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Daugavet Centers

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An operator G: $X \to Y$ is said to be a Daugavet center if $\|G + T\| = \|G\| + \|T\|$ for every rank-1 operator T: $X \to Y$. The main result of the paper is: if G: $X \to Y$ is a Daugavet center, Y is a subspace of a Banach space E, and $J : Y \to E$ is the natural embedding operator, then E can be equivalently renormed in such a way that $J \circ G : X \to E$ is also a Daugavet center. This result was previously known for the particular case $X = Y$, $G = \mathrm{Id}$ and only in separable spaces. The proof of our generalization is based on an idea completely different from the original one. We also give some geometric characterizations of the Daugavet centers, present a number of examples, and generalize (mostly in straightforward manner) to Daugavet centers some results known previously for spaces with the Daugavet property.

Key words: Daugavet center, Daugavet property, renorming.

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1. Introduction

A Banach space X is said to have the Daugavet property if all the operators T: $X \to X$ of rank-1 satisfy the Daugavet equation

$$\|\mathrm{Id} + T\| = 1 + \|T\|. \tag{1.1}$$

Several classical spaces have the Daugavet property: $C(K)$, where K is perfect [1], $L_1(\mu)$, where μ has no atoms [2], and certain functional algebras such as the disk algebra $A(\mathbb{D})$ or the algebra of bounded analytic functions H^∞ ([12, 14]).

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Geometric and linear-topological properties of such spaces were studied intensively during the last two decades (see the survey paper [13] and most recent developments in [8, 3, 4]). In particular, if X is a space with the Daugavet property, then every weakly compact operator, even every strong Radon–Nikodým operator on X, and every operator on X not fixing a copy of ℓ_1, fulfill (1.1) as well ([6, 11]). These spaces contain subspaces isomorphic to ℓ_1, cannot have the Radon–Nikodým property, never have an unconditional basis and even never embed into a space having an unconditional basis. The key to the later embedding property is the following theorem:

Theorem 1.1. *[6, Th. 2.5]. Let X be a subspace of a separable Banach space Y, $J : X \to Y$ be the natural embedding operator, and suppose X has the Daugavet property. Then Y can be renormed so that the new norm coincides with the original one on X and in the new norm $\|J + T\| = 1 + \|T\|$ for every rank-1 operator $T\colon X \to Y$.*

The aim of our paper is to remove the separability condition in the above theorem. On this way we introduce and study the following concept:

Definition 1.2. *Let X and Y be Banach spaces. A linear continuous nonzero operator $G\colon X \to Y$ is said to be a Daugavet center if the norm identity*

$$\|G + T\| = \|G\| + \|T\| \tag{1.2}$$

is fulfilled for every rank-1 operator $T\colon X \to Y$.

Our main result is more general than just the nonseparable version of Theorem 1.1. Namely, we prove the following:

Theorem 1.3. *If $G\colon X \to Y$ is a Daugavet center, Y is a subspace of a Banach space E, and $J : Y \to E$ is the natural embedding operator, then E can be equivalently renormed in such a way that the new norm coincides with the original one on Y, and $J \circ G : X \to E$ is also a Daugavet center.*

Let us explain the structure of the paper. In Section 2. of this paper we collect some straightforward generalizations to Daugavet centers of the properties known for Id in the spaces with the Daugavet property. We also study the properties of the unit ball images under Daugavet centers. In Section 3 we give some examples of Daugavet centers quite different from the known before identity operator or isometric embedding. Finally, Section 4. is devoted to the proof of the main result.

In this paper we deal with real Banach spaces. We use the letters X, Y, E to denote Banach spaces and their subspaces. $L(X, Y)$ stands for the space of all linear bounded operators acting from X to Y. B_X denotes the closed unit ball of a Banach space X and S_X denotes its unit sphere. For a bounded closed convex set $A \subset X$ and for $x^* \in X^*$ we denote

$$S(A, x^*, \varepsilon) = \{x \in A \colon x^*(x) \geq \sup x^*(A) - \varepsilon\}$$

the slice of A, generated by x^*. We use the notation

$$S(x^*, \varepsilon) = \{x \in B_X \colon x^*(x) \geq 1 - \varepsilon\}$$

for the slice of B_X determined by a functional $x^* \in S_{X^*}$ and $\varepsilon > 0$.

We say that an element $x \in A$ is a denting point of the set A if for every $\varepsilon > 0$ there is a slice of A which contains x and has a diameter smaller than ε.

A set A is said to have the Radon-Nikodým property if every closed convex subset $B \subset A$ is the closed convex hull of its denting points.

The operator $T \in L(X, Y)$ is said to be a strong Radon–Nikodým operator if the closure of $T(B_X)$ has the Radon–Nikodým property.

2. Basic Properties of Daugavet Centers

Definition 1.2 implies the equality $\|aG+bT\| = a\|G\|+b\|T\|$ for every $a, b \geq 0$. This means that an operator G is a Daugavet center if and only if $G/\|G\|$ is. Therefore below we mostly consider the case $\|G\| = 1$, and by the same reason, when it is convenient, we require $\|T\| = 1$.

Theorem 2.1. *For an operator $G \in L(X, Y)$ with $\|G\| = 1$ the following assertions are equivalent:*

(i) *G is a Daugavet center.*

(ii) *For every $y_0 \in S_Y$ and every slice $S(x_0^*, \varepsilon_0)$ of B_X there is another slice $S(x_1^*, \varepsilon_1) \subset S(x_0^*, \varepsilon_0)$ such that for every $x \in S(x_1^*, \varepsilon_1)$ the inequality $\|Gx + y_0\| > 2 - \varepsilon_0$ holds.*

(iii) *For every $y_0 \in S_Y$, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$ there is $x \in B_X$ such that $x_0^*(x) \geq 1 - \varepsilon$ and $\|Gx + y_0\| > 2 - \varepsilon$.*

(iv) *For every $x_0^* \in S_{X^*}$ and every weak* slice $S(B_{Y^*}, y_0, \varepsilon_0)$ (where $y_0 \in S_Y \subset S_{Y^{**}}$) there is another weak* slice $S(B_{Y^*}, y_1, \varepsilon_1) \subset S(B_{Y^*}, y_0, \varepsilon_0)$ such that for every $y^* \in S(B_{Y^*}, y_1, \varepsilon_1)$ the inequality $\|G^*y^* + x_0^*\| > 2 - \varepsilon_0$ holds.*

(v) *For every $x_0^* \in S_{X^*}$ and every weak* slice $S(B_{Y^*}, y_0, \varepsilon_0)$ there is $y^* \in S(B_{Y^*}, y_0, \varepsilon_0)$ which satisfies the inequality $\|G^*y^* + x_0^*\| > 2 - \varepsilon_0$.*

