$$|(\mathcal{B}_{\lambda,\mu}(t)g,h)| \leqslant k \left\|A^{1/2}(t)g\right\| \left\|A^{1/2}(t)h\right\| \quad (k>0). \tag{10}$$

It follows from (10) that the operator $\mathcal{B}^{(0)}_{\lambda,\mu}(t) : H(t) \to H(t)$ possesses the continuous extension $\tilde{\mathcal{B}}^{(0)}_{\lambda,\mu}(t) : H_{-1/2}(t) \to H(t)$ (this fact also follows from Statement 2). Let $\tilde{\mathcal{B}}^{+}_{\lambda,\mu}(t)$ be the adjoint operator for $\tilde{\mathcal{B}}^{(0)}_{\lambda,\mu}(t)$. Then $\tilde{\mathcal{B}}^{+}_{\lambda,\mu}(t)$ maps $H(t)$ into $H_{1/2}(t)$ continuously. We claim that $\tilde{\mathcal{B}}^{+}_{\lambda,\mu}(t) = \mathcal{B}^{(0)}_{\lambda,\mu}(t)$. Since the operator $\mathcal{B}_{\lambda,\mu}(t)$ is self-adjoint, we have $(\mathcal{B}_{\lambda,\mu}(t)x_1,x_2) = (x_1,\mathcal{B}_{\lambda,\mu}(t)x_2)$ for all $x_1,x_2 \in H(t)$. On the other hand,

$$(\mathcal{B}_{\lambda,\mu}(t)x_1,x_2) = (\tilde{\mathcal{B}}^{0}_{\lambda,\mu}(t)x_1,x_2) = (x_1,\tilde{\mathcal{B}}^{+}_{\lambda,\mu}(t)x_2).$$

Hence, $\mathcal{B}^{(0)}_{\lambda,\mu}(t) = \tilde{\mathcal{B}}^{+}_{\lambda,\mu}(t)$. Thus, $\mathcal{B}^{(0)}_{\lambda,\mu}(t)$ maps $H(t)$ into $H_{1/2}(t)$ continuously. From this, we obtain that inequality (10) holds for all $h \in H_{-1/2}(t)$. Consequently,

$$\begin{aligned}
\left\|\mathcal{B}^{(0)}_{\lambda,\mu}(t)g\right\|^2_{H_{1/2}(t)} &= (\tilde{A}_0^{-1/2}(t)\mathcal{B}_{\lambda,\mu}(t)g, \tilde{A}_0^{-1/2}(t)\mathcal{B}_{\lambda,\mu}(t)g) \\
&\leqslant k \left\|\tilde{A}_0^{1/2}(t)g\right\| \left\|\tilde{A}_0^{-1/2}(t)\mathcal{B}^{(0)}_{\lambda,\mu}(t)g\right\| = k\,\|g\|_{H_{-1/2}(t)} \left\|\mathcal{B}^{(0)}_{\lambda,\mu}(t)g\right\|_{H_{1/2}(t)}
\end{aligned}$$

for all $g \in H(t)$. Hence,

$$\left\| \mathcal{B}^{(0)}_{\lambda,\mu}(t) g \right\|_{H_{1/2}(t)} \leqslant k \, \|g\|_{H_{-1/2}(t)} \,, \tag{11}$$

where $k > 0$ is independent of $t \in \mathcal{I}_0$. All assertions of the lemma follow from (11). Lemma 1 is proved.

Let $\tilde{\mathcal{B}}_{\lambda,\mu}(t)$ denote the operator defined on $H_{-1/2}(t) \oplus G(t)$ which is equal to $\tilde{\mathcal{B}}^{(0)}_{\lambda,\mu}(t)$ on $H_{-1/2}(t)$ and to zero on $G(t)$. We set $\tilde{\mathcal{C}}_{\lambda,\mu}(t) = \tilde{\mathcal{B}}_{\lambda,\mu}(t) + i\tilde{\mathcal{A}}_\lambda(t) - i\tilde{\mathcal{A}}_\mu(t)$. It follows from (7), (11) that $\tilde{\mathcal{C}}_{\lambda,\mu}(t)$, $\tilde{\mathcal{B}}_{\lambda,\mu}(t)$, $\tilde{\mathcal{A}}_\lambda(t)$ are continuous mappings of $H_{-1/2}(t) \oplus G(t)$ into $H_{1/2}(t)$. By $\widetilde{\mathbf{C}}_{\lambda,\mu}$, $\widetilde{\mathbf{B}}_{\lambda,\mu}$, $\widetilde{\mathbf{A}}_\lambda$ denote the operators

$$y(t) \to \tilde{A}_0^{-1}(t)\tilde{\mathcal{C}}_{\lambda,\mu}(t)y(t), \; y(t) \to \tilde{A}_0^{-1}(t)\tilde{\mathcal{B}}_{\lambda,\mu}(t)y(t), \; y(t) \to \tilde{A}_0^{-1}(t)\tilde{\mathcal{A}}_\lambda(t)y(t),$$

respectively. The operators $\widetilde{\mathbf{C}}_{\lambda,\mu}$, $\widetilde{\mathbf{B}}_{\lambda,\mu}$, $\widetilde{\mathbf{A}}_\lambda$ are considered in the space $\mathfrak{H}$. Thus, with equalities (7), (11) being taken into account, we obtain the following corollaries from Lemma 1. (Corollary 1 is the generalization of Statement 2.)

Corollary 1. *For all fixed λ, $\mu \in \mathbb{C}_0$, $t \in \mathcal{I}_0$, the operator $\tilde{\mathcal{C}}_{\lambda,\mu}(t)$ maps $H_{-1/2}(t)$ into $H_{1/2}(t)$ continuously and the inequality $\left\| \tilde{\mathcal{C}}_{\lambda,\mu}(t) x \right\|_{H_{1/2}(t)} \leqslant k \, \|x\|_{H_{-1/2}(t)}$ holds, where $x \in H_{-1/2}(t)$, the constant k is independent of $t \in \mathcal{I}_0$.*

Corollary 2. *For all fixed $\lambda, \mu \in \mathbb{C}_0$, the operator $\widetilde{\mathbf{C}}_{\lambda,\mu}$ is bounded in $\mathfrak{H}$. For fixed λ or for fixed μ, the operator function $\widetilde{\mathbf{C}}_{\lambda,\mu}$ is holomorphic with respect to the other variable.*

Corollary 3. *For all fixed $\lambda, \mu \in \mathbb{C}_0$, the operators $\widetilde{\mathbf{A}}_\lambda$, $\widetilde{\mathbf{B}}_{\lambda,\mu}$ are bounded in $\mathfrak{H}$. The operator functions $\widetilde{\mathbf{A}}_\lambda$, $\widetilde{\mathbf{B}}_{\lambda,\mu}$ are continuous with respect to $\lambda, \mu \in \mathbb{C}_0$ in the uniform operator topology.*

2. Families of Maximal and Minimal Relations

Let $\mathbf{B}_1$, $\mathbf{B}_2$ be Banach spaces. The linear relation T is understood as any linear manifold $\mathrm{T} \subset \mathbf{B}_1 \times \mathbf{B}_2$. Terminology on the linear relations can be found, for example, in [8], [10], [11]. From now onwards, the following notation are used: $\{\cdot,\cdot\}$ is an ordered pair; $\ker \mathrm{T}$ is a set of the elements $x \in \mathbf{B}_1$ such that $\{x, 0\} \in \mathrm{T}$; KerT is a set of ordered pairs of the form $\{x, 0\} \in \mathrm{T}$; $\mathcal{D}(\mathrm{T})$ is the domain of T; $\mathcal{R}(\mathrm{T})$ is the range of T. The relation T is called invertible if T^{-1} is an operator and T is called continuously invertible if T^{-1} is a bounded everywhere defined operator. The linear operators are regarded as linear relations, and so the notations $\mathrm{S} \subset \mathbf{B}_1 \times \mathbf{B}_2$, $\{x_1, x_1\} \in \mathrm{S}$ are used for the operators S. Let $\mathbf{B}_1 = \mathbf{B}_2$

be a Hilbert space. A linear relation T is called accumulative (dissipative) if for all pairs $\{x_1, x_2\} \in \mathrm{T}$ the inequality $\mathrm{Im}(x_2, x_1) \leqslant 0$ ($\mathrm{Im}(x_2, x_1) \geqslant 0$) holds. The accumulative (dissipative) relation is called maximum if it does not have proper accumulative (dissipative) extensions. Since all the relations considered further are linear, the word "linear" will often be omitted.

A family of the linear manifolds in a Banach space $\mathbf{B}$ is understood as a function $\lambda \to \mathcal{T}(\lambda)$, where $\mathcal{T}(\lambda)$ is a linear manifold, $\mathcal{T}(\lambda) \subset \mathbf{B}$. A family of (closed) subspaces $\mathcal{T}(\lambda)$ is called holomorphic at the point $\lambda_1 \in \mathbb{C}$ if there exists a Banach space $\mathbf{B}_0$ and a family of the bounded linear operators $\mathcal{K}(\lambda) : \mathbf{B}_0 \to \mathbf{B}$ such that the operator $\mathcal{K}(\lambda)$ bijectively maps $\mathbf{B}_0$ onto $\mathcal{T}(\lambda)$ for any fixed λ and the family $\lambda \to \mathcal{K}(\lambda)$ is holomorphic in some neighborhood of λ_1. A family of subspaces is called holomorphic on the domain $\mathcal{D}$ if it is holomorphic at all points belonging to $\mathcal{D}$. Since the closed relation $\mathrm{T}(\lambda)$ is the subspace in $\mathbf{B}_1 \times \mathbf{B}_2$, the definition of holomorphic families is applied to the families of linear relations. This definition generalizes the corresponding definition of holomorphic families of closed operators [12, Ch. 7].

