Journal of Mathematical Physics, Analysis, Geometry
2010, vol. 6, No. 3, pp. 266–276

Higher Order Differential Operators with Finite Number of $\delta-$Interactions in Multivariate Case

E.H. Eyvazov

Applied Mathematics and Cybernetics Faculty, Baku State University
23 Z. Khalilov Str., Baku, AZ1148, Azerbaijan
E-mail:eyvazovelshad@mail.ru

Received March 5, 2009

A formula for calculating the defect number of a multi-center differential operator of higher order is obtained. The spectral properties of selfadjoint operators describing the finite number pointwise interactions are studied.

Key words: Schrödinger operator, δ-interaction, pointwise interactions, higher order differential operators, deficiency index of the operators, quantum mechanics, selfadjoint extensions.

Mathematics Subject Classification 2000: 35J10, 35P05.

The author devotes the paper to the memory of the academician of NAS of Azerbaijan prof. M.G. Gasymov

Since the time the book [1] was published, the number of papers on spectral theory of singular differential operators initiated by E. Fermi and developed by Ya.B. Zeldovich, F.A. Berezin, R.A. Minlosov, L.D. Faddeev (references to the papers by these authors can be found in [1]) has considerably increased. Notice that in the mentioned monograph only the classes of solvable models of quantum mechanics with pointwise interaction in the spaces of no more than three dimensions were studied.

1. The goal of our paper is to give a strict sense to the formal operator

$$l(D) + \sum_{j=1}^{l} C_j \delta\left(x - t_j\right), \tag{1}$$

where $t_1, t_2, \dots, t_l$ is a set of l pairwise different points in R_n, $x = (x_1, x_2, \dots, x_n) \in R_n$,

$$l(D) = \sum_{|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n \le m} a_\alpha D^\alpha \left(P_0(\xi) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} a_\alpha \xi^\alpha \in R_1, \quad \forall \xi \in \widehat{R}_n \right)$$

is an elliptic differential expression of the m-th order with constant coefficients, $\delta(x)$ is Dirac's $\delta-$function, $c_1, c_2, \dots, c_l$ are real numbers.

Following Berezin and Faddeev's method [3], at first we consider the operator

$$\widetilde{H}_0 := l(D)\Big|_{C_0^\infty(R_n\backslash S)}, \; S = \{t_k\}_{k=1}^l \subset R_n,$$

in the space $L_2(R_n)$ and denote by $H_0 = \overline{\widetilde{H}}_0$ its closure in $L_2(R_n)$, i.e.

$$D(H_0) = \overset{\circ}{W}{}_2^{m} (R_n\backslash S) \;\; \text{(Sobolev space) and}$$

$$H_0\psi(x) = l(D)\psi(x), \quad \text{for } \; \forall\psi(x) \in D(H_0).$$

Then, after having found the deficiency index of the operator H_0, by using the theory of self-adjoint extensions of symmetric operators (von Neumann's theory) [4] and the method of renormalization of relation constants $c_1, c_2, \dots, c_l$, standing before δ-function [3], we describe the pointwise interactions with (with the center at the points of the set S) l-centers.

The following conjecture is crucial in proving a theorem on the deficiency index of the operator H_0.

Conjecture 1 (see [5, p. 154]). *Let $\mathcal{E}_{n,m}(x)$ be a fundamental solution for an elliptic operator $l(D)$. Then the following estimate is valid in a neighborhood of the point $x = 0$:*

$$|D^\alpha \mathcal{E}_{n,m}(x)| \le C_0 + C_1 |x|^{m-n-|\alpha|}, \quad |\alpha| = 0, 1, 2, \dots, m-1; \;\; |\alpha| \ne m-n,$$

$$|D^\alpha \mathcal{E}_{n,m}(x)| \le C_0 + C_1 \ln\left(\frac{1}{|x|}\right), \;\; |\alpha| = m-n.$$

Let us denote by $H_{j,0}$, $j = 0, 1, \dots, l$, a closure of minimal operators $\widetilde{H}_{j,0}$, $j = 0, 1, 2, \dots, l$, defined in the space $L_2(R_n)$ by the formulas

$$\widetilde{H}_{j,0} = l(D), \;\; D\left(\widetilde{H}_{j,0}\right) = C_0^\infty\;(R_n\backslash\{t_j\}), \; t_j \in S, \; j = 1, 2, \dots, l,$$

$$\widetilde{H}_{0,0} = l(D), \;\; D\left(\widetilde{H}_{0,0}\right) = C_0^\infty\;(R_n\backslash\{0\}).$$

The following two lemmas are used in the proof of the deficiency index theorem.