P r o o f. $(i) \Rightarrow (ii)$. Define T by $Tx = x_0^*(x)y_0$. Then $\|G^* + T^*\| = \|G+T\| = 2$, so there exists a functional $y^* \in S_{Y^*}$ such that $\|G^*y^*+T^*y^*\| \geq 2-\varepsilon_0$ and $y^*(y_0) \geq 0$. Put

$$x_1^* = \frac{G^*y^* + T^*y^*}{\|G^*y^* + T^*y^*\|}, \quad \varepsilon_1 = 1 - \frac{2 - \varepsilon_0}{\|G^*y^* + T^*y^*\|}.$$

Then for every $x \in S(x_1^*, \varepsilon_1)$ we have

$$\langle (G^* + T^*)y^*, x\rangle \geq (1 - \varepsilon_1)\|G^*y^* + T^*y^*\| = 2 - \varepsilon_0;$$

hence

$$1 + x_0^*(x) \geq y^*(Gx) + y^*(y_0)x_0^*(x) \geq 2 - \varepsilon_0,$$

which implies that $x_0^*(x) \geq 1 - \varepsilon_0$, i.e., $x \in S(x_0^*, \varepsilon_0)$, and

$$2 - \varepsilon_0 \leq y^*(Gx) + y^*(y_0) = y^*(Gx + y_0) \leq \|Gx + y_0\|.$$

The implication $(ii) \Rightarrow (iii)$ is evident. Let us prove $(iii) \Rightarrow (i)$. If T is a rank-1 operator and $\|T\| = 1$, then T can be represented as $Tx = x_0^*(x)y_0$ with $y_0 \in S_Y$, $x_0^* \in S_{X^*}$. Fix an $\varepsilon > 0$ and let $x \in B_X$ be the corresponding element from (iii). Then

$$\begin{aligned} 2 - \varepsilon &\leq \|Gx + y_0\| \leq \|Gx + x_0^*(x)y_0\| + \|(1 - x_0^*(x))y_0\| \\ &\leq \|(G + T)x\| + \varepsilon\|y_0\| \leq \|G + T\| + \varepsilon. \end{aligned}$$

So we have proved the equivalence $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$. The remaining equivalence $(i) \Leftrightarrow (iv) \Leftrightarrow (v)$ can be proved in the same way by using the adjoint operators T^* and G^* instead of T and G.

For a bounded subset $A \subset Y$ denote $r_y(A) = \sup\{\|y - a\|\colon a \in A\}$.

Definition 2.2. *A bounded subset $A \subset Y$ is said to be a quasiball if for every $y \in Y$*

$$r_y(A) = \|y\| + r_0(A). \tag{2.3}$$

Definition 2.3. *A bounded subset $A \subset Y$ is called antidentable if for every $y \in Y$ and for every $r \in [0, r_y(A))$*

$$\overline{\text{conv}}\,(A \setminus B_Y(y, r)) \supset A. \tag{2.4}$$

Theorem 2.4. *If an operator $G \in S_{L(X,Y)}$ is a Daugavet center, then $A := G(B_X)$ is a quasiball.*

P r o o f. From Theorem 2.1, item (iii) it follows in particular that for every $y \in Y$ and for every $\varepsilon > 0$ there is an $x \in B_X$ with $\|y - Gx\| > \|y\| + 1 - \varepsilon$. So, $r_y(A) > \|y\| + 1 - \varepsilon \geq \|y\| + r_0(A) - \varepsilon$.

Theorem 2.5. *If an operator $G \in S_{L(X,Y)}$ is a Daugavet center, then for every $y \in Y$ and for every $r \in [0, r_y(G(B_X)))$*

$$V := \overline{\mathrm{conv}}\left(B_X \setminus G^{-1}(B_Y(y,r))\right) \supset B_X. \tag{2.5}$$

P r o o f. Assume it is not true. Then there is an $y \in Y$ and an $r \in [0, r_y(G(B_X))$ such that the corresponding V does not contain the whole B_X. Consider a slice $S = S(x^*, \varepsilon_0)$ of B_X which does not intersect V. For this slice we have $S \subset G^{-1}(B_Y(y,r))$. Select such a small $\delta > 0$ that $\|y\| + 1 - \delta > r$. By Theorem 2.1, item (iii) applied to $x_0^* = x^*$, $y_0 = -y$ and $\varepsilon = \min\{\varepsilon, \delta\}$, there is an $x \in S \subset G^{-1}(B_Y(y,r))$ with $\|Gx - y\| > \|y\| + 1 - \delta > r$. But in such a case $Gx \notin B_Y(y,r)$, i.e., $x \notin G^{-1}(B_Y(y,r))$, which leads to contradiction.

Corollary 2.6. *If an operator $G \in S_{L(X,Y)}$ is a Daugavet center, then $A := G(B_X)$ is antidentable.*

P r o o f. According to the previous theorem for every $y \in Y$ and for every $r \in [0, r_y(A))$ the inclusion (2.5) holds true. So

$$A \subset G(V) \subset \overline{\mathrm{conv}}\, G\left(B_X \setminus G^{-1}(B_Y(y,r))\right) = \overline{\mathrm{conv}}\left(A \setminus B_Y(y,r)\right).$$

Now we prove that the properties from Theorems 2.4 and 2.5 together give a characterization of Daugavet centers. In fact, we prove even more:

Theorem 2.7. *An operator $G \in S_{L(X,Y)}$ is a Daugavet center if and only if it satisfies the following two conditions:*

1. *the set $A := G(B_X)$ is a quasiball;*
2. *the condition (2.5) holds true for all $y \in Y$ and for all $r \in [0, r_y(G(B_X)))$.*

Moreover, if G is a Daugavet center, then equation (1.2) holds true for every strong Radon–Nikodým operator $T \in L(X,Y)$.