In [13], [14], the statement below is proved for the families of closed relations. However, the proof remains valid for the families of subspaces.

Statement 4. *Let $\mathcal{T}(\lambda) \subset \mathbf{B}$ be a family of subspaces in a Banach space $\mathbf{B}$. Suppose there exists a subspace $\mathfrak{N} \subset \mathbf{B}$ such that the decomposition in the direct sum $\mathbf{B} = \mathcal{T}(\lambda_1) \dot{+} \mathfrak{N}$ holds for some fixed point λ_1. The family $\mathcal{T}(\lambda)$ is holomorphic at λ_1 if and only if the space $\mathbf{B}$ is decomposed in the direct sum of subspaces $\mathcal{T}(\lambda)$ and $\mathfrak{N}$ for all λ belonging to some neighborhood of λ_1 and the family $\mathcal{P}(\lambda)$ is holomorphic at λ_1, where $\mathcal{P}(\lambda)$ is the projection of space $\mathbf{B}$ onto $\mathcal{T}(\lambda)$ in parallel $\mathfrak{N}$.*

The proof of the following statement is given in [15].

Statement 5. *Let $\mathcal{T}(\lambda)$ be a family of linear relations holomorphic at the point λ_1, and $\mathcal{S}(\lambda)$ be an operator function holomorphic at the point λ_1 whose values are the bounded everywhere defined operators. Then the family of relations $\mathcal{T}(\lambda) + \mathcal{S}(\lambda)$ is holomorphic at λ_1.*

Let Q_0 be the set of elements $c \in H^r$ such that the function $W_\mu(t)c$ ($\mu \in \mathbb{C}_0$) is identified with zero in $\mathfrak{H}$, i.e., the equality $\|A(t)W_\mu(t)c\| = 0$ holds almost everywhere. Using (3), we get

$$W_\mu(t)c = W_\lambda(t)c$$
$$+ W_\lambda(t)J_r^{-1}(t_0)\int_{t_0}^{t} W_{\bar\lambda}^*(s)(\mathcal{B}_\mu(s) - \mathcal{B}_\lambda(s) + i\mathcal{A}_\mu(s) - i\mathcal{A}_\lambda(s))W_\mu(s)c\,ds. \quad (12)$$

It follows from (12) and Statement 1 that the set Q_0 does not change if we substitute W_λ for W_μ ($\lambda, \mu \in \mathbb{C}_0$).

R e m a r k 1. If we change λ to μ and μ to λ in (12), then we obtain the true equality ($\lambda, \mu \in \mathbb{C}_0$).

Further, in this section and Section 3, we assume that the boundaries a, b are regular.

Let Q be the orthogonal complement of Q_0 in H^r, i.e., $Q = H^r \ominus Q_0$. In Q, we introduce the norm

$$\|c\|_- = \left(\int_a^b \left\| A^{1/2}(s) W_\mu(s) c \right\|^2 ds \right)^{1/2} \leqslant k \|c\|, \quad c \in Q, \ k > 0. \tag{13}$$

We denote the completion of Q with respect to this norm by Q_-. By (12), we get

$$\left\| A^{1/2}(t) W_\lambda(t) c \right\|$$
$$\leqslant \left\| A^{1/2}(t) W_\mu(t) c \right\| + k \left\| A^{1/2}(t) \right\| \int_a^b \|(\mathcal{C}_\lambda(s) - \mathcal{C}_\mu(s)) W_\mu(s) c\| \, ds.$$

We will obtain the analogous estimate for the norm $\left\| A^{1/2}(t) W_\mu(t) c \right\|$ if we change W_λ to W_μ and W_μ to W_λ. It follows from this and Statement 2 that the replacement of W_μ by W_λ in (13) leads to the same set Q_- with the equivalent norm. Thereby the space Q_- is independent of $\lambda \in \mathbb{C}_0$. The space Q_- can be treated as a space with negative norm with respect to Q [8, Ch. 2]. By Q_+ denote the corresponding space with positive norm.

The operator $c \to W_\lambda(t) c$ is a continuous one-to-one mapping of Q_- into $\mathfrak{H}$. We denote this operator by $\mathcal{W}_\lambda$. Its range is closed in $\mathfrak{H}$. Consequently, the adjoint operator $\mathcal{W}_\lambda^*$ continuously maps $\mathfrak{H}$ onto Q_+. We find the form $\mathcal{W}_\lambda^*$. For any elements $x \in Q$ and $f \in \mathfrak{H}$, we have

$$(f, \mathcal{W}_\lambda x)_A = \int_a^b (\tilde{A}(s) f(s), W_\lambda(s) x) ds = \int_a^b (W_\lambda^*(s) \tilde{A}(s) f(s), x) ds = (\mathcal{W}_\lambda^* f, x).$$

Hence, taking into account that Q can be densely embedded in Q_-, we obtain

$$\mathcal{W}_\lambda^* f = \int_a^b W_\lambda^*(s) \tilde{A}(s) f(s) ds.$$

Here we replace λ by $\bar{\lambda}$ and summarize the properties of operators $\mathcal{W}_\lambda$, $\mathcal{W}_{\bar{\lambda}}^*$ in the following statement.

Lemma 2. *For fixed $\lambda \in \mathbb{C}_0$, the operator $\mathcal{W}_\lambda$ is a continuous one-to-one mapping of Q_- into $\mathfrak{H}$, and the range $\mathcal{R}(\mathcal{W}_\lambda)$ is closed. The operator $\mathcal{W}^*_{\bar\lambda}$ maps continuously $\mathfrak{H}$ onto Q_+. The operator functions $\mathcal{W}_\lambda$, $\mathcal{W}^*_{\bar\lambda}$ are holomorphic on $\mathbb{C}_0$.*

In the same way as in [1], [3], we define the minimal and maximal relations generated by the expression l_λ ($\lambda \in \mathbb{C}_0$) and function $A(t)$. Let D'_λ be a set of functions $y \in \mathfrak{H}$ satisfying the following conditions: (a) the quasiderivatives $y^{[k]}$ exist, they are absolutely continuous up to the order $r-1$ inclusively, and $l_\lambda[y](t) \in H_{1/2}(t)$ almost everywhere; (b) the function $\tilde{A}_0^{-1}(t) l_\lambda[y] \in \mathfrak{H}$. To each class of the functions identified in $\mathfrak{H}$ with $y \in D'_\lambda$ we assign the class of functions identified in $\mathfrak{H}$ with $\tilde{A}_0^{-1}(t) l_\lambda[y]$. Thus, we get a linear relation $L'(\lambda) \subset \mathfrak{H} \times \mathfrak{H}$. Denote its closure by $L(\lambda)$ and call it a maximal relation. We define the minimal relation $L_0(\lambda)$ as a restriction of $L(\lambda)$ to the set of functions $y \in D'_\lambda$ satisfying the conditions $y^{[k]}(a) = y^{[k]}(b) = 0$ ($k = 0, 1, \ldots, r-1$).

The following statements can be proved by analogy with the corresponding assertions in [3], [16].

Lemma 3. *For fixed $\lambda \in \mathbb{C}_0$, the relation $L(\lambda)$ consists of all pairs $\{y, f\} \in \mathfrak{H} \times \mathfrak{H}$ for which*

$$y = W_\lambda(t) c_\lambda + F_\lambda, \tag{14}$$

where $c_\lambda \in Q_-$, $F_\lambda(t) = W_\lambda(t) J_r^{-1}(t_0) \int_a^t W^*_{\bar\lambda}(s) \tilde{A}(s) f(s) ds$. The pair $\{y, f\} \in L(\lambda)$ belongs to $L_0(\lambda)$ if and only if $c_\lambda = 0$ and

$$\int_a^b W^*_{\bar\lambda}(s) \tilde{A}(s) f(s) ds = 0. \tag{15}$$

Corollary 4. *The range $\mathcal{R}(L_0(\lambda))$ is closed and consists of the elements $f \in \mathfrak{H}$ with property* (15). *The relation $L_0^{-1}(\lambda)$ is a bounded operator on its domain of definition.*

Corollary 5. *The relation $L_0(\lambda)$ is closed.*

Corollary 6. *The operator $\mathcal{W}_\lambda$ is a continuous one-to-one mapping of Q_- onto* $\ker L(\lambda)$.

R e m a r k 2. In equality (14), the element $c_\lambda \in Q_-$ and the function $F_\lambda \in \mathfrak{H}$ are uniquely determined by the pair $\{y, f\} \in L(\lambda)$.

Theorem 1. *For all $\lambda, \mu \in \mathbb{C}_0$, the equality $L(\mu) = L(\lambda) + \widetilde{\mathbf{C}}_{\lambda,\mu}$ holds.*

P r o o f follows from the definition of $L(\lambda)$ and from Corollary 2.

Corollary 7. *The families of relations $L(\lambda)$, $L_0(\lambda)$ are holomorphic on $\mathbb{C}_0$.*

Proof follows from Theorem 1, Corollary 2, Statement 5.

Lemma 4. *The equality $(L_0(\lambda))^* = L(\bar{\lambda})$ holds for each $\lambda \in \mathbb{C}_0$.*

Proof. Let $\widetilde{L}(\lambda)$ be the maximal relation and $\widetilde{L}_0(\lambda)$ be the minimal relation generated by the formally self-adjoint expression $l[y] - \mathcal{B}_\lambda(t)y$. In [1], [16], it was shown that $(\widetilde{L}_0(\lambda))^* = \widetilde{L}(\lambda) = \widetilde{L}(\bar{\lambda})$. Using the equalities $L(\bar{\lambda}) = \widetilde{L}(\bar{\lambda}) - i\widetilde{\mathbf{A}}_{\bar{\lambda}}$, $L_0(\lambda) = \widetilde{L}_0(\lambda) - i\widetilde{\mathbf{A}}_\lambda$ and Corollary 3, we obtain the desired assertion. Lemma 4 is proved.