Lemma 1.

$$def(H_0) = \sum_{j=1}^{l} def\left(H_{j,0}\right) = l \cdot def\left(H_{0,0}\right).$$

The proof of this lemma follows from the local nature of interactions.

Lemma 2. *Let $f(x) \equiv f(x_1, x_2, \ldots, x_n)$ be a sufficiently smooth function, and*

$$A_{n,m}(f) = \{D^\alpha f(x)\}_{|\alpha| \le m}$$

be a totality of all its different derivatives (in Schwartz's sense) up to the order m inclusively. Then the equality

$$\nu\left(A_{n,m}(f)\right) = \begin{cases} C^m_{n+m}, & m \in N, \\ 1, & m = 0 \end{cases} \tag{2}$$

is true for number $\nu\left(A_{n,m}(f)\right)$ of elements of the set $A_{n,m}(f)$.

P r o o f. Since the number of different derivatives (in Schwartz's sense) of the m-th order of the function $f(x)$ equals

$$C^{m-1}_{m-1} + C^{m-1}_{m} + \cdots + C^{m-1}_{n+m-2},$$

then formula (2) follows from the equalities

$$C^{m-1}_{m-1} + C^{m-1}_{m} + \cdots + C^{m-1}_{n+m-2} = C^m_{n+m-1},$$

and

$$C^m_{n+m-1} + C^{m-1}_{n+m-1} = C^m_{n+m}.$$

In the following theorem we study the dependence of deficiency numbers of the operator $H_{0,0}$ on the dimension of space and on the order of elliptic differential expression $l(D)$.

Theorem 1. *a) If $n \ge 2m$, then the operator $H_{0,0}$ has deficiency indices $(0,0)$;*

b) if $n = 2m - 2p + j$, $j = 0, 1$; $p = 1, 2, \ldots, m$, then deficiency indices of the operator $H_{0,0}$ are (r_p, r_p), where

$$r_p = \begin{cases} C^{p-1}_{n+p-1}, & \text{if} \quad p = 2, 3, \ldots, m, \\ 1, & \text{if} \quad p = 1. \end{cases}$$

P r o o f. We denote a fundamental solution of the differential operator $l(D) - \lambda$ (here λ is an arbitrary nonreal complex number) by $\mathcal{E}_{n,m}(x, \lambda)$. Since the function $\mathcal{E}_{n,m}(x,\lambda)$ and its derivatives of any order are square integrable near ∞ (see [6, p. 287]), the amount of functions $D^\alpha \mathcal{E}_{n,m}(x,\lambda)$ belonging to $L_2(R_n)$, depends on the amount of functions $D^\alpha \mathcal{E}_{n,m}(x,\lambda)$ lying in L_2 near zero. From Lemma 2 and Conjecture 1 it follows that all the functions

$$D^\alpha \mathcal{E}_{n,m}(x,\lambda), \ \ |\alpha| \le \frac{n+2p-j}{2}, \ \ p=1,2,\dots,m; \ j=0,1,$$

belong to $L_2(R_n)$. Since the system of functions $\{D^\alpha \mathcal{E}_{n,m}(x,\lambda)\}$ is linear independent, then the dimension of the deficiency subspace

$$M_\lambda = L\left(\{D^\alpha \mathcal{E}_{n,m}(x,\lambda)\}_{|\alpha|\le \frac{n+2p-j}{2};\, p=1,2,\dots,m;\ j=0,1}\right) \tag{3}$$

of the operator $H_{0,0}$ for $n=2m-2p+j$, $p=1,2,\dots,m$; $j=0,1$, equals r_p, and for $n \ge 2m$, equals zero, and thus the theorem is proved.