P r o o f. What remains to prove is that conditions (1) and (2) imply equation (1.2) for every strong Radon–Nikodým operator $T \in L(X,Y)$. Fix an $\varepsilon > 0$. Let $x \in S_X$ be an element for which $\|Tx\| > \|T\| - \varepsilon$, and Tx belongs to a slice $\tilde{S}$ of $T(B_X)$ with the diameter smaller than ε. Put $r = r_{Tx}(G(B_X)) - \varepsilon$. Then $T^{-1}\tilde{S}$ is a slice of B_X, so

$$T^{-1}\tilde{S} \bigcap \left(B_X \setminus G^{-1}(B_Y(Tx,r))\right) \neq \emptyset.$$

Hence there is an $x_0 \in B_X$ such that $Tx_0 \in \tilde{S}$ (and, consequently, $\|Tx_0 - Tx\| < \varepsilon$), but $Tx_0 \notin B_Y(Tx, r)$, i.e, $\|Gx_0 - Tx\| > r$. Then

$$\|G - T\| \geq \|Gx_0 - Tx_0\| \geq \|Gx_0 - Tx\| - \varepsilon > r - \varepsilon = r_{Tx}(G(B_X)) - 2\varepsilon$$

$$= \|Tx\| + r_0(G(B_X)) - 2\varepsilon \geq \|Tx\| + \|G\| - 2\varepsilon \geq \|T\| + \|G\| - 3\varepsilon.$$

R e m a r k 2.8. *Theorem 2.7 will not hold true if we require $G(B_X)$ to be antidentable instead of condition (2) of this theorem. Consider $G\colon C[0,1] \oplus_1 \mathbb{R} \to C[0,1]$, $G(f,a) = f$. It is obvious that $G(B_{C[0,1]\oplus_1\mathbb{R}}) = B_{C[0,1]}$. Since $C[0,1]$ has the Daugavet property, $B_{C[0,1]}$ is an antidentable quasiball. Let us show that G is not a Daugavet center. Consider a rank-1 operator $T\colon C[0,1] \oplus_1 \mathbb{R} \to C[0,1]$, $T(f,a) = a \cdot y_0$ for some $y_0 \in S_{C[0,1]}$. So $\|G\| + \|T\| = 2$, but*

$$\begin{aligned}\|G + T\| &= \sup_{(f,a)\in S_X} \|(G + T)(f,a)\| \\ &= \sup_{(f,a)\in S_X} \|f + a\, y_0\| \leq \sup_{(f,a)\in S_X} (\|f\| + |a|) = 1.\end{aligned}$$

For a set Γ denote by $\mathrm{FIN}(\Gamma)$ the set of all finite subsets of Γ. Recall that a (maybe uncountable) series $\sum_{n\in\Gamma} x_n$ in a Banach space X is said to be unconditionally convergent to $x \in X$ if for every $\varepsilon > 0$ there is an $A \in \mathrm{FIN}(\Gamma)$ such that for every $B \in \mathrm{FIN}(\Gamma)$, $B \supset A$

$$\|x - \sum_{n\in B} x_n\| < \varepsilon.$$

Theorem 2.9. *Let $G \in L(X,Y)$. Suppose that inequality $\|G+T\| \geq C + \|T\|$ with $C > 0$ holds for every operator T from a subspace $\mathcal{M} \subset L(X,Y)$ of operators. Let $\widetilde{T} = \sum_{n\in\Gamma} T_n$ be a (maybe uncountable) pointwise unconditionally convergent series of operators $T_n \in \mathcal{M}$. Then $\|G - \widetilde{T}\| \geq C$.*

P r o o f. Pointwise unconditional convergence of $\sum_{n\in\Gamma} T_n$ implies that for every $x \in X$

$$\sup\left\{\left\|\sum_{n\in A} T_n x\right\| \colon A \in \mathrm{FIN}(\Gamma)\right\} < \infty.$$

Consequently, by the Banach-Steinhaus theorem, the quantity

$$\alpha = \sup\left\{\left\|\sum_{n\in A} T_n\right\| \colon A \in \mathrm{FIN}(\Gamma)\right\}$$

is finite, and whenever $B \subset \Gamma$, then

$$\Big\| \sum_{n \in B} T_n \Big\| \le \sup \left\{ \Big\| \sum_{n \in A} T_n \Big\| \colon A \in \mathrm{FIN}(\Gamma),\ A \subset B \right\} \le \alpha.$$

Let $\varepsilon > 0$ and pick $A_0 \in \mathrm{FIN}(\Gamma)$ such that $\| \sum_{n \in A_0} T_n \| \ge \alpha - \varepsilon$. Then we have

$$\|G - \widetilde{T}\| \ge \Big\| G - \sum_{n \in A_0} T_n \Big\| - \Big\| \sum_{n \notin A_0} T_n \Big\| \ge C + \Big\| \sum_{n \in A_0} T_n \Big\| - \alpha \ge C - \varepsilon,$$

which proves the theorem.

R e m a r k 2.10. *Let* $G \colon X \to Y$ *be a Daugavet center. Since by Theorem 2.7 every weakly compact operator satisfies (1.2), the above theorem means, in particular, that* G *cannot be represented as a pointwise unconditionally convergent series of weakly compact operators. So, neither* X *nor* Y *can have an unconditional basis (countable or uncountable) or be represented as unconditional sum of reflexive subspaces.*

Lemma 2.11. *Let* $G \colon X \to Y$ *be a Daugavet center,* $\|G\| = 1$. *Then for every finite-dimensional subspace* Y_0 *of* Y, *every* $\varepsilon_0 > 0$ *and every slice* $S(x_0^*, \varepsilon_0)$ *of* B_X *there is a slice* $S(x_1^*, \varepsilon_1) \subset S(x_0^*, \varepsilon_0)$ *of* B_X *such that*

$$\|y + tGx\| \ge (1 - \varepsilon_0)(\|y\| + |t|) \qquad \forall y \in Y_0,\ x \in S(x_1^*, \varepsilon_1),\ \forall t \in \mathbb{R}. \tag{2.6}$$

P r o o f. Let $\delta = \varepsilon_0/2$ and pick a finite δ-net $\{y_1, \dots, y_n\}$ in S_{Y_0}. By a repeated application of Theorem 2.1, item (*ii*), we obtain a sequence of slices $S(x_0^*, \varepsilon_0) \supset S(u_1^*, \delta_1) \supset \dots \supset S(u_n^*, \delta_n)$ such that one has

$$\|y_k + Gx\| \ge 2 - \delta \tag{2.7}$$

for all $x \in S(u_k^*, \delta_k)$. Put $x_1^* = u_n^*$ and $\varepsilon_1 = \delta_n$; then (2.7) is valid for every $x \in S(x_1^*, \varepsilon_1)$ and $k = 1, \dots, n$. This implies that for every $x \in S(x_1^*, \varepsilon_1)$ and every $y \in S_{Y_0}$ the condition

$$\|y + Gx\| \ge 2 - 2\delta = 2 - \varepsilon_0$$

holds.

Let $0 \le t_1, t_2 \le 1$ with $t_1 + t_2 = 1$. If $t_1 \ge t_2$, we have for x and y as above

$$\begin{aligned} \|t_1 Gx + t_2 y\| &= \|t_1(Gx + y) + (t_2 - t_1)y\| \ \ge\ t_1\|Gx + y\| - |t_2 - t_1|\,\|y\| \\ &\ge\ t_1(2 - \varepsilon_0) + t_2 - t_1 \ =\ t_1 + t_2 - t_1\varepsilon_0 \ \ge\ 1 - \varepsilon_0, \end{aligned}$$

and an analogous argument shows this estimate in the case $t_1 < t_2$.