Remark 3. Taking into account Theorem 1 and the proof of Lemma 4, we get $\mathcal{D}(L(\mu)) = \mathcal{D}(L(\lambda)) = \mathcal{D}(\widetilde{L}(\lambda))$ and $\mathcal{D}(L_0(\mu)) = \mathcal{D}(L_0(\lambda)) = \mathcal{D}(\widetilde{L}_0(\lambda))$ for all $\lambda, \mu \in \mathbb{C}_0$.

Lemma 5. *For fixed λ $(\mathrm{Im}\lambda \neq 0)$ and for all pairs $\{y_0, g\} \in L_0(\lambda)$, the inequality $|\mathrm{Im}\lambda|^{-1}\,\mathrm{Im}(g, y_0)_A \leqslant -k\,\|y_0\|_A^2$, $k > 0$, holds.*

Proof. Suppose $\mathrm{Im}\lambda > 0$. Using (1) and the definition of $L_0(\lambda)$, we get

$$\mathrm{Im}(g, y_0)_A = -\int_a^b (\mathcal{A}_\lambda(t)y_0(t), y_0(t))dt. \tag{16}$$

Now the desired assertion follows from (16) and Statement 3. The case $\mathrm{Im}\lambda < 0$ is considered analogously. Lemma 5 is proved.

So, the relation $L_0(\lambda)$ is accumulative in the upper half-plane and $L_0(\lambda)$ is dissipative in the lower half-plane.

3. Holomorphic Families of Invertible Restrictions of Maximal Relations

Let $\mathbf{B}_1$, $\mathbf{B}_2$, $\tilde{B}_1$, $\tilde{B}_2$ be Banach spaces, $T \subset \mathbf{B}_1 \times \mathbf{B}_2$ be a closed linear relation, $\gamma\colon T \to \tilde{B}_1 \times \tilde{B}_2$ be a linear operator, $\gamma_i = P_i\gamma$ $(i = 1, 2)$, where P_i is the projection $\tilde{B}_1 \times \tilde{B}_2$ onto $\tilde{B}_i$, i.e., $P_i\{x_1, x_2\} = x_i$. The following definition is given in [17] for the operators and in [18] for the relations.

Definition 1. *The quadruple $(\tilde{B}_1, \tilde{B}_2, \gamma_1, \gamma_2)$ is called a space of boundary values (SBV) for a closed relation T if the operator γ is a continuous mapping of T onto $\tilde{B}_1 \times \tilde{B}_2$ and the restriction of the operator γ_1 to $\mathrm{Ker}T$ is a one-to-one mapping of $\mathrm{Ker}T$ onto $\tilde{B}_1$.*

We define an operator $\Phi_\gamma : \tilde{B}_1 \to \tilde{B}_2$ by the equality $\Phi_\gamma = \gamma_2\beta$, where $\beta = (\gamma_1|_{\mathrm{Ker}T})^{-1}$ is the operator inverse to the restriction of γ_1 to Ker$T$. We denote $T_0 = \ker\gamma$. It follows from the definition of SBV that between the relations $\theta \subset \tilde{B}_1 \times \tilde{B}_2$ and the relations $\tilde{T}$ with property $T_0 \subset \tilde{T} \subset T$ there is a one-to-one correspondence determined by the equality $\gamma\tilde{T} = \theta$. In this case we denote $\tilde{T} = T_\theta$. The relation θ is called a boundary relation. The following assertion is proved in [18].

Statement 6. *The relation T_θ is continuously invertible if and only if the relation $\theta - \Phi_\gamma$ is continuously invertible.*

Suppose $\{y, f\} \in L(\lambda)$ $(\lambda \in \mathbb{C}_0)$. Then (14) holds. To each pair $\{y, f\} \in L(\lambda)$ we assign a pair of boundary values $\{\mathcal{Y}_\lambda, \mathcal{Y}'_\lambda\} \in Q_- \times Q_+$ by the formulas

$$\mathcal{Y}'_\lambda = \mathcal{W}^*_{\bar{\lambda}} f = \int_a^b W^*_{\bar{\lambda}}(s)\tilde{A}(s)f(s)ds, \quad \mathcal{Y}_\lambda = c_\lambda + (1/2)J_r^{-1}(t_0)\mathcal{Y}'_\lambda. \tag{17}$$

It follows from Remark 2 that the pair $\{\mathcal{Y}_\lambda, \mathcal{Y}'_\lambda\}$ of boundary values is uniquely determined for each pair $\{y, f\} \in L(\lambda)$. Let $\gamma(\lambda)$, $\gamma_1(\lambda)$, $\gamma_2(\lambda)$ be the operators defined by the equalities $\gamma(\lambda)\{y, f\} = \{\mathcal{Y}_\lambda, \mathcal{Y}'_\lambda\}$, $\gamma_1(\lambda)\{y, f\} = \mathcal{Y}_\lambda$, $\gamma_2(\lambda)\{y, f\} = \mathcal{Y}'_\lambda$. Lemmas 2, 3 and Corollary 6 imply the following assertion.

Theorem 2. *The quadruple $\{Q_-, Q_+, \gamma_1(\lambda), \gamma_2(\lambda)\}$ is SBV for the relation $L(\lambda)$. The corresponding operator $\Phi_{\gamma(\lambda)}$ equals zero. Moreover,* $\ker\gamma(\lambda) = L_0(\lambda)$.

Theorem 3. *The range $\mathcal{R}(\gamma(\lambda))$ of the operator $\gamma(\lambda)$ coincides with $Q_- \times Q_+$, and for all pairs $\{y, f\} \in L(\lambda)$, $\{z, g\} \in L(\bar{\lambda})$ $(\lambda \in \mathbb{C}_0)$ "the Green formula" is valid*

$$(f, z)_A - (y, g)_A = (\mathcal{Y}'_\lambda, \mathcal{Z}_{\bar{\lambda}}) - (\mathcal{Y}_\lambda, \mathcal{Z}'_{\bar{\lambda}}), \tag{18}$$

where $\{\mathcal{Y}_\lambda, \mathcal{Y}'_\lambda\} = \gamma(\lambda)\{y, f\}$, $\{\mathcal{Z}_{\bar{\lambda}}, \mathcal{Z}'_{\bar{\lambda}}\} = \gamma(\bar{\lambda})\{z, g\}$.

P r o o f. Using Lemmas 2, 3, we get $\mathcal{R}(\gamma(\lambda)) = Q_- \times Q_+$. According to Lemma 3, $y = W_\lambda(t)c_\lambda + F_\lambda$, $z = W_{\bar{\lambda}}(t)d_{\bar{\lambda}} + G_{\bar{\lambda}}$, where $c_\lambda, d_{\bar{\lambda}} \in Q_-$,

$$F_\lambda(t) = W_\lambda(t)J_r^{-1}(t_0)\int_a^t W^*_{\bar{\lambda}}(s)\tilde{A}(s)f(s)ds,$$

$$G_{\bar{\lambda}}(t) = W_{\bar{\lambda}}(t)J_r^{-1}(t_0)\int_a^t W^*_{\lambda}(s)\tilde{A}(s)g(s)ds.$$

The operations l_λ, $l_{\bar\lambda}$ can be applied to the functions F_λ, $G_{\bar\lambda}$ and $l_\lambda[F_\lambda]=\tilde A(t)f$, $l_{\bar\lambda}[G_{\bar\lambda}]=\tilde A(t)g$. From this, (1), (2), and (17), we obtain

$$(f,G_{\bar\lambda})_A-(F_\lambda,g)_A=(J_r(b)\widehat{W}_\lambda(b)J_r^{-1}(t_0)\mathcal{Y}'_\lambda,\widehat{W}_{\bar\lambda}(b)J_r^{-1}(t_0)\mathcal{Z}'_{\bar\lambda})=(\mathcal{Y}'_\lambda,J_r^{-1}(t_0)\mathcal{Z}'_{\bar\lambda}). \tag{19}$$

We take two sequences $\{c_{\lambda,n}\}$, $\{d_{\bar\lambda,n}\}$ such that $c_{\lambda,n},d_{\lambda,n}\in Q$ and $\{c_{\lambda,n}\}$, $\{d_{\bar\lambda,n}\}$ converge to $c_\lambda\in Q_-$ and $d_{\bar\lambda}\in Q_-$ in Q_-, respectively. We denote $v_n(t)=W_\lambda(t)c_{\lambda,n}$, $u_n(t)=W_{\bar\lambda}(t)d_{\bar\lambda,n}$. Then the sequences $\{v_n(t)\}$, $\{u_n(t)\}$ converge to $v(t)=W_\lambda(t)c_\lambda$, $u(t)=W_{\bar\lambda}(t)d_{\bar\lambda}$ in $\mathfrak{H}$, respectively. Using (1), (2), (17), we obtain

$$(f,u_n)_A=(f,u_n)_A-(F_\lambda,0)_A=(J_r(t_0)\widehat{W}_\lambda(b)J_r^{-1}(t_0)\mathcal{Y}'_\lambda,\widehat{W}_{\bar\lambda}(b)d_{\bar\lambda,n})=(\mathcal{Y}'_\lambda,d_{\bar\lambda,n}),$$

$$(v_n,g)_A=(v_n,g)_A-(0,G_{\bar\lambda})_A=(J_r(t_0)\widehat{W}_\lambda(b)c_{\lambda,n},\widehat{W}_{\bar\lambda}(b)J_r^{-1}(t_0)\mathcal{Z}'_{\bar\lambda})=-(c_{\lambda,n},\mathcal{Z}'_{\bar\lambda}).$$

In these equalities, we pass to the limit as $n\to\infty$. We get

$$(f,u)_A=(\mathcal{Y}'_\lambda,d_{\bar\lambda}),\quad (v,g)_A=-(c_\lambda,\mathcal{Z}'_{\bar\lambda}). \tag{20}$$

Using (19), (20), and the equality $J_r^*(t_0)=-J_r(t_0)$, we obtain

$$(f,z)_A-(y,g)_A=(\mathcal{Y}'_\lambda,d_{\bar\lambda})-(c_\lambda,\mathcal{Z}'_{\bar\lambda})+(\mathcal{Y}'_\lambda,J_r^{-1}(t_0)\mathcal{Z}'_{\bar\lambda})=$$
$$=(\mathcal{Y}'_\lambda,d_{\bar\lambda}+(1/2)J_r^{-1}(t_0)\mathcal{Z}'_{\bar\lambda})-(c_\lambda+(1/2)J_r^{-1}(t_0)\mathcal{Y}'_{\bar\lambda},\mathcal{Z}'_{\bar\lambda})=(\mathcal{Y}'_\lambda,\mathcal{Z}_{\bar\lambda})-(\mathcal{Y}_\lambda,\mathcal{Z}'_{\bar\lambda}).$$

Theorem 3 is proved.