Remarks to Theorem 1:

1) for $m=2$, statement a) of the theorem is proved in [2, p. 184];

2) physical sense of statement a) of the theorem is that for $n \ge 2m$ there are no pointwise interactions;

3) for $n=1$, $m=2$, the operator $H_{0,0}$ has deficiency indices (2,2). Since the operator $H_{0,0}$ has four parametric families (see Th. 3) of selfadjoint extensions in the space $L_2(R_1)$, then in addition to δ-interaction there exist supplementary types of pointwise interactions (for example, δ'-interactions (see [1, p. 121])).

Now, we formulate the main theorem on the deficiency index of the operator H_0.

Theorem 2. *If $n=2m-2p+j$, $p=1,2,\dots,m$; $j=0,1$, then the operator H_0 has deficiency indices (lr_p, lr_p).*

The proof of Theorem 2 follows from Lemma 1 and Theorem 1.

Remarks to Theorem 2:

1) in Theorems 1 and 2, $j=0$ should be taken for even n, and $j=1$, for odd n;

2) theorem 2 for a single-center Schrödinger operator in three-dimensional case was proved in the paper [3], and for multicenter Schrödinger operator, in [9].

2. Additional remarks. Notice that Theorem 2 can be generalized for an elliptic operator acting in the space of vector-functions.

Conjecture 2. *Let $A : C_0^\infty(R_n \backslash \{t_k\}_{k=1}^l, C^d) \to L_2(R_n, C^d)$ (here C^d is a d-dimensional complex field) be a minimal operator generated by an elliptic differential operator of the m-th order with constant coefficients acting in the space of vector-functions. Then the operator A has a deficiency index (lr_pd, lr_pd).*

Thus, the operator A may be represented in the form of direct sum d of scalar operators of the type considered in Theorem 2.

In Svendsen's paper [8] a more general situation is considered. In particular, the deficiency numbers (see [8, Cor. 3.2, p. 538]) of the operator $A: C_0^\infty(\Omega\backslash M, C^d) \to L_2(\Omega, C^d)$ (here M is C^∞-manifold with $codimM > 0$), acting in the space of vector-functions that was generated by an elliptic differential expression with C^∞-coefficients, were studied. Notice that if $codimM > 0$ and the number of the elements of the set M is infinite, then the deficiency numbers of the operator A equal ∞, i.e. in this case Corollary 3.2 is obvious, but if $codimM > 0$ and the number of the elements of the set M is finite, Conjecture 2 agrees with Svendsen's Corollary 3.2. Svendsen himself notes (see [8, the example after Th. 3.1, p. 558]) that ellipticity conditions mast not be rejected. Theorem 3.2 (consequently, Cor. 3.2) of the paper [8] is obliquely based on Theorem 2.1, where C^∞-manifold M (the manifolds consisting of finite elements can hardly be considered as a C^∞-manifold, if only we suppose that) locally straightens out to some finite–dimensional space R_k. In general, if we study only deficiency numbers, then Svendsen's technique on which a specially constructed approximate sequence, the Sobolev inequality, the "straightening" mentioned above, and property (2) are based (see [8, p. 561] for its proof see [7, p. 196–199]), is very complicated to be used. One can easily notice some misprints and inaccuracies in the paper [8]. For example, in formulation of Theorem 3.2 (see. [8, p. 558]) in the first statement there should be the sign of inequality, and in the second one, the sign of equality. The elements $\mathcal{E}(3.2), \mathcal{E}(3.3), \mathcal{E}(3.4), \mathcal{E}(3.5), \mathcal{E}(6.6), \mathcal{E}(6.7)$ of Table 1 (see. [8, p. 560]) should be equal to $3, 4, 1, 1, 28, 36$, respectively.

3. The obvious forms of the basis elements of deficiency subspaces M_λ and $M_{\overline{\lambda}}$ (see. (3)) allow to describe all selfadjoint extensions of the operator H_0 and to choose the selfadjoint extension that corresponds to the dynamics of the described system.