This implies (2.6) by the homogeneity of the norm and the symmetry of S_{Y_0}.

Theorem 2.12. *Let $G\colon X \to Y$ be a Daugavet center. Then G fixes a copy of ℓ_1.*

P r o o f. Using Lemma 2.11 inductively, we construct the sequences of the vectors $\{x_n\}_{n=1}^{\infty} \subset X$ and $\{y_n\}_{n=1}^{\infty} \subset Y$ and a sequence of the slices $S(x_n^*, \varepsilon_n)$, $\varepsilon_n \le 2^{-n}$, $n \in \mathbb{N}$, such that $y_n = Gx_n$, $x_n \in S(x_n^*, \varepsilon_n)$ and for every $y \in \operatorname{lin}\{y_1, \dots, y_n\}$ and every $t \in \mathbb{R}$ the inequality

$$\|y + ty_{n+1}\| \ge (1 - \varepsilon_n)(\|y\| + |t|\|y_{n+1}\|)$$

holds true. Hence the sequences $\{x_n\}_{n=1}^{\infty} \subset X$ and $\{y_n\}_{n=1}^{\infty} \subset Y$ are equivalent to the canonical basis in ℓ_1, and G fixes a copy of ℓ_1.

3. Some Examples of Daugavet Centers

Proposition 3.1. *Let $G\colon X \to Y$ be a Daugavet center. Then for all surjective linear isometries $V\colon X \to X$ and $U\colon Y \to Y$ the operator UGV is also a Daugavet center.*

P r o o f. Let $T\colon X \to Y$ have rank one. Then

$$\begin{aligned} \|UGV + T\| &= \|U(GV + U^{-1}T)\| = \|U\|\|GV + U^{-1}T\| \\ &= \|GV + U^{-1}T\| = \|(G + U^{-1}TV^{-1})V\| \\ &= \|G + U^{-1}TV^{-1}\|\|V\| = \|G + U^{-1}TV^{-1}\|. \end{aligned}$$

The operator T can be represented as $Tx = x_0^*(x)y_0$ with $y_0 \in Y$, $x_0^* \in X^*$ hence $U^{-1}TV^{-1}x = x_0^*(V^{-1}x)U^{-1}y_0$ is also a rank-1 operator. Since G is a Daugavet center,

$$\begin{aligned} \|UGV + T\| &= \|G + U^{-1}TV^{-1}\| = \|G\| + \|U^{-1}TV^{-1}\| \\ &= \|G\| + \|T\| = \|UGV\| + \|T\|. \end{aligned}$$

Proposition 3.2. *Let $G\colon X \to Y$ be a Daugavet center. Then $\widetilde{G} : X/\operatorname{Ker} G \to Y$ (the natural injectivization of G) is also a Daugavet center.*

P r o o f. We will prove this proposition using Definition 1.2 of the Daugavet center. Let $T \in L(X/\operatorname{Ker} G, Y)$ be a rank-1 operator and $q\colon X \to X/\operatorname{Ker} G$ be the corresponding quotient mapping. Then the composition $T \circ q\colon X \to Y$ is a linear continuous rank-1 operator. Since G is a Daugavet center, the identity

$$\|G + T \circ q\| = \|G\| + \|T \circ q\|$$

holds true. The operator $\widetilde{G}$ is the natural injectivization of G hence $\|G\| = \|\widetilde{G}\|$. It is well known that $q(\mathring{B}_X) = \mathring{B}_{X/\operatorname{Ker} G}$, where $\mathring{B}_X$ and $\mathring{B}_{X/\operatorname{Ker} G}$ are the open unit balls of X and $X/\operatorname{Ker} G$, respectively. This implies that $\|T \circ q\| = \|T\|$ and $\|\widetilde{G} + T\| = \|(\widetilde{G} + T) \circ q\| = \|G + T \circ q\|$. So we have

$$\|\widetilde{G} + T\| = \|G + T \circ q\| = \|G\| + \|T \circ q\| = \|\widetilde{G}\| + \|T\|,$$

which proves the proposition.

Lemma 3.3. *If $G_1\colon X_1 \to Y_1$ and $G_2\colon X_2 \to Y_2$ are Daugavet centers, $\|G_1\| = \|G_2\| = 1$. Then the operator $G\colon X_1 \oplus_\infty X_2 \to Y_1 \oplus_\infty Y_2$ ($G\colon X_1 \oplus_1 X_2 \to Y_1 \oplus_1 Y_2$), which maps every (x_1, x_2) into (G_1x_1, G_2x_2), is a Daugavet center.*

P r o o f. We first prove that $G\colon X_1 \oplus_\infty X_2 \to Y_1 \oplus_\infty Y_2$ is a Daugavet center. Consider $x_j^* \in X_j^*$, $y_j \in Y_j$ $(j = 1, 2)$ with $\|(y_1, y_2)\| = \max\{\|y_1\|, \|y_2\|\} = 1$, $\|(x_1^*, x_2^*)\| = \|x_1^*\| + \|x_2^*\| = 1$. Assume without loss of generality that $\|y_1\| = 1$. We will use the characterization of the Daugavet centers from item (iii) of Theorem 2.1. For a given $\varepsilon > 0$ there is an $x_1 \in X_1$ satisfying

$$\|x_1\| = 1, \quad x_1^*(x_1) \geq \|x_1^*\|(1 - \varepsilon), \quad \|G_1x_1 + y_1\| \geq 2 - \varepsilon.$$

Also, pick $x_2 \in X_2$ such that

$$\|x_2\| = 1, \quad x_2^*(x_2) \geq \|x_2^*\|(1 - \varepsilon).$$

Then $\|(x_1, x_2)\| = 1$, $\langle (x_1^*, x_2^*), (x_1, x_2)\rangle \geq 1 - \varepsilon$ and

$$\|G(x_1, x_2) + (y_1, y_2)\| \geq \|Gx_1 + y_1\| \geq 2 - \varepsilon.$$

Thus, G is a Daugavet center.

A similar calculation, based on item (v) of Theorem 2.1, proves that $G\colon X_1 \oplus_1 X_2 \to Y_1 \oplus_1 Y_2$ is a Daugavet center.