Notice that for the first time the linear relations were applied to describe self-adjoint extensions of differential operators in [19]. Further bibliography can, for example, be found in [8], [10]. The boundary values in form (17) are suggested in [16].

For fixed $\lambda\in\mathbb{C}_0$, between the relations $\widehat{L}(\lambda)$ with the property $L_0(\lambda)\subset\widehat{L}(\lambda)\subset L(\lambda)$ and relations $\theta\subset Q_-\times Q_+$ there is a one-to-one correspondence determined by the equality $\gamma(\lambda)\widehat{L}(\lambda)=\theta$. In this case we denote $\widehat{L}(\lambda)=L_\theta(\lambda)$. Thus, a pair $\{y,f\}\in L_\theta(\lambda)$ if and only if $\{y,f\}\in L(\lambda)$ and $\{\mathcal{Y}_\lambda,\mathcal{Y}'_\lambda\}\in\theta$, where $\{\mathcal{Y}_\lambda,\mathcal{Y}'_\lambda\}=\gamma(\lambda)\{y,f\}$. Using Lemma 4, we get $L^*(\lambda)=L_0(\bar\lambda)$, $(L_0(\lambda))^*=L(\bar\lambda)$. Suppose $L_0(\lambda)\subset L_\theta(\lambda)\subset L(\lambda)$. Then $L_0(\bar\lambda)\subset(L_\theta(\lambda))^*\subset L(\bar\lambda)$. Hence, taking (18) into account, we obtain

$$(L_\theta(\lambda))^*=L_{\theta^*}(\bar\lambda),\quad \lambda\in\mathbb{C}_0. \tag{21}$$

Lemma 6. *Let $\theta(\lambda)\subset Q_-\times Q_+$ be a family of linear relations defined on a set $\mathcal{D}_1\subset\mathbb{C}_0$ symmetric with respect to the real axis. Then the equality $\theta^*(\lambda)=\theta(\bar\lambda)$ holds if and only if the equality $(L_{\theta(\lambda)}(\lambda))^*=L_{\theta(\bar\lambda)}(\bar\lambda)$ holds.*

P r o o f follows from equality (21).

Theorem 4. *For fixed $\lambda \in \mathbb{C}_0$, the relation $L_{\theta(\lambda)}(\lambda)$ is continuously invertible if and only if so is the relation $\theta(\lambda)$. In this case, the operator $R_\lambda = L^{-1}_{\theta(\lambda)}(\lambda)$ is integral*

$$R_\lambda g = \int_a^b K_\lambda(t,s)\tilde{A}(s)g(s)ds, \quad g \in \mathfrak{H},$$

where $K_\lambda(t,s) = W_\lambda(t)(\theta^{-1}(\lambda) - (1/2)\mathrm{sgn}(s-t)J_r^{-1}(t_0))W^*_{\bar{\lambda}}(s)$.

Proof. The first part of the theorem follows from Statement 6 and Theorem 2. Let us prove the second part of the theorem. For any pair $\{y,g\} \in L(\lambda)$, we transform equality (14) to the form

$$y(t) = W_\lambda(t)\left(\tilde{c}_\lambda + (1/2)J_r^{-1}(t_0)\int_a^t W^*_{\bar{\lambda}}(s)\tilde{A}(s)g(s)ds \right.$$
$$\left. -(1/2)J_r^{-1}(t_0)\int_t^b W^*_{\bar{\lambda}}(s)\tilde{A}(s)g(s)ds\right), \quad (22)$$

where $\tilde{c}_\lambda = c_\lambda + (1/2)J_r^{-1}(t_0)\int_a^b W^*_{\bar{\lambda}}(s)\tilde{A}(s)g(s)ds = \mathcal{Y}_\lambda = \gamma_1(\lambda)\{y,g\}$.

Using (17), (22), we get

$$y(t) = W_\lambda(t)\left(\theta^{-1}(\lambda)\mathcal{Y}'_\lambda + (1/2)J_r^{-1}(t_0)\int_a^t W^*_{\bar{\lambda}}(s)\tilde{A}(s)g(s)ds \right.$$
$$\left. -(1/2)J_r^{-1}(t_0)\int_t^b W^*_{\bar{\lambda}}(s)\tilde{A}(s)g(s)ds\right).$$

This implies the desired assertion. Theorem 4 is proved.

Corollary 8. *The equalities $\theta^*(\lambda) = \theta(\bar{\lambda})$ and $R^*_\lambda = R_{\bar{\lambda}}$ hold together.*

Theorem 5. *The function R_λ is holomorphic at the point $\lambda_1 \in \mathbb{C}_0$ if and only if the function $\theta^{-1}(\lambda)$ is holomorphic at the same point.*

Proof. According to Lemma 2 and Theorem 4, if the function $\theta^{-1}(\lambda)$ is holomorphic at λ_1, then the function R_λ has the same property. Now suppose that the function R_λ is holomorphic at λ_1. Using Lemma 2 and Theorem 4, we obtain that the function $\mathcal{W}_\lambda\theta^{-1}(\lambda)\mathcal{W}^*_{\bar{\lambda}}g$ is holomorphic for every $g \in \mathfrak{H}$. It follows from Remark 1 and equality (12) with the element c replaced by $\theta^{-1}(\lambda)\mathcal{W}^*_{\bar{\lambda}}g$ that the functions $\mathcal{W}_\lambda\theta^{-1}(\lambda)\mathcal{W}^*_{\bar{\lambda}}g$ and $\mathcal{W}_\mu\theta^{-1}(\lambda)\mathcal{W}^*_{\bar{\lambda}}g$ are holomorphic together for fixed $\mu \in \mathbb{C}_0$. From Corollary 6 we conclude that the function $\theta^{-1}(\lambda)\mathcal{W}^*_{\bar{\lambda}}g$ is holomorphic at λ_1 for any $g \in \mathfrak{H}$. Now the property that the function $\theta^{-1}(\lambda)$ is holomorphic at λ_1 follows from Statement 7 proved in [20]. In this statement it should be taken that $\mathfrak{B}_1 = \mathfrak{H}$, $\mathfrak{B}_2 = Q_+$, $\mathfrak{B}_3 = Q_-$, $S_1(\lambda) = \mathcal{W}^*_{\bar{\lambda}}$, $S_2(\lambda) = \theta^{-1}(\lambda)$, $S_3(\lambda) = \theta^{-1}(\lambda)\mathcal{W}^*_{\bar{\lambda}}$. Theorem 5 is proved.

Statement 7. *Let $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_3$ be Banach spaces. Suppose the bounded operators $S_3(\lambda) : \mathfrak{B}_1 \to \mathfrak{B}_3$, $S_1(\lambda) : \mathfrak{B}_1 \to \mathfrak{B}_2$, $S_2(\lambda) : \mathfrak{B}_2 \to \mathfrak{B}_3$ satisfy the equality $S_3(\lambda) = S_2(\lambda)S_1(\lambda)$ for every fixed λ belonging to some neighborhood of a point λ_1; moreover, $\mathcal{R}(S_1(\lambda_1)) = \mathfrak{B}_2$. If the functions $S_1(\lambda)$, $S_3(\lambda)$ are strongly differentiable at the point λ_1, then at the same point the function $S_2(\lambda)$ is strongly differentiable.*

Let $\mathfrak{N}_\mu = \mathfrak{H} \ominus \mathcal{R}(L_0(\mu))$, where $\mathrm{Im}\mu > 0$. Using Lemmas 3, 4 and Corollary 6, we get $\mathfrak{N}_\mu = \ker L(\bar{\mu})$. It follows from Theorem 1 that $\mathfrak{N}_\mu \subset \mathcal{D}(L(\lambda))$ and the pair

$$\{w(t), \tilde{A}_0^{-1}(t)\tilde{\mathcal{C}}_{\bar{\mu},\lambda}(t)w(t)\} \in L(\lambda), \tag{23}$$

where $w(t) \in \mathfrak{N}_\mu$. The set $\mathcal{D}(L_0(\lambda)) = \mathcal{D}(L_0(\bar{\mu}))$ and $\mathfrak{N}_\mu$ are linearly independent in $\mathfrak{H}$. For $\mathrm{Im}\lambda > 0$, we define the relation $\mathcal{L}(\lambda)$ as the restriction of $L(\lambda)$ to $\mathcal{D}(L_0(\lambda)) \dotplus \mathfrak{N}_\mu$. By (23) and the equality $\mathcal{A}_{\bar{\mu}} = -\mathcal{A}_\mu$, it follows that the relation $\mathcal{L}(\lambda)$ consists of all pairs of the form

$$\{y_{0,\lambda}(t) + w(t),\ y_{1,\lambda}(t) + \tilde{A}_0^{-1}(t)(\tilde{\mathcal{B}}_{\mu,\lambda}(t) - i(\tilde{\mathcal{A}}_\lambda(t) + \tilde{\mathcal{A}}_\mu(t)))w(t)\}, \tag{24}$$

where $\{y_{0,\lambda}, y_{1,\lambda}\} \in L_0(\lambda)$, $w(t) \in \mathfrak{N}_\mu$. For $\mathrm{Im}\lambda < 0$, we set $\mathcal{L}(\lambda) = \mathcal{L}^*(\bar{\lambda})$.