Take some orthonormed bases $\left\{e_k^+(x)\right\}_{k=1}^{lr_p}$ and $\left\{e_k^-(x)\right\}_{k=1}^{lr_p}$ in the subspaces M_λ and $M_{\overline{\lambda}}$, respectively. Let $U = (u_{kj})$ be a unitary matrix of order lr_p. We denote a selfadjoint extension of the operator H_0, that corresponds to unitary matrix U, by H_u. Applying the general theory of extensions of symmetric operators (see [4, p. 166]), we arrive at the following theorem.

Theorem 3. *Selfadjoint extensions H_u of the operator H_0 are given by the formulae*

$$D\left(H_u\right) = \Big\{\psi(x) = \varphi(x) + \sum_{j=1}^{lr_p} d_j \Big[e_j^+(x) + \sum_{k=1}^{lr_p} u_{kj} e_j^-(x)\Big] :$$

$$\varphi(x) \in D(H_0),\ d_j \in \mathbb{C},\ j = 1, 2, \ldots, lr_p\Big\},$$

$$H_u\psi(x) = H_0\varphi(x) + \overline{\lambda}\sum_{j=1}^{lr_p} d_j e_j^+(x) + \lambda \sum_{j=1}^{lr_p}\sum_{k=1}^{lr_p} u_{kj} e_j^-(x).$$

The following theorem describes the spectral properties of the operator H_u.

Theorem 4. *i) A continuous part of the spectrum of any selfadjoint extension H_u of the operator H_0 is absolutely continuous and covers the range of values $E\left(P_0(\xi)\right)$ of the function $P_0(\xi)$.*

ii) If

$$\min\left[E\left(P_0(\xi)\right)\right] = P_0 > -\infty,$$

then a part of the spectrum of any selfadjoint extension H_u lying to the left of P_0 may consist of only finite numbers of eigenvalues whose sum of multiplicities does not exceed lr_p.

P r o o f. Taking into account that Fridrich's extension H_{-E} with absolutely continuous spectrum is among the selfadjoint extensions of the operator H_0 and the continuous spectrum does not depend on the choice of extension, we get the validity of statement i). Statement ii) follows directly from M.G. Krein's theorem (see [4, Th. 16, p. 179]).

4. Now, we will show that the selfadjoint extension of the operator H_0, that corresponds to a finite number of δ-interactions, is of the form $H_{u(\theta_1,\theta_2,\ldots,\theta_l)}$, where $u\left(\theta_1,\theta_2,\ldots,\theta_l\right) = diag\left(u_1,u_2,\ldots,u_l\right)$ is a diagonal block matrix of order lr_p, $u_i = diag\left(\theta_i,-1,\ldots,-1\right)$ are diagonal matrices of order r_p, θ_i are complex numbers satisfying the condition $|\theta_i| = 1$, $i = 1,2,\ldots,l$. Taking into account the local nature of interactions, it suffices to consider the operator

$$L = l\left(D\right) + \varepsilon\delta\left(x\right). \tag{4}$$

Consider the Fourier transformation of expression (4)

$$\hat{L}\hat{\psi}\left(\xi\right) = P\left(-i\xi\right)\hat{\psi}\left(\xi\right) + \frac{\varepsilon}{(2\pi)^n}\int\limits_{R_n}\hat{\psi}\left(\xi\right)d\xi. \tag{5}$$

It is easy to get the expressions for H_0 and H_u in ξ-representation

$$D\left(\hat{H}_0\right) = \left\{\int\limits_{R_n}\left|P\left(-i\xi\right)\hat{\varphi}\left(\xi\right)\right|^2 d\xi < +\infty,\ \int\limits_{R_n}\hat{\varphi}\left(\xi\right)\xi^\alpha d\xi = 0,\ |\alpha| \le r_p - 1\right\}, \tag{6}$$

$$\hat{H}_0\hat{\varphi}\left(\xi\right) = P\left(-i\xi\right)\hat{\varphi}\left(\xi\right);$$

for $\hat{\psi}(\xi) \in D\left(\hat{H}_u\right)$ for $\lambda = i$ we have

$$\hat{\psi}(\xi) = \hat{\varphi}(\xi) + \sum_{k=1}^{r_p} d_k \frac{Q_k(-i\xi)}{P(-i\xi)+i} + \sum_{k=1}^{r_p}\left(\sum_{j=1}^{r_p} u_{kj} d_j\right) \frac{Q_k(-i\xi)}{P(-i\xi)-i}, \qquad (7)$$