Lemma 3.4. *Let $G\colon X \to Y$ be a Daugavet center, $\|G\| = 1$. Denote $\widetilde{G}\colon X \to Y \oplus_1 Y_1$, $\widetilde{G}x = (Gx, 0)$, and $\hat{G}\colon X_1 \oplus_\infty X \to Y$, $\hat{G}(x_1, x_2) = Gx_2$. Then:*

(a) *the operator $\widetilde{G}$ is a Daugavet center;*

(b) *the operator $\hat{G}$ is a Daugavet center.*

P r o o f. Part (b) can be proved in a similar manner as Lemma 3.3, so we present only the proof of (a). Consider $x^* \in S_{X^*}$, $y_j \in Y_j$ $(j = 0, 1)$ with $\|(y_0, y_1)\| = \|y_0\| + \|y_1\| = 1$. By Theorem 2.1 there is, given $\varepsilon > 0$, some $x_0 \in S(x^*, \varepsilon)$ satisfying

$$\left\| Gx_0 + \frac{y_0}{\|y_0\|} \right\| \geq 2 - \varepsilon.$$

Then we have

$$\begin{aligned}
\|\widetilde{G}x_0 + (y_0, y_1)\| &= \|Gx_0 + y_0\| + \|y_1\| \\
&\geq \left\| Gx_0 + \frac{y_0}{\|y_0\|} + y_0\left(1 - \frac{1}{\|y_0\|}\right)\right\| + \|y_1\| \\
&\geq \left\| Gx_0 + \frac{y_0}{\|y_0\|}\right\| + \|y_0\|\left(1 - \frac{1}{\|y_0\|}\right) + \|y_1\| \\
&\geq 2 - \varepsilon
\end{aligned}$$

which proves the lemma.

Let K be a compact space without isolated points. Then $C(K)$ has the Daugavet property and this means that the identity operator is a Daugavet center. Therefore, by Proposition 3.1, every surjective linear isometry $V\colon C(K) \to C(K)$ is a Daugavet center.

In particular, if we consider any bijective continuous function $\varphi\colon K \to K$, then the operator $G_\varphi\colon C(K) \to C(K)$, $G_\varphi f = f \circ \varphi$ is a surjective linear isometry and hence a Daugavet center.

Our next aim is to prove that for every continuous function $\varphi\colon K \to K$ such that $\varphi^{-1}(t)$ is nowhere dense in K for all $t \in K$ the corresponding operator G_φ is a Daugavet center as well.

Lemma 3.5. *For an operator $G\colon X \to C(K)$, $\|G\| = 1$, the following assertions are equivalent:*

(i) *G is a Daugavet center.*

(ii) *For every $\varepsilon > 0$, every open set $U \subset K$, every $x^* \in S_{X^*}$ and $s = \pm 1$ there is $f \in S(x^*, \varepsilon)$ such that*

$$\sup_{t \in U} s \cdot (Gf)(t) > 1 - \varepsilon.$$

P r o o f. $(i) \Rightarrow (ii)$ Let us consider a function $g \in S_{C(K)}$ such that $\operatorname{supp} g \subset U$ and $s \cdot g \geq 0$. By Theorem 2.1, for every $\varepsilon > 0$ and every $x^* \in S_{X^*}$ there is an element $f \in S(x^*, \varepsilon)$ such that

$$\sup_{t \in K} |(Gf + g)(t)| > 2 - \varepsilon.$$

Notice that $|Gf + g| = |Gf| \leq 1$ on $K \setminus U$ and hence $|Gf + g|$ attains its supremum in U. Then there is a point $t_0 \in U$ which fulfills the inequality $|(Gf + g)(t_0)| > 2 - \varepsilon$. Since $s \cdot g \geq 0$, then $s \cdot (Gf)(t_0) \geq 0$ and

$$\begin{aligned}
|(Gf + g)(t_0)| &= s \cdot (Gf + g)(t_0) \leq \sup_{t \in U} s \cdot (Gf + g)(t) \\
&\leq \sup_{t \in U} s \cdot (Gf)(t) + \sup_{t \in U} s \cdot g(t) \leq \sup_{t \in U} s \cdot (Gf)(t) + 1.
\end{aligned}$$

Therefore

$$\sup_{t\in U} s\cdot (Gf)(t) > 1-\varepsilon.$$

$(ii) \Rightarrow (i)$ Consider $g \in S_{C(K)}$, pick some $\tau \in K$ with $|g(\tau)| = 1$ and put $s = g(\tau)$. Then for every $\varepsilon > 0$ there is an open neighborhood U of τ such that $s \cdot g > 1-\varepsilon$ on U. For every $x^* \in S_{X^*}$ there is an element $f \in S(x^*, \varepsilon)$ which satisfies the inequality

$$\sup_{t\in U} s\cdot (Gf)(t) > 1-\varepsilon.$$

Then we have

$$\begin{aligned}\|Gf+g\| &= \sup_{t\in K} |s\cdot (Gf+g)(t)| \ \geq\ \sup_{t\in U}(s\cdot (Gf)(t) + s\cdot g(t))\\ &\geq\ \sup_{t\in U} s\cdot (Gf)(t) + 1-\varepsilon \ \geq\ 1-\varepsilon+1-\varepsilon = 2-2\varepsilon.\end{aligned}$$

By Theorem 2.1, G is a Daugavet center.

In Lemma 3.5 consider $X = C(K_1)$. By the Riesz representation theorem, for any linear functional x^* on $C(K_1)$ there is a unique Borel regular signed measure σ on K_1 such that

$$x^*(f) = \int_{K_1} f\, d\sigma$$

for all $f \in C(K_1)$, and $\|x^*\| = |\sigma|(K_1)$. So every slice

$$\begin{aligned}S(x^*,\varepsilon) &= \{f \in B_{C(K_1)}\colon \int_{K_1} f\, d\sigma \geq |\sigma|(K_1) - \varepsilon\}\\ &= \{f \in B_{C(K_1)}\colon \int_{K_1} (1 - f\,(\mathbf{1}_{K_1^+} - \mathbf{1}_{K_1^-}))\, d|\sigma| \leq \varepsilon\}.\end{aligned}$$

Here $K_1 = K_1^+ \sqcup K_1^-$ is a Hahn decomposition of K_1 for σ, and $\mathbf{1}_A$ denotes a characteristic function of the set A.