Theorem 6. *The family of relations $\mathcal{L}(\lambda)$ is holomorphic for* $\mathrm{Im}\lambda \neq 0$.

P r o o f. First, we consider the case $\mathrm{Im}\lambda > 0$. By Corollary 7, the family of relations $L_0(\lambda)$ is holomorphic at every point λ_1 ($\mathrm{Im}\lambda_1 > 0$). It follows from the definition of holomorphic family that there exists a Banach space $\mathfrak{B}$ and a family of bounded linear operators $\mathcal{K}_0(\lambda) : \mathfrak{B} \to \mathfrak{H}$ such that the operator $\mathcal{K}_0(\lambda)$ bijectively maps $\mathfrak{B}$ onto $L_0(\lambda)$ for any fixed λ and the family $\lambda \to \mathcal{K}_0(\lambda)$ is holomorphic in some neighborhood of λ_1. By $\mathcal{K}_1(\lambda)$ we denote an operator taking every element $w \in \mathfrak{N}_\mu$ to the pair $\{w, \widetilde{\mathbf{C}}_{\bar{\mu},\lambda}w\} \in \mathfrak{H} \times \mathfrak{H}$. By Corollary 2, the operator $\mathcal{K}_1(\lambda)$ is bounded for fixed $\lambda \in \mathbb{C}_0$ and the family of operators $\mathcal{K}_1(\lambda)$ is holomorphic on $\mathbb{C}_0$. Consider the space $\widetilde{\mathfrak{B}} = \mathfrak{B} \times \mathfrak{N}_\mu$ and define the operator $\widetilde{\mathcal{K}}(\lambda) : \widetilde{\mathfrak{B}} \to \mathfrak{H} \times \mathfrak{H}$ by the formula $\widetilde{\mathcal{K}}(\lambda)\{u, w\} = \mathcal{K}_0(\lambda)u + \mathcal{K}_1(\lambda)w$, where $u \in \mathfrak{B}$, $w \in \mathfrak{N}_\mu$. For fixed λ, the operator $\widetilde{\mathcal{K}}(\lambda)$ is a continuous one-to-one mapping of $\widetilde{\mathfrak{B}}$ onto $\mathcal{L}(\lambda)$ and the family $\lambda \to \widetilde{\mathcal{K}}(\lambda)$ is holomorphic in some neighborhood of λ_1. So, the theorem is valid for $\mathrm{Im}\lambda > 0$.

Suppose Im$\lambda < 0$. Let the operator $\mathcal{J} : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H} \times \mathfrak{H}$ be given by the formula $\mathcal{J}\{x, x_1\} = \{-x_1, x\}$. Then $\mathcal{L}^*(\bar{\lambda}) = \mathcal{J}\mathcal{L}^{\perp}(\bar{\lambda})$ (Here and further, we denote $\mathcal{N}^{\perp} = \mathcal{H} \ominus \mathcal{N}$, where $\mathcal{N}$ is a subspace of a Hilbert space $\mathcal{H}$.) The family of subspaces $\mathcal{J}\mathcal{L}^{\perp}(\bar{\lambda})$ is holomorphic if and only if the family $\mathcal{L}^{\perp}(\bar{\lambda})$ is holomorphic. Now the desired assertion is obtained from Lemma 7 below. Theorem 6 is proved.

Corollary 9. *The family of relations $\mathcal{L}^{-1}(\lambda)$ is holomorphic for* Im$\lambda \neq 0$.

Lemma 7. *Let $\mathcal{H}$ be a Hilbert space. If the family of subspaces $\mathcal{T}(\lambda) \subset \mathcal{H}$ is holomorphic at the point λ_1, then the family $\mathcal{T}^{\perp}(\bar{\lambda})$ is holomorphic at $\bar{\lambda}_1$.*

Proof. By Statement 4, $\mathcal{H} = \mathcal{T}(\lambda) \dotplus \mathfrak{N}$ and the family $\mathcal{P}(\lambda)$ is holomorphic at λ_1, where $\mathcal{P}(\lambda)$ is the projection of space $\mathcal{H}$ onto $\mathcal{T}(\lambda)$ in parallel $\mathfrak{N}$. Then $\mathcal{Q}(\lambda) = E - \mathcal{P}(\lambda)$ is the projection $\mathcal{H}$ onto $\mathfrak{N}$ in parallel $\mathcal{T}(\lambda)$. Therefore, $\ker \mathcal{Q}^*(\lambda) = \mathfrak{N}^{\perp}$, $\mathcal{R}(\mathcal{Q}^*(\lambda)) = \mathcal{T}^{\perp}(\lambda)$, and $(\mathcal{Q}^*(\lambda))^2 = \mathcal{Q}^*(\lambda)$. Consequently, $\mathcal{Q}^*(\lambda)$ is the projection of $\mathcal{H}$ onto $\mathcal{T}^{\perp}(\lambda)$ in parallel $\mathfrak{N}^{\perp}$. The corresponding decomposition has the form $\mathcal{H} = \mathcal{T}^{\perp}(\lambda) \dotplus \mathfrak{N}^{\perp}$. Since the family $\mathcal{Q}^*(\bar{\lambda})$ is holomorphic, the family of subspaces $\mathcal{T}^{\perp}(\bar{\lambda})$ is holomorphic at the point $\bar{\lambda}_1$. Lemma 7 is proved.

Theorem 7. *For fixed λ, the relation $\mathcal{L}(\lambda)$ is maximal accumulative if* Im$\lambda > 0$ *and it is maximal dissipative if* Im$\lambda < 0$. *The relation $\mathcal{L}^{-1}(\lambda)$ is a bounded everywhere defined operator.*

Proof. Let Im$\lambda > 0$, $\{z, z_1\} \in \mathcal{L}(\lambda)$. Then $\{z, z_1\}$ has form (24). Using (1), (24) and the equalities $l_{\bar{\mu}}[w] = 0$, $l[y_{0,\lambda}] = \tilde{A}_0 y_{1,\lambda} + i\mathcal{A}_\lambda y_{0,\lambda} + \mathcal{B}_\lambda y_{0,\lambda}$, we get

$$
\begin{aligned}
(z_1, z)_A &= (y_{1,\lambda}, y_{0,\lambda})_A + (y_{1,\lambda}, w)_A + (\widetilde{\mathbf{B}}_{\mu,\lambda} w, y_{0,\lambda})_A + (\widetilde{\mathbf{B}}_{\mu,\lambda} w, w)_A \\
&\qquad - i((\widetilde{\mathbf{A}}_\lambda + \widetilde{\mathbf{A}}_\mu) w, y_{0,\lambda})_A - i((\widetilde{\mathbf{A}}_\lambda + \widetilde{\mathbf{A}}_\mu) w, w)_A \\
&= (y_{1,\lambda}, y_{0,\lambda})_A + (y_{1,\lambda}, w)_A + (w, y_{1,\lambda})_A + (\widetilde{\mathbf{B}}_{\mu,\lambda} w, w)_A \\
&\qquad - 2i(\widetilde{\mathbf{A}}_\lambda w, y_{0,\lambda})_A - i((\widetilde{\mathbf{A}}_\lambda + \widetilde{\mathbf{A}}_\mu) w, w)_A.
\end{aligned}
$$

Hence, taking (16) into account, we obtain

$$
\begin{aligned}
\mathrm{Im}(z_1, z)_A &= \mathrm{Im}(y_{1,\lambda}, y_{0,\lambda})_A - 2\mathrm{Re}(\widetilde{\mathbf{A}}_\lambda w, y_{0,\lambda})_A - ((\widetilde{\mathbf{A}}_\lambda + \widetilde{\mathbf{A}}_\mu) w, w)_A \\
&= -(\widetilde{\mathbf{A}}_\lambda (y_{0,\lambda} + w), y_{0,\lambda} + w)_A - (\widetilde{\mathbf{A}}_\mu w, w)_A = -(\widetilde{\mathbf{A}}_\lambda z, z)_A - (\widetilde{\mathbf{A}}_\mu w, w)_A \\
&= -\int_a^b (\tilde{\mathcal{A}}_\lambda(t) z(t), z(t)) dt - \int_a^b (\tilde{\mathcal{A}}_\mu(t) w(t), w(t)) dt.
\end{aligned}
$$

Therefore,

$$
\mathrm{Im}(z_1, z)_A \leqslant -\int_a^b (\tilde{\mathcal{A}}_\lambda(t) z(t), z(t)) dt \leqslant -k(z, z)_A \quad (k > 0). \tag{25}
$$

Thus, the relation $\mathcal{L}(\lambda)$ is accumulative. It follows from (25) that the range $\mathcal{R}(\mathcal{L}(\lambda))$ is closed and the relation $\mathcal{L}^{-1}(\lambda)$ is the operator. Moreover, $\mathcal{R}(\mathcal{L}(\mu)) = \mathfrak{H}$. The equality $\mathcal{R}(\mathcal{L}(\lambda)) = \mathfrak{H}$ is obtained from Lemma 8 and Corollary 10 below. Consequently, the relation $\mathcal{L}(\lambda)$ is maximal accumulative if $\mathrm{Im}\lambda > 0$. For $\mathrm{Im}\lambda < 0$, the desired assertion is obtained from the equality $\mathcal{L}(\lambda)=\mathcal{L}^*(\bar{\lambda})$. Theorem 7 is proved.