$$\hat{H}_u\hat{\psi}(\xi) = P(-i\xi)\hat{\varphi}(\xi) - i\sum_{k=1}^{r_p} d_k \frac{Q_k(-i\xi)}{P(-i\xi)+i}$$

$$+ i\sum_{k=1}^{r_p}\left(\sum_{j=1}^{r_p} u_{kj} d_j\right) \frac{Q_k(-i\xi)}{P(-i\xi)-i}, \qquad (8)$$

where $\hat{\varphi}(\xi) \in D\left(\hat{H}_0\right)$, $Q_k(-i\xi) = (-i\xi)^{\alpha}$, moreover $Q_1(-i\xi) = 1$.

Using (7) and (8), we get

$$\hat{H}_u\hat{\psi}(\xi) = P(-i\xi)\hat{\psi}(\xi) - d_1(1+u_{11}) - \sum_{k=2}^{r_p}\left(d_k + \sum_{j=1}^{r_p} u_{kj} d_j\right) Q_k(-i\xi). \quad (9)$$

For (9) to get the form of (5), we must assume

$$d_k + \sum_{j=1}^{r_p} u_{kj} d_j = 0, \quad k = 2, 3, \ldots, r_p.$$

Due to the arbitrariness of the coefficients d_j, $j = 1, 2, \ldots, r_p$, we have

$$u_{kk} = -1, \;\; u_{kj} = 0, \quad k = 2, 3, \ldots, r_p, \;\; j \neq k. \qquad (10)$$

It follows from (10) and the unitarity of matrix U that

$$u_{11} = \theta \;\; (\theta \text{is a complex number with } \; |\theta| = 1), \;\; u_{1j} = 0, \;\; j = 2, 3, \ldots, r_p. \quad (11)$$

Thus, if (10) and (11) are fulfilled, then we have

$$\hat{\psi}(\xi) = \hat{\varphi}(\xi) + d_1\left(\frac{1}{P(-i\xi)+i} + \theta\frac{1}{P(-i\xi)-i}\right)$$

$$+\sum_{k=2}^{r_p} d_k\left(\frac{1}{P(-i\xi)+i} - \frac{1}{P(-i\xi)-i}\right) Q_k(-i\xi) \qquad (12)$$

and

$$\hat{H}_{(\theta,-1,-1,\ldots,-1)}\hat{\psi}(\xi) \equiv H_\theta\hat{\psi}(\xi) = P(-i\xi)\hat{\psi}(\xi) - d_1(1+\theta). \qquad (13)$$

The functional $d_1\left(\hat{\psi},\theta\right)$ is obtained from the asymptotic formulae for the integrals

$$\int\limits_{R_n} \chi_N(\xi)\ \hat{\psi}(\xi)\, Q_k(-i\xi)\ d\xi \text{ as } N \to +\infty,$$

where $\chi_N(\xi)$ is a characteristic function of the ball with the center at zero and with the radius N. Comparing (13) and (5), we find relation between ε and θ. In some cases this dependence is expressed directly and in some cases via the renormalization of relation constant ε. We explain briefly (since this theme is a subject of a separate paper) what the renormalization of the relation constant ε means. The equation

$$P(-i\xi)\,\hat{\psi}(\xi) + \frac{\varepsilon}{(2\pi)^n}\int\limits_{R_n} \hat{\psi}(\xi)\ d\xi = \lambda\hat{\psi}, \quad \lambda > 0 \tag{14}$$

is replaced by

$$P(-i\xi)\,\hat{\psi}_N(\xi) + \varepsilon_N(\theta)\,\chi_N(\xi)\left(\int\limits_{R_n} \chi_N(t)\,\hat{\psi}_N(t)dt\right) = \lambda\hat{\psi}_N(\xi)\,.$$

The form of $\varepsilon_N(\theta)$ is determined from the equation

$$d_1(1+\theta) = \lim_{N\to+\infty} \varepsilon_N(\theta)\int\limits_{R_n} \chi_N(t)\ \hat{\psi}(\xi)\,d\xi.$$

Thus, the limit $\hat{\psi}(\xi)$ of the sequence $\hat{\psi}_N(\xi)$ as $N \to +\infty$ is a solution of the problem of scattering theory for the operator H_θ. Thereby, the mathematical background of equation (14) is in the replacement of the expression

$$|\xi|^2 \cdot + \frac{\varepsilon}{(2\pi)^n}\int\limits_{R^n} \cdot\, d\xi$$

by the operator H_θ.