So, in the case of $X = C(K_1)$ Lemma 3.5 can be reformulated as follows:

Lemma 3.6. *For an operator $G\colon C(K_1) \to C(K_2)$, $\|G\| = 1$, the following assertions are equivalent:*

(i) *G is a Daugavet center.*

(ii) *For every $\varepsilon > 0$, every open set $U \subset K_2$ and every Borel regular signed measure σ on K_1 and $s = \pm 1$ there is an $f \in B_{C(K_1)}$ such that*

$$\int_{K_1} (1 - f\,(\mathbf{1}_{K_1^+} - \mathbf{1}_{K_1^-}))\, d|\sigma| \leq \varepsilon \tag{3.8}$$

and

$$\sup_{t\in U} s\cdot (Gf)(t) > 1-\varepsilon. \tag{3.9}$$

Theorem 3.7. *Let K_1 and K_2 be compact spaces without isolated points, φ: $K_2 \to K_1$ be a continuous function such that for every $t \in K_1$ the set $\varphi^{-1}(t)$ is nowhere dense in K_2. Suppose that an operator G_φ: $C(K_1) \to C(K_2)$ maps every $f \in C(K_1)$ into the composition $f \circ \varphi$. Then G_φ is a Daugavet center.*

P r o o f. Consider an $\varepsilon > 0$, an open set $U \subset K_2$, and a Borel regular signed measure σ on K_1, and put $s = 1$. We will construct a function $f \in B_{C(K_1)}$ satisfying (3.8) and (3.9).

The measure σ can have at most countable set of atoms. Let us show that for every open $U \subset K_2$ the set $\varphi(U)$ is uncountable. Assume that there exists an open set $U \subset K_2$ for which it is not true. Then $\varphi^{-1}(\varphi(U))$ is a countable union of nowhere dense sets in K_2 because for every $t \in \varphi(U) \subset K_1$ the set $\varphi^{-1}(t)$ is nowhere dense in K_2 by the condition of this theorem. This contradicts the Baire category theorem.

So, we can pick a point $t_0 \in U$ such that $\varphi(t_0)$ is not an atom of σ, i.e., $|\sigma|(\varphi(t_0)) = 0$. Moreover, since σ is a Borel regular measure, there is an open neighborhood $V \subset \varphi(U)$ of the point $\varphi(t_0)$ such that $|\sigma|(V) < \varepsilon/4$.

Now we pass on to the construction of f. To satisfy (3.9) we select f in such a way that $f(\varphi(t_0)) > 1-\varepsilon$. First, we pick a function $\tilde f \in S(\sigma, \varepsilon/2)$. If $\tilde f(\varphi(t_0)) > 1-\varepsilon$, then we can simply put $f = \tilde f$.

If $\tilde f(\varphi(t_0)) \le 1-\varepsilon$, we put $f = \tilde f$ in $K_1 \setminus V$ and $f(\varphi(t_0)) = 1$. Since $K_1 \setminus V \cup \{\varphi(t_0)\}$ is closed, we can use the Tietze extension theorem to construct a continuous extension f on $V \setminus \varphi(t_0)$ and keep the condition $\|f\| = 1$. Now we show that (3.8) also holds for this f:

$$\begin{aligned}
\int_{K_1} (1 - f(\mathbf{1}_{K_1^+} - \mathbf{1}_{K_1^-}))\, d|\sigma| &= \int_{K_1\setminus V} (1 - \tilde f(\mathbf{1}_{K_1^+} - \mathbf{1}_{K_1^-}))\, d|\sigma| \\
&+ \int_V (1 - f(\mathbf{1}_{K_1^+} - \mathbf{1}_{K_1^-}))\, d|\sigma| \\
&\le \int_{K_1} (1 - \tilde f(\mathbf{1}_{K_1^+} - \mathbf{1}_{K_1^-}))\, d|\sigma| + \varepsilon/2 \\
&\le \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

So, for every $\varepsilon > 0$, every Borel regular measure σ on K_1, every open set $U \subset K_2$ and $s = 1$ we have a function $f \in B_{C(K_1)}$ satisfying inequalities (3.8) and (3.9). The case $s = -1$ can be proved in a very similar way. Thus, by Lemma 3.6, G_φ is a Daugavet center.

Let us give an example of a Daugavet center on $C(K)$ of a very different nature.

Proposition 3.8. *Consider* $K_1 = [-1; 1]$ *and define* $G\colon C(K_1) \to C(K_1)$ *as* $(Gf)(x) = \frac{f(x)+f(-x)}{2}$. *Then* G *is a Daugavet center.*

P r o o f. We will use Lemma 3.6 to prove this proposition. Let us fix an $\varepsilon > 0$, a Borel regular signed measure σ, an open set $U \subset K_1$, $s = 1$ and a function $\tilde{f} \in S(\sigma, \varepsilon/2)$. If there is a point $t_0 \in U$ such that $\frac{\tilde{f}(t_0)+\tilde{f}(-t_0)}{2} > 1 - \varepsilon$, then

$$\sup_{t \in U} s \cdot (G\tilde{f})(t) > 1 - \varepsilon.$$

Otherwise we pick a point $t_1 \in U$ such that neither t_1 nor $-t_1$ is an atom of σ. Consider disjoint segments $[a_1, b_1], [a_2, b_2] \subset K_1$ such that $|\sigma|([a_1, b_1]) < \varepsilon/8$, $t_1 \in [a_1, b_1]$ and $|\sigma|([a_2, b_2]) < \varepsilon/8$, $-t_1 \in [a_2, b_2]$. Let $\tilde{f}_1\colon [a_1, b_1] \to K_1$ be a continuous function such that $\tilde{f}_1(a_1) = \tilde{f}(a_1)$, $\tilde{f}_1(b_1) = \tilde{f}(b_1)$ and $\tilde{f}_1(t_1) = 1$. Let $\tilde{f}_2\colon [a_2, b_2] \to K_1$ be a continuous function such that $\tilde{f}_2(a_2) = \tilde{f}(a_2)$, $\tilde{f}_2(b_2) = \tilde{f}(b_2)$ and $\tilde{f}_2(-t_1) = 1$. Then denote $\Delta := K_1 \setminus \{[a_1, b_1] \cup [a_2, b_2]\}$, put

$$f := \mathbf{1}_{[a_1,b_1]}\tilde{f}_1 + \mathbf{1}_{[a_2,b_2]}\tilde{f}_2 + \mathbf{1}_\Delta \tilde{f}.$$

Then $f \in B_{C(K_1)}$ and

$$\begin{aligned} \int_{K_1} (1 - f\,(\mathbf{1}_{K_1^+} - \mathbf{1}_{K_1^-}))\,d|\sigma| &= \int_{\Delta} (1 - \tilde{f}\,(\mathbf{1}_{K_1^+} - \mathbf{1}_{K_1^-}))\,d|\sigma| \\ + \int_{[a_1,b_1]} (1 - \tilde{f}_1\,(\mathbf{1}_{K_1^+} - \mathbf{1}_{K_1^-}))\,d|\sigma| &+ \int_{[a_2,b_2]} (1 - \tilde{f}_2\,(\mathbf{1}_{K_1^+} - \mathbf{1}_{K_1^-}))\,d|\sigma| \\ \le \int_{K_1} (1 - \tilde{f}\,(\mathbf{1}_{K_1^+} - \mathbf{1}_{K_1^-}))\,d|\sigma| &+ \varepsilon/4 + \varepsilon/4 \le \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $f \in S(\sigma, \varepsilon)$ and

$$\sup_{t \in U} s \cdot (Gf)(t) \ge \frac{f(t_1) + f(-t_1)}{2} = 1.$$

If we put $s = -1$, the analogous conclusions prove the proposition.