Lemma 8. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{S}(\lambda) \subset \mathcal{H} \times \mathcal{H}$ be a family of linear relations holomorphic at the point μ. Suppose that the relation $\mathcal{S}(\mu)$ has the following properties: $\mathcal{R}(\mathcal{S}(\mu))$ is a closed subspaces in $\mathcal{H}$ and the relation $\mathcal{S}^{-1}(\mu)$ is an operator. Then there exists a neighborhood of μ such that the relations $\mathcal{S}(\lambda)$ have the same properties for all points λ belonging to this neighborhood. Moreover,* $\dim \mathcal{H} \ominus \mathcal{R}(\mathcal{S}(\lambda)) = \dim \mathcal{H} \ominus \mathcal{R}(\mathcal{S}(\mu))$.

P r o o f. We denote $\mathcal{H}_0 = \mathcal{H} \ominus \mathcal{R}(\mathcal{S}(\mu))$, $\mathfrak{N} = \mathcal{H} \times \mathcal{H}_0$. We claim that the decomposition in the direct sum $\mathcal{H} \times \mathcal{H} = \mathcal{S}(\mu) \dotplus \mathfrak{N}$ holds. Indeed, any pair $\{x_1, x_2\} \in \mathcal{H} \times \mathcal{H}$ is uniquely produced by the form

$$\{x_1, x_2\} = \{\mathcal{S}^{-1}(\mu)Px_2, Px_2\} + \{x_1 - \mathcal{S}^{-1}(\mu)Px_2, P^{\perp}x_2\},$$

where P is the orthogonal projection $\mathcal{H}$ onto $\mathcal{R}(\mathcal{S}(\mu))$, $P^{\perp} = E - P$.

According to Statement 4, the decomposition in the direct sum

$$\mathcal{H} \times \mathcal{H} = \mathcal{S}(\lambda) \dotplus \mathfrak{N} \tag{26}$$

holds for all λ belonging to some neighborhood of μ. Using (26), we obtain $\{x_1, x_2\}=\{f_1, f_2\}+\{x_1-f_1, x_2-f_2\}$ for all $\{x_1, x_2\}\in\mathcal{H}\times\mathcal{H}$, where $\{f_1, f_2\}\in\mathcal{S}(\lambda)$, $\{x_1-f_1, x_2-f_2\}\in\mathfrak{N}$. If $f_2=0$, then $x_2\in\mathcal{H}_0$. Hence, $\{x_1, x_2\}\in\mathfrak{N}$. The uniqueness of decomposition (26) implies $f_1 = 0$. So, $\mathcal{S}^{-1}(\lambda)$ is the operator.

Let us prove that $\mathcal{R}(\mathcal{S}(\lambda))$ is closed. For this purpose, we consider the operator $U(\lambda) : \mathcal{R}(\mathcal{S}(\mu)) \to \mathcal{H}$ defined by the equality $U(\lambda)x = P_2\mathcal{P}(\lambda)\{\mathcal{S}^{-1}(\mu)x, x\}$, where $x \in \mathcal{R}(\mathcal{S}(\mu))$, $\mathcal{P}(\lambda)$ is the projection of space $\mathcal{H} \times \mathcal{H}$ onto $\mathcal{S}(\lambda)$ in parallel $\mathfrak{N}$ corresponding to (26), P_2 is the projection $\mathcal{H} \times \mathcal{H}$ onto the second factor, i.e., $P_2\{x_1, x_2\} = x_2$. For fixed λ, the operator $U(\lambda)$ is a continuous one-to-one mapping of $\mathcal{R}(\mathcal{S}(\mu))$ onto $\mathcal{R}(\mathcal{S}(\lambda))$. The operator function $U(\lambda)$ is holomorphic in some neighborhood of μ and $U(\mu)x = x$. Therefore the set $\mathcal{R}(\mathcal{S}(\lambda))$ is closed for all λ from some neighborhood of μ. Lemma 8 is proved.

Corollary 10. *Suppose the family of relations $\mathcal{S}(\lambda)$ is holomorphic in a connected domain $\mathcal{D}$, and $\mathcal{S}(\mu)$ satisfies the assumptions of Lemma 8 for all points $\mu \in \mathcal{D}$. Then* $\dim \mathcal{H} \ominus \mathcal{R}(\mathcal{S}(\lambda))$ *is constant in $\mathcal{D}$.*

P r o o f follows from the fact that this dimension is constant in a neighborhood of every point belonging to $\mathcal{D}$.

4. On the Characteristic Operator

In the space $\mathfrak{H} = L_2(H, A(t); a, b)$, the norm and the scalar product are defined by equalities (5), (6). According to Statements 1, 3, after replacing $A(t) = \mathfrak{a}_{\lambda_0}(t)$ by $\mathfrak{a}_\lambda(t)$, we obtain a space with the equivalent norm. By $\|\cdot\|_{\mathfrak{a}_\lambda}$ ($(\cdot,\cdot)_{\mathfrak{a}_\lambda}$) we denote the norm (the scalar product, respectively) in the space $L_2(H, \mathfrak{a}_\lambda(t); a, b)$.

We consider the equation

$$l[y] - \mathcal{B}_\lambda(t)y - i\mathcal{A}_\lambda(t)y = \mathfrak{a}_\lambda(t)f(t) \quad (f \in \mathfrak{H}) \tag{27}$$

on the finite or infinite interval (a, b) and use the following definition (see [4], [5]).

Definition 2. *Let $M(\lambda) = M^*(\bar{\lambda})$ be a function holomorphic for* $\mathrm{Im}\lambda \neq 0$ *whose values are bounded linear operators such that $\mathcal{D}(M(\lambda)) = Q_+$ and $\mathcal{R}(M(\lambda)) \subset Q_-$. This function $M(\lambda)$ is called a characteristic operator of equation* (27) *if for any function $f \in \mathfrak{H}$ with compact support the corresponding solution $y_\lambda(t)$ of* (27)

$$y_\lambda(t) = (\mathbf{R}_\lambda f)(t)$$

$$= \int_a^b W_\lambda(t)(M(\lambda) - (1/2)\mathrm{sgn}(s-t)J_r^{-1}(t_0))W_{\bar{\lambda}}^*(s)\mathfrak{a}_\lambda(s)f(s)ds \tag{28}$$

satisfies the inequality

$$\|\mathbf{R}_\lambda f\|_{\mathfrak{a}_\lambda}^2 \leqslant \mathrm{Im}(\mathbf{R}_\lambda f, f)_{\mathfrak{a}_\lambda}/\mathrm{Im}\lambda, \quad \mathrm{Im}\lambda \neq 0. \tag{29}$$

(For the case of the singular boundary a or b, the spaces Q_-, Q_+ are defined bellow.)

In [4], [5], the existence of the characteristic operator is proved. In the case of the axis, the characteristic operators appear that are unbounded in Q [4, pp. 167, 172]. In [5, p. 209], there is proposed another method allowing to connect a holomorphic family of dissipative operators with the expression l_λ. (In [4], [5], the case $r = 1$ is considered.)

Suppose that the boundaries a, b are regular. Let $L_0(\lambda) \subset \mathfrak{L}(\lambda) \subset L(\lambda)$ for $\mathrm{Im}\lambda \neq 0$. We say that a family of linear relations $\mathfrak{L}(\lambda)$ generates a characteristic operator $M(\lambda)$ of equation (27) if $\gamma(\lambda)\mathfrak{L}(\lambda) = M^{-1}(\lambda)$.

Theorem 8. *The family of closed relations $\mathfrak{L}(\lambda)$* ($\mathrm{Im}\lambda \neq 0$) *generates the characteristic operator of equation* (27) *if and only if $\mathfrak{L}(\lambda)$ satisfies the following conditions:* 1) $\mathfrak{L}(\bar{\lambda}) = \mathfrak{L}^*(\lambda)$; 2) *the family $\mathfrak{L}(\lambda)$ is holomorphic;* 3) *for fixed λ,* *the relation* $\mathfrak{L}^{-1}(\lambda)$ *is an everywhere defined operator* ($\mathfrak{L}^{-1}(\lambda)$ *is bounded since* $\mathfrak{L}^{-1}(\lambda)$ *is closed*); 4) *the inequality*

$$\mathrm{Im}(g_\lambda, z_\lambda)_A \leqslant -(\tilde{A}_0^{-1}\tilde{\mathcal{A}}_\lambda z_\lambda, z_\lambda)_A = -\int_a^b (\tilde{\mathcal{A}}_\lambda(t) z_\lambda(t), z_\lambda(t))dt \tag{30}$$

holds for all pairs $\{z_\lambda, g_\lambda\} \in \mathfrak{L}(\lambda)$ *and all* λ *such that* $\mathrm{Im}\lambda > 0$.