The following examples show that this generalization is well agreed for second order differential operators. All the examples are cited in ξ-representation, since it is easy to get x-representation from them.

Example 1. $n = 1;\ m = 2$. Consider the Schrodinger equation

$$\hat{L}\hat{\psi}(\xi) \equiv \xi^2\hat{\psi}(\xi) + \frac{\varepsilon}{2\pi}\int\limits_{-\infty}^{\infty} \hat{\psi}(\xi)\,d\xi = \lambda\hat{\psi}(\xi)\,. \tag{15}$$

By $\hat{H}_0$ we denote an operator of multiplication by ξ^2 in $L_2(R_1)$ with the domain of definition

$$D(H_0) = \left\{ \int\limits_{-\infty}^{\infty} \xi^4 \left|\hat{\varphi}(\xi)\right|^2 d\xi < +\infty, \quad \int\limits_{-\infty}^{\infty} \hat{\varphi}(\xi)\, \xi^j d\xi = 0, \;\; j = 0,1 \right\}.$$

The operator $\hat{H}_0$ is a closed symmetric operator in $L_2(-\infty,\infty)$ with deficiency indices (2,2). All selfadjoint extensions are given by the formula

$$\hat{H}_\theta \hat{\psi}(\xi) = \xi^2 \hat{\psi}(\xi) - d_1(1+\theta), \tag{16}$$

where

$$\hat{\psi}(\xi) = \hat{\varphi}(\xi) + d_1 \frac{1}{\xi^2+i} + d_2 \frac{\xi}{\xi^2+i} +$$
$$+ d_1 \theta \frac{1}{\xi^2-i} - d_2 \frac{\xi}{\xi^2-i}, \quad \hat{\varphi}(\xi) \in D\left(\hat{H}_0\right). \tag{17}$$

From (16) and (17), we get directly

$$\hat{H}_\theta \hat{\psi}(\xi) = \xi^2 \hat{\psi}(\xi) + \frac{1+\theta}{i\sqrt{i} + \theta\, i\sqrt{-i}} \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \hat{\psi}(\xi)\, d\xi. \tag{18}$$

Comparing (18) and (15), we have

$$\varepsilon = \frac{2(1+\theta)}{i\sqrt{i} + \theta\, i\sqrt{-i}}.$$

(18) in x-representation means that the operator $H_\varepsilon \psi = -\psi'' + \varepsilon\delta(x)\psi$ with the domain of definition

$$D(H_\varepsilon) = \{\psi(x) \in L_2 \bigcap C(-\infty,+\infty) : \psi'(+0) - \psi'(-0) = \varepsilon\psi(0),$$
$$-\psi''(x) + \varepsilon\delta(x)\psi(x) \in L_2(-\infty,+\infty)\}$$

is a selfadjoint operator in $L_2(-\infty,\infty)$.

Example 2. $n = 2$; $m = 2$. Consider the Schrödinger equation

$$\hat{L}\hat{\psi}(\xi) = |\xi|^2 \hat{\psi}(\xi) + \frac{\varepsilon}{4\pi^2} \int\limits_{R^2} \hat{\psi}(\xi)\, d\xi = \lambda \hat{\psi}(\xi). \tag{19}$$

By $\hat{H}_0$ we denote an operator of multiplication by $|\xi|^2$ in $L_2(R_2)$ with the domain of definition

$$D\left(\hat{H}_0\right) = \left\{ \int\limits_{R_2} |\xi|^4 \left|\hat{\varphi}(\xi)\right|^2 d\xi < +\infty, \quad \int\limits_{R^n} \hat{\varphi}(\xi)\, d\xi = 0 \right\}. \tag{20}$$