A rather nontrivial class of the Daugavet centers was discovered in [9], where every isometric embedding $G\colon L_1[0,1] \to L_1[0,1]$ was a Daugavet center. Let us show that the analogous result for $C[0,1]$ is false. This will answer in negative a question from [10].

E x a m p l e. *Consider* $T\colon C[0,1] \to C[0,1]$,

$$Tf = \begin{cases} f(2t) & \text{if} \quad t \in \left[0, \frac{1}{2}\right], \\ 2f(1)(1-t) & \text{if} \quad t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Let us prove that T *is an isometric embedding. It is obvious that* T *is a linear operator. Notice that* $|Tf|$ *attains its supremum in* $\left[0, \frac{1}{2}\right]$ *because for every* $t \in \left(\frac{1}{2}, 1\right]$ *we have* $|Tf(t)| = |2f(1)(1-t)| < |f(1)| = |Tf\left(\frac{1}{2}\right)|$. *Hence for every* $f \in C[0,1]$

$$\|Tf\| = \sup_{t \in \left[0, \frac{1}{2}\right]} |Tf(t)| = \sup_{t \in [0,1]} |f(t)| = \|f\|.$$

Now we show with the help of Lemma 3.5 that T *is not a Daugavet center. Our aim is to find an* $\varepsilon > 0$, *an open set* $U \subset [0,1]$ *and an* $x^* \in S_{C^*[0,1]}$ *such that every* $f \in S(x^*, \varepsilon)$ *satisfies* $\sup_{t \in U} Tf(t) \le 1 - \varepsilon$. *If we put* $\varepsilon := \frac{1}{4}$ *and* $U := \left(\frac{3}{4}, 1\right]$, *then for every* $f \in B_{C[0,1]}$ *we have*

$$\sup_{t \in U} Tf(t) = \sup_{t \in \left(\frac{3}{4}, 1\right]} 2f(1)(1-t) \le 2|f(1)|\left(1 - \frac{3}{4}\right) = \frac{|f(1)|}{2} \le \frac{1}{2} < 1 - \varepsilon.$$

4. The Main Result

Definition 4.1. *Let* E *be a seminormed space,* $A \subset B_E$, $\mathcal{U}$ *be a free ultrafilter on a set* Γ, *and* $f : \Gamma \to A$ *be a function. The triple* $(\Gamma, \mathcal{U}, f)$ *is said to be an* A*-valued* E*-atom if for every* $w \in E$

$$\lim_{\mathcal{U}} \|f + w\| = 1 + \|w\|. \tag{4.1}$$

The following characterization of Daugavet centers is a consequence of Theorem 2.1 and Lemma 2.11.

Theorem 4.2. *Let* X, Y *be Banach spaces. An operator* $G \in S_{L(X,Y)}$ *is a Daugavet center if and only if for every slice* S *of* B_X *there is a* $G(S)$*-valued* Y*-atom.*

P r o o f. Let us start with the "if" part. We are going to prove that G satisfies condition (iii) of Theorem 2.1. Fix $y_0 \in S_Y$, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$. Denote $S = S(x_0^*, \varepsilon)$. Due to our assumption there is a $G(S)$-valued Y-atom $(\Gamma, \mathcal{U}, f)$. Plugging $w = y_0$ in (4.1), we get, in particular, that $\|f(t) + y_0\| > 2 - \varepsilon$ for some $t \in \Gamma$. Since $f(t) \in G(S)$, there is an $x \in S$ such that $f(t) = Gx$. This x fulfills the required conditions $x_0^*(x) \ge 1 - \varepsilon$ and $\|Gx + y_0\| > 2 - \varepsilon$. The "if" part is proved.

Let us demonstrate the "only if" part. Fix a slice S of B_X. Put $\Gamma = \mathrm{FIN}(Y)$ and take the natural filter $\mathcal{F}$ on Γ whose base is formed by the collection of subsets $\hat{A} \subset \mathrm{FIN}(Y)$, $A \in \mathrm{FIN}(Y)$, where $\hat{A} := \{B \in \mathrm{FIN}(Y)\colon A \subset B\}$. According to Lemma 2.11 for every $A \in \mathrm{FIN}(Y)$ there is an element $x(A) \in S$ such that for all $y \in A$

$$\|y + G(x(A))\| > \left(1 - \frac{1}{|A|}\right)(\|y\| + 1).$$

Define $f(A) := G(x(A))$. It is easy to see that for every ultrafilter $\mathcal{U} \succ \mathcal{F}$ the triple $(\Gamma, \mathcal{U}, f)$ is the required $G(S)$-valued Y-atom.

It is clear that if $A \subset B$, then every A-valued E-atom is at the same time a B-valued E-atom. A B_E-valued E-atom will be called just the E-atom.

Lemma 4.3. *Let (E, p) be a seminormed space, Y be a subspace of E, and $(\Gamma, \mathcal{U}, f)$ be a Y-atom. Define*

$$p_r(x) = \mathcal{U}\text{-}\lim_t p(x + rf(t)) - r$$

for $x \in E$ and $r > 0$. Then:

(a) $0 \le p_r(x) \le p(x)$ *for all $x \in E$,*
(b) $p_r(y) = p(y)$ *for all $y \in Y$,*
(c) $x \mapsto p_r(x)$ *is convex for each r,*
(d) $r \mapsto p_r(x)$ *is convex for each x,*
(e) $p_r(tx) = tp_{r/t}(x)$ *for each $t > 0$.*

P r o o f. The only thing that is not obvious is that $p_r \ge 0$; note that (b) is just the definition of Y-atom. Now, given $\varepsilon > 0$, pick t_ε such that $p(f(t_\varepsilon)) > 1 - \varepsilon$, and

$$p(x + rf(t_\varepsilon)) \le \mathcal{U}\text{-}\lim_t p(x + rf(t)) + \varepsilon.$$

Then

$$\begin{aligned}
\mathcal{U}\text{-}\lim_t p(x + rf(t)) &\ge \mathcal{U}\text{-}\lim_t p(-rf(t_\varepsilon) + rf(t)) - p(x + rf(t_\varepsilon)) \\
&= rp(f(t_\varepsilon)) + r - p(x + rf(t_\varepsilon)) \\
&\ge 2r - r\varepsilon - \mathcal{U}\text{-}\lim_t p(x + rf(t)) - \varepsilon;
\end{aligned}$$

hence $\mathcal{U}\text{-}\lim_t p(x + rf(t)) \ge \frac{1}{2}(2r - \varepsilon - r\varepsilon)$ and $p_r(x) \ge 0$.