P r o o f. Assume that the family $\mathfrak{L}(\lambda)$ possesses the properties 1)–4). It follows from Theorems 4, 5 and Lemma 6 that there exists a holomorphic function $M(\lambda) = M^*(\bar{\lambda})$ ($\mathrm{Im}\lambda \neq 0$) whose values are the bounded operators such that $\mathcal{D}(M(\lambda)) = Q_+$, $\mathcal{R}(M(\lambda)) \subset Q_-$, and the equality

$$z_\lambda(t) = (\mathfrak{L}^{-1}(\lambda)g)(t)$$
$$= \int_a^b W_\lambda(t)(M(\lambda) - (1/2)\mathrm{sgn}(s-t)J_r^{-1}(t_0))W_{\bar{\lambda}}^*(s)\tilde{A}(s)g(s)ds \tag{31}$$

holds for all $g \in \mathfrak{H}$. The function $M(\lambda)$ is defined by the equality

$$\gamma(\lambda)\mathfrak{L}(\lambda) = M^{-1}(\lambda). \tag{32}$$

We set $f(t) = (\mathrm{Im}\lambda)\tilde{\mathcal{A}}_{0,\lambda}^{-1}(t)\tilde{A}(t)g(t)$. It follows from the arguments to the definition of the spaces $H_{1/2}(t)$, $H_{-1/2}(t)$ and Corollary 3 that the functions f and g belong or do not belong to the space $\mathfrak{H}$ together. Moreover, the equalities

$$g(t) = \tilde{A}_0^{-1}(t)\mathfrak{a}_\lambda(t)f(t), \quad \mathfrak{a}_\lambda(t)f(t) = \tilde{A}(t)g(t) \tag{33}$$

hold. We fix the function $f \in \mathfrak{H}$ and denote $g_\lambda(t) = \tilde{A}_0^{-1}(t)\mathfrak{a}_\lambda(t)f(t)$, $y_\lambda = \mathfrak{L}^{-1}(\lambda)g_\lambda = \mathbf{R}_\lambda f$. By (31), (33), it follows that y_λ is the solution of equation (27) and y_λ has the form (28).

Now we claim that inequality (29) holds. Indeed, (30) implies

$$\mathrm{Im}(f, \mathbf{R}_\lambda f)_{\mathfrak{a}_\lambda} = \mathrm{Im}(g_\lambda, y_\lambda)_A \leqslant -\mathrm{Im}\lambda\, \|y_\lambda\|^2_{\mathfrak{a}_\lambda} = -\mathrm{Im}\lambda\, \|\mathbf{R}_\lambda f\|^2_{\mathfrak{a}_\lambda}$$

for $\mathrm{Im}\lambda > 0$. If $\mathrm{Im}\lambda < 0$, then (29) follows from the equality $\mathfrak{L}(\bar{\lambda}) = \mathfrak{L}^*(\lambda)$.

Thus, if the family $\mathfrak{L}(\lambda)$ satisfies the conditions 1)–4), then the function $M(\lambda)$ defined by (32) is the characteristic operator. The proof is convertible. From this there follows the converse. Theorem 8 is proved.

Using Theorems 6, 7, 8, formula (25) and the equality $\mathcal{L}(\bar{\lambda})=\mathcal{L}^*(\lambda)$, we obtain

Corollary 11. *The family* $M(\lambda) = (\gamma(\lambda)\mathcal{L}(\lambda))^{-1}$ *is a characteristic operator.*

R e m a r k 4. In (14), suppose that $c_\lambda \in Q$. By Lemma 2, y is the function ranging in H. Using (14), (4), we get $\widehat{y}(a) = \widehat{W}_\lambda(a)c_\lambda$, $\widehat{y}(b) = \widehat{W}_\lambda(b)c_\lambda + \widehat{F}_\lambda(b)$. From this and (17), by straightforward calculations, we obtain

$$\mathcal{Y}'_\lambda = J_r(t_0)(\widehat{W}_\lambda^{-1}(b)\widehat{y}(b) - \widehat{W}_\lambda^{-1}(a)\widehat{y}(a)),\ \mathcal{Y}_\lambda = 2^{-1}(\widehat{W}_\lambda^{-1}(b)\widehat{y}(b) + \widehat{W}_\lambda^{-1}(a)\widehat{y}(a)).$$

These equalities correspond to Remark 1.1 in [4]. For $\mathcal{L}(\lambda)$, the values $\widehat{y}(a)$, $\widehat{y}(b)$ are calculated by formula (12), where μ is changed to $\bar{\mu}$.

R e m a r k 5. Let $\widetilde{L}(\mu_0)$ be the maximal relation and $\widetilde{L}_0(\mu_0)$ be the minimal relation generated by the formally self-adjoint expression $l[y]-\mathcal{B}_{\mu_0}(t)y$ in $\mathfrak{H}$, where $\mu_0 \in \mathbb{C}_0$ is fixed. It follows from Theorem 8 that the family $\mathfrak{L}(\lambda)$ generates a characteristic operator if and only if the family $\Lambda(\lambda) = \mathfrak{L}(\lambda) + \widetilde{\mathbf{B}}_{\lambda,\mu_0} + i\widetilde{\mathbf{A}}_\lambda$ has the properties: 1) $\widetilde{L}_0(\mu_0) \subset \Lambda(\lambda) \subset \widetilde{L}(\mu_0)$; 2) $\Lambda^*(\lambda) = \Lambda(\bar{\lambda})$; 3) the family $\Lambda(\lambda)$ is holomorphic for $\mathrm{Im}\lambda \neq 0$; 4) $\Lambda(\lambda)$ is a maximal accumulative relation for $\mathrm{Im}\lambda > 0$. Thus, $(\Lambda(\lambda) - \lambda E)^{-1}$ is a generalized resolvent of $\widetilde{L}_0(\mu_0)$. This statement is established by the other method in [4].

Below we assume that $a = -\infty$, $b = \infty$. So, $\mathfrak{H} = L_2(H, A(t); -\infty, \infty)$. We define maximal and minimal relations in the following way. Let $D'_{0,\lambda}$ ($\lambda \in \mathbb{C}_0$) be a set of finite functions $y \in \mathfrak{H}$ satisfying the conditions: (a) the quasiderivatives $y^{[k]}$ exist, they are absolutely continuous up to the order $r-1$ inclusively; (b) $l_\lambda[y](t) \in H_{1/2}(t)$ almost everywhere; (c) $\tilde{A}_0^{-1}(t)l_\lambda[y] \in \mathfrak{H}$. We introduce a correspondence between each class of functions identified with $y \in D'_{0,\lambda}$ in $\mathfrak{H}$ and the class of functions identified with $\tilde{A}_0^{-1}(t)l_\lambda[y]$ in $\mathfrak{H}$. Thus, in the space $\mathfrak{H}$, we obtain a linear relation $L'_0(\lambda)$, denote its closure by $L_0(\lambda)$, and call $L_0(\lambda)$ the minimal relation. (In the regular case, one can define the minimal relation in the same way. One can show that this definition and the definition of minimal relation from Section 2 are equivalent. We do not use this fact.) We denote $L(\lambda) = (L_0(\bar{\lambda}))^*$. The relation $L(\lambda)$ is called maximal. (According to Lemma 4, in the regular case the maximal relation can be defined in the same way.)

Using the equality $L'_0(\mu) = L'_0(\lambda) + \widetilde{\mathbf{C}}_{\lambda,\mu}$ ($\lambda, \mu \in \mathbb{C}_0$) and Corollary 2, we get $L_0(\mu) = L_0(\lambda) + \widetilde{\mathbf{C}}_{\lambda,\mu}$. This implies that the family $L_0(\lambda)$ is holomorphic on $\mathbb{C}_0$. By Lemma 7, it follows that the family $L(\lambda)$ is holomorphic on $\mathbb{C}_0$. Using Corollary 2, we get $L(\mu) = L(\lambda) + \widetilde{\mathbf{C}}_{\lambda,\mu}$. Thus, Theorem 1 and Corollary 7 are valid for the relations $L(\lambda)$ and $L_0(\lambda)$. For the relation $L'_0(\lambda)$ ($\mathrm{Im}\lambda \neq 0$), the proof of Lemma 5 is the same as the above proof for the regular case. By the limit passage, we obtain that Lemma 5 is valid for relation $L_0(\lambda)$. Consequently, the range $\mathcal{R}(L_0(\lambda))$ is closed in $\mathfrak{H}$ and the relation $L_0^{-1}(\lambda)$ is a bounded everywhere defined operator on $\mathcal{R}(L_0(\lambda))$ ($\mathrm{Im}\lambda \neq 0$).

In Corollary 10, we set $\mathcal{S}(\lambda) = L_0(\lambda)$. Then we obtain that the dimension $\dim \mathfrak{H} \ominus \mathcal{R}(L_0(\lambda)) = \dim \ker L(\bar{\lambda})$ is constant in the half-planes $\mathrm{Im}\lambda > 0$, $\mathrm{Im}\lambda < 0$. In [4, 6, 7, 21–23] this assertion was proved by various methods for various cases.

We fix some μ ($\mathrm{Im}\mu > 0$) and denote $\mathfrak{N}_\mu = \mathfrak{H} \ominus \mathcal{R}(L_0(\mu))$ (it is not inconceivable that $\mathfrak{N}_\mu = \{0\}$). Obviously, $\mathfrak{N}_\mu = \ker(L_0(\mu))^* = \ker L(\bar{\mu})$. The definition of the relation $\mathcal{L}(\lambda)$ is the same as the above for the regular case, i.e, $\mathcal{L}(\lambda)$ is the restriction $L(\lambda)$ to $\mathcal{D}(L_0(\lambda)) \dot{+} \mathfrak{N}_\mu$. The relation $\mathcal{L}(\lambda)$ consists of the ordered pairs of the form (24). We set $\mathcal{L}(\lambda) = \mathcal{L}^*(\bar{\lambda})$ for $\mathrm{Im}\lambda < 0$. Theorems 6, 7 are obtained by the literal repetition of the proof used for the regular case.