The operator $\hat{H}_0$ is a closed symmetric operator with deficiency indices $(1,1)$. All selfadjoint extensions are given by the formula

$$\hat{H}_\theta \hat{\psi}(\xi) = |\xi|^2 \hat{\psi}(\xi) - d_1 (1+\theta), \tag{21}$$

where

$$\hat{\psi}(\xi) = \hat{\varphi}(\xi) + d_1 \frac{1}{|\xi|^2 + i} + d_1 \theta \frac{1}{|\xi|^2 - i}, \quad \hat{\varphi}(\xi) \in D\left(\hat{H}_0\right). \tag{22}$$

From formulae (21) and (22), we have

$$\hat{H}_\alpha \hat{\psi}(\xi) = |\xi|^2 \hat{\psi}(\xi) + \frac{1}{2\pi} \lim_{N \to +\infty} \frac{\alpha}{1 - \alpha \ln N} \int\limits_{R_2} \chi_N(\xi)\ \hat{\psi}(\xi)\, d\xi, \tag{23}$$

where

$$\alpha = \frac{4(1+\theta)}{\pi i (1-\theta)}; \quad |\theta| = 1. \tag{24}$$

Here, under the renormalization of the relation constants ε we mean that instead of (19) we consider the equation

$$|\xi|^2 \hat{\psi}_N(\xi) + \varepsilon_N \chi_N(\xi) \int\limits_{R_2} \chi_N(t)\ \hat{\psi}_N(t)\, dt = \lambda \hat{\psi}_N(\xi), \tag{25}$$

where

$$\varepsilon_N = \frac{2\pi\alpha}{1 - \alpha \ln N},$$

and α is determined from formula (24).

The eigenfunctions of the continuous spectrum of the operator $\hat{H}_\alpha$ appear to be the limits of the sequence $\hat{\psi}_N(\xi)$ as $N \to +\infty$ in the sense of distributions. Thus, the mathematical background of equation (19) is in the replacement of the expression

$$|\xi|^2 \cdot + \frac{\varepsilon}{4\pi^2} \int\limits_{R_2} \cdot\, d\xi$$

by the operator H_α.

Another instructive example is in [3]. It must be noted that in formula $(12')$ in [3] instead of ε there should stand α, and in place of delta-sequence we can take

$$-\frac{1}{2\pi^2} \frac{1}{|x|} \frac{d}{d|x|} \left(\frac{\sin N|x|}{|x|} \right),$$

since in a three-dimensional space the sequence $\dfrac{\sin N|x|}{|x|}$ converges in the sense of distributions to zero but not to $\delta(x)$.

In conclusion I express my deep gratitude to Doctor of Physico-Mathematical Sciences Araz Rafig ogly Aliyev for his suggestions on writing this paper as well as to the referee for his valuable remarks and his referring us to the paper [8].

References

[1] *S. Albeverio, F. Gesztesy, Høegh-Krohn, and H. Holden*, Solvable Models in Quantum Mechanics. Mir, Moscow, 1991.

[2] *M. Reed and B. Saimon*, Methods of Modern Mathematical Physics. **2**. Mir, Moscow, 1978.

[3] *F.A. Berezin and L.D. Faddeev*, Remark on a Schrödinger Equation with a Singular Potential. — *DAN SSSR* **137** (1961), No. 5, 1011–1014.

[4] *M.A. Naimark*, Linear Differential Operators. Nauka, Moscow, 1969.

[5] *S. Misohata*, Theory of Partial Equations. Mir, Moscow, 1977.

[6] *L. Hörmander*, The Analysis of Linear Partial Differential Operators. **2**. Mir, Moscow, 1986.

[7] *R. Narasimhan*, Analysis on Real and Complex Manifolds. Mir, Moscow, 1971.

[8] *E.C. Svendsen*, The Effect of Submanifolds upon Essential Self-Adjointness and Deficiency Indices. — *J. Math. Anal. Appl.* **80** (1981), 551–565.

[9] *E.H. Eyvazov*, Investigation of Spectral Properties of Some Differential Operators with Generalized Coefficients. Thesis for PhD. Institute of Mathematics and Mechanics of AS AzerbSSR, Baku, 1979.