Lemma 4.4. *Assume the conditions of Lemma 4.3. Then $r \mapsto p_r(x)$ is decreasing for each x. The quantity*

$$\bar{p}(x) := \lim_{r \to \infty} p_r(x) = \inf_r p_r(x)$$

satisfies (a)–(c) of Lemma 4.3 and, moreover,

$$\bar{p}(tx) = t\bar{p}(x) \qquad \text{for } t > 0,\ x \in X. \tag{4.2}$$

P r o o f. By Lemma 4.3(a) and (d), $r \mapsto p_r(x)$ is bounded and convex, hence decreasing. Therefore, $\bar{p}$ is well defined. Clearly, (4.2) follows from (e) above.

P r o o f of the main theorem (Theorem 1.3). Let $G: X \to Y$ be a Daugavet center, Y be a subspace of a Banach space E, and $J : Y \to E$ be the natural embedding operator.

Let $\mathcal{P}$ be the family of all seminorms q on E that are dominated by the norm of E and for which $q(y) = \|y\|$ for $y \in Y$. By Zorn's lemma, $\mathcal{P}$ contains a minimal element, say, p.

Claim. *Every Y-atom $(\Gamma, \mathcal{U}, f)$ is at the same time an (E, p)-atom, i.e., for every $w \in E$*

$$\lim_{\mathcal{U}} p(f + w) = 1 + p(w). \tag{4.3}$$

P r o o f. To prove the claim, associate the functional $\bar{p}$ from Lemma 4.4 to p and $(\Gamma, \mathcal{U}, f)$. Notice that $0 \le \bar{p} \le p$, but $\bar{p}$ need not be a seminorm. However,

$$q(x) = \frac{\bar{p}(x) + \bar{p}(-x)}{2}$$

defines a seminorm, and $q \le p$. By Lemma 4.3(b) and by minimality of p, we get that

$$q(x) = p(x) \qquad \forall x \in X. \tag{4.4}$$

Now, since $p(x) \ge \bar{p}(x)$ and $p(x) = p(-x) \ge \bar{p}(-x)$, (4.4) implies that $p(x) = \bar{p}(x)$. Finally, by Lemma 4.3(a) and the definition of $\bar{p}$, we have $p(x) = p_r(x)$ for all $r > 0$; in particular $p(x) = p_1(x)$, which is our claim (4.3).

Now let us introduce a new norm on E as

$$|||x||| := p(x) + \|[x]\|_{E/Y};$$

and let us show that this is the equivalent norm that we need. Indeed, clearly $|||x||| \le 2\|x\|$. On the other hand, $|||x||| \ge \frac{1}{3}\|x\|$. To see this assume $\|x\| = 1$. If $\|[x]\|_{E/Y} \ge \frac{1}{3}$, there is nothing to prove. If not, pick $y \in Y$ such that $\|x - y\| < \frac{1}{3}$. Then $p(y) = \|y\| > \frac{2}{3}$, and

$$|||x||| \ge p(x) \ge p(y) - p(x - y) > \frac{2}{3} - \|x - y\| > \frac{1}{3}.$$

Therefore, $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms. Also evidently for $y \in Y$

$$|||y||| = p(y) = \|y\|.$$

What remains to prove is that $J \circ G : X \to E$ is a Daugavet center. This can be done easily with the help of Theorem 4.2 and Claim. Namely, let S be an arbitrary slice of B_X. Due to Theorem 4.2 it is sufficient to demonstrate the existence of a $G(S)$-valued $(E, |||\cdot|||)$-atom. Since $G : X \to Y$ is a Daugavet center, the same Theorem 4.2 ensures the existence of a $G(S)$-valued Y-atom $(\Gamma, \mathcal{U}, f)$. But according to Claim, $(\Gamma, \mathcal{U}, f)$ is also an (E, p)-atom. Consequently, for every $w \in E$

$$\begin{aligned} \lim_{\mathcal{U}} |||f + w||| &= \lim_{\mathcal{U}} \left(p(f + w) + \|[f + w]\|_{E/Y}\right) \\ &= \lim_{\mathcal{U}} p(f + w) + \|[w]\|_{E/Y} = 1 + p(w) + \|[w]\|_{E/Y} = 1 + |||w|||. \end{aligned}$$

This means that $(\Gamma, \mathcal{U}, f)$ is the required $G(S)$-valued $(E, |||\cdot|||)$-atom. **The main theorem is proved**. The same renorming idea is applicable to the theory of ℓ_1-types [5].

The next corollary improves the statement of Remark 2.10.

Corollary 4.5. *If G: $X \to Y$ is a nonzero Daugavet center, then neither X nor Y can be embedded into a space E, in which the identity operator Id_E has a representation as a pointwise unconditionally convergent series of weakly compact operators. In particular, neither X nor Y can be embedded into a space E having an unconditional basis (countable or uncountable) or having a representation as unconditional sum of reflexive subspaces.*

P r o o f. Let $\mathrm{Id}_E = \sum_{n\in\Gamma} T_n$, where the series is pointwise unconditionally convergent, and all the T_n: $E \to E$ are weakly compact. At first, assume $Y \subset E$, and denote $J \in L(Y, E)$ the natural embedding operator. Equip E with the equivalent norm from Theorem 1.3 making $J \circ G$ a Daugavet center. Then $J \circ G = \sum_{n\in\Gamma} T_n \circ J \circ G$, the series is pointwise unconditionally convergent, and all the operators $T_n \circ J \circ G$ are weakly compact. This contradicts Theorem 2.9.

Now assume $X \subset E$. Recall that for a set Δ of big cardinality (say, for $\Delta = B_{Y^*}$), there is an isometric embedding J: $Y \to \ell_\infty(\Delta)$. Since $\ell_\infty(\Delta)$ is an injective space (i.e., the Hahn–Banach extension theorem holds true for $\ell_\infty(\Delta)$-valued operators), there is an operator U: $E \to \ell_\infty(\Delta)$ such that $U|_X = J \circ G$. Then

$$J \circ G = (U \circ \mathrm{Id}_E)|_X = \sum_{n\in\Gamma} U \circ (T_n)|_X.$$

This representation leads to contradiction in the same way as in the previous case.

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