We construct the domain and the range of the characteristic operator. By Q_0 denote a set of elements $c \in H^r$ such that the norm $\|A(t)W_\mu(t)c\| = 0$ almost everywhere on $(-\infty, \infty)$, $Q = H^r \ominus Q_0$. The sets Q_0, Q do not depend on $\mu \in \mathbb{C}_0$. (This assertion can be proved by analogy with the corresponding assertion for the regular case.)

Let $[\alpha_n, \beta_n]$ be a sequence of intervals such that $[\alpha_n, \beta_n] \subset (\alpha_{n+1}, \beta_{n+1})$, and $\alpha_n \to -\infty$, $\beta_n \to \infty$ as $n \to \infty$ ($n \in \mathbb{N}$). In Q, we introduce the system of seminorms

$$p_n(x) = \left(\int_{\alpha_n}^{\beta_n} \left\| A^{1/2}(s) W_\mu(s) x \right\|^2 ds \right)^{1/2}, \quad \mu \in \mathbb{C}_0, \ \ x \in Q. \tag{34}$$

We denote the completion of Q with respect to this system by Q_-. The space Q_- is locally convex. Arguing as above, we see that the replacement of μ by $\lambda \in \mathbb{C}_0$ leads to an isomorphic space [25, Ch. 1]. The space Q_- is isomorphic to a projective limit of a family of Banach spaces $Q_-(n)$ (see [25, Ch. 2, proof 5.4]). The spaces $Q_-(n)$ are constructed bellow. Let $Q_+(n)$ be the space adjoint to $Q_-(n)$, Q_+ be an inductive limit of the spaces $Q_+(n)$. We claim that $\mathcal{D}(M(\lambda)) = Q_+$, $\mathcal{R}(M(\lambda)) \subset Q_-$. We describe this construction more detail for the proof of properties of the operator $M(\lambda)$. This construction is little distinguished from the corresponding construction in [24]. Therefore we omit the details of argumentation.

We denote $\mathfrak{H}_n = L_2(H, A(t); \alpha_n, \beta_n)$, $n \in \mathbb{N}$. Let $Q_0(n)$ be a set of elements $c \in Q$ such that the function $W_\lambda(t)c$ is identified with zero in the space $\mathfrak{H}_n$, $Q(n) = Q \ominus Q_0(n)$. Then $Q_0(n) \subset Q_0(m)$ and $Q(n) \supset Q(m)$ for $n > m$. In $Q(n)$, we introduce the seminorm $p_n(x)$ by formula (34). This seminorm is the norm on $Q(n)$. We denote it by $\|\cdot\|_-^{(n)}$. If $m < n$, $c \in Q(m)$, then $\|c\|_-^{(m)} \leqslant \|c\|_-^{(n)}$. Let $Q_-(n)$ be the completion of $Q(n)$ with respect to the norm $\|\cdot\|_-^{(n)}$. We define mappings $h_{mn} : Q_-(n) \to Q_-(m)$ ($n > m$) in the following way. Let $Q_-^{(n)}(n,m)$ be the completion of $Q(n,m) = Q(n) \ominus Q(m)$ with respect to the norm $\|\cdot\|_-^{(n)}$ and $Q_-^{(n)}(m)$ be the completion of $Q(m)$ with respect to this norm. Then $Q_-(n) = Q_-^{(n)}(m) \dot{+} Q_-^{(n)}(n,m)$. We define $h_{m,n}c = 0$ whenever $c \in Q_-^{(n)}(n,m)$, and we define $h_{m,n}c = j_{mn}c$ whenever $c \in Q_-^{(n)}(m)$, where j_{mn} is the inclusion map of $Q_-^{(n)}(m)$ into $Q_-(m)$. We set $h_{nn}c = c$ for $m = n$. The mappings h_{mn} are continuous.

We consider the projective limit $\lim(pr)h_{mn}Q_-(n)$ of the family of the spaces $\{Q_-(n); n \in \mathbb{N}\}$ with respect to the mappings h_{mn} $(m, n \in \mathbb{N}, m \leqslant n)$ [25, Ch. 2]. By repeating the corresponding arguments from [25, Ch. 2], one can show that this projective limit is isomorphic to the space Q_- introduced after formula (34). We do not use this fact and thereby omit its proof. Further, by Q_- we denote the projective limit $\lim(pr)h_{mn}Q_-(n)$. It follows from the definition of projective limit [25, Ch. 2] that Q_- is a subspace of the product $\prod_n Q_-(n)$ and Q_- consists of all elements $c = \{c_n\}$ such that $c_m = h_{mn}c_n$ for all m, n, where $m \leqslant n$.

The space $Q_-(n)$ can be a treated as a negative one with respect to $Q(n)$ [8, Ch. 2]. By $Q_+(n)$ we denote the corresponding space with positive norm. Then $Q_+(m) \subset Q_+(n)$ for $m \leqslant n$, and the inclusion map of $Q_+(m)$ into $Q_+(n)$ is continuous. This inclusion map coincides with h^+_{mn}, where $h^+_{mn}: Q_+(m) \to Q_+(n)$ is the adjoint mapping of h_{mn}. By Q_+ we denote the inductive limit [25, Ch. 2] of the family $\{Q_+(n); n \in \mathbb{N}\}$ with respect to the mappings h^+_{mn}, i.e., $Q_+ = \lim(ind)h^+_{mn}Q_+(n)$. According to [25, Ch. 4], Q_+ is the adjoint space of Q_-. The space Q_+ can be treated as the union $Q_+ = \cup_n Q_+(n)$ with the strongest topology such that all inclusion maps of $Q_+(n)$ into Q_+ are continuous [25, Ch. 2].

By Lemma 2, the operator $c_n \to W_\lambda(t)c_n$ is a continuous one-to-one mapping of $Q_-(n)$ into $\mathfrak{H}_n$ and it has the closed range. Denote this operator by $\mathcal{W}_\lambda(n)$. It follows from Lemma 2 that the adjoint operator $\mathcal{W}^*_\lambda(n)$ maps continuously $\mathfrak{H}_n$ onto $Q_+(n)$, and $\mathcal{W}^*_\lambda(n)f = \int_{\alpha_n}^{\beta_n} W^*_\lambda(s)\tilde{A}(s)f(s)ds$. Hence, for each function $f \in \mathfrak{H}$ and all finite α, β, we have $\int_\alpha^\beta W^*_\lambda(s)\tilde{A}(s)f(s)ds \in Q_+$.

Let $c = \{c_n\} \in Q_-$. Then $c_m = h_{mn}c_n$ $(m \leqslant n)$. Consequently, the restriction of the function $\tilde{A}^{1/2}(t)W_\lambda(t)c_n$ to the segment $[\alpha_m, \beta_m]$ coincides with the function $\tilde{A}^{1/2}(t)W_\lambda(t)c_m$ in the space $L_2(H; \alpha_m, \beta_m)$. By $\tilde{A}^{1/2}(t)W_\lambda(t)c$ we denote the function that is equal to $\tilde{A}^{1/2}(t)W_\lambda(t)c_n$ on each segment $[\alpha_n, \beta_n]$. Now by $W_\lambda(t)c$ denote the function ranging in $H_{-1/2}(t) \oplus G(t)$ and coinciding with $W_\lambda(t)c_n$ in the space $\mathfrak{H}_n$ for each $n \in \mathbb{N}$. So, for all m, n $(m \leqslant n)$, the equality $W_\lambda(t)c_n = W_\lambda(t)c_m$ holds in the space $\mathfrak{H}_m$.

Lemma 9. *If the pair $\{y, f\} \in L(\lambda)$, then y has the form*

$$y(t) = W_\lambda(t)c + W_\lambda(t)J_r^{-1}(t_0)\int_{t_0}^{t} W^*_{\bar{\lambda}}(s)\tilde{A}(s)f(s)ds,$$

where $c \in Q_-$.

P r o o f follows from Lemma 3 and the definition of the function $W_\lambda(t)c$.

Theorem 9. *For fixed* λ $(\mathrm{Im}\lambda \neq 0)$*, the operator* $\mathcal{L}^{-1}(\lambda)$ *is integral*

$$\mathcal{L}^{-1}(\lambda)g = \int_{-\infty}^{\infty} K_\lambda(t,s)\tilde{A}(s)g(s)ds, \quad g \in \mathfrak{H},$$

where $K_\lambda(t,s) = W_\lambda(t)(M(\lambda) - (1/2)\mathrm{sgn}(s-t)J_r^{-1}(t_0))W^*_{\bar{\lambda}}(s)$ *and* $M(\lambda): Q_+ \to Q_-$ *is a continuous operator such that* $M(\bar{\lambda}) = M^*(\lambda)$*. The operator function* $M(\lambda)x$ *is holomorphic for* $\mathrm{Im}\lambda \neq 0$ *and each* $x \in Q_+$*.*

R e m a r k 6. In Theorem 9, the integral converges at least weakly in $\mathfrak{H}$.

P r o o f. As established above, the relation $\mathcal{L}^{-1}(\lambda)$ is a bounded everywhere defined operator in the space $\mathfrak{H}$ and the operator function $\mathcal{L}^{-1}(\lambda)$ is holomorphic for $\mathrm{Im}\lambda \neq 0$. By this fact, the further proof of the theorem and the remark repeats word-for-word the proof of the main theorem in [24]. Theorem 9 is proved.

Theorem 10. *In Theorem* 9*, the operator function* $M(\lambda)$ *is the characteristic operator of equation* (27) *on the axis* $(-\infty, \infty)$*.*

P r o o f. It follows from Theorem 9 that for any finite function $g \in \mathfrak{H}$ the equality (31) holds with the replacement of a by $-\infty$ and of b by $+\infty$. The family of relations $\mathcal{L}(\lambda)$ satisfies all conditions of Theorem 8. We obtain the desired assertion by repeating the proof of this theorem. Theorem 10 is proved.

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