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Isometric Expansions of Quantum Algebra of Linear Bounded Operators

V.A. Zolotarev

Department of Mechanics and Mathematics, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61077, Ukraine
E-mail:Vladimir.A.Zolotarev@univer.kharkov.ua

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The isometric expansion $\left\{V_s, \overset{+}{V}_s\right\}_{s=1}^{2}$ for a quantum algebra of linear bounded nonunitary operators $\{T_1, T_2\}$, which is given by commutative relation $T_1T_2 = qT_2T_1$ ($|q| = 1$), is constructed. Basic properties of characteristic function $S(z)$ corresponding to isometric expansion $\left\{V_s, \overset{+}{V}_s\right\}_{s=1}^{2}$ for given quantum algebra $\{T_1, T_2\}$ are described.

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Spectral decompositions of the selfadjoint operators have its origin in quantum mechanics and are among the major achievements of the functional analysis. They play the key role in many fields of mathematics and physics. It is common to consider functional models [3, 4] as an analogue of such spectral decompositions for non-selfadjoint and nonunitary operators. Construction of these models is based on the theory of isometric and unitary expansions (dilations) of the operator semigroups. Given field of analysis is well investigated and has a number of nontrivial and substantial applications. Solution of the problem of construction of the spectral decomposition for the linear selfadjoint operator systems $\{A_k\}_1^n$ satisfying certain commutative relations (e.g. that of the Lie algebra), that also emerged in quantum mechanics, stimulated mutually enriching development of different fields of mathematics, [1]. For the commutative systems of linear non-selfadjoint operators $\{A_k\}_1^n$, the construction of the functional models is based [5, 6] on the basic idea of M.S. Livšic [5] who showed that construction of the isometric (unitary) expansions of the semigroups for such operator systems is based on the study of the consistency conditions for the systems of differential equations. Given consistency conditions signify closedness of the differential forms and are written in terms of the external parameters of the expansion. The author [7] has successfully generalized this approach for the Lie algebras of the linear non-selfadjoint operators $\{A_k\}_1^n$. This fact led to the problems of the harmonic analysis on the Lie groups and also to the noncommutative Lax–Fillips scattering scheme on groups. At the same time, the important connections with theory of functions on the Riemann surface have been established.

The formation [8] of the similar constructions for the commutative systems of linear operators $\{T_k\}_1^n$ that are close to the unitary ones (e.g. contractions) allowed to find the constructive approach to the solution of the problem (which is more than 30 years old) of the construction of the unitary dilation and of the corresponding functional model for the commutative system of the contractive operators$\{T_k\}_1^n$, [3]. The search for the sensible discrete noncommutative analogue of the Lie algebras for the operator systems $\{T_k\}_1^n$ that are close to the unitary ones has led the author to the quantum operator algebras [10, 11]. In this work, the simplest quantum algebra of the linear operators $\{T_1, T_2\}$, that represent the so-called Rieffel torus [9–11], the commutative relation for which is $T_1T_2 = qT_2T_1$, $q \in \mathbb{C}$, $|q| = 1$, is studied. Really, if $q = e^{ih}$, $h \in \mathbb{R}$, and $T_2 = \exp\{hA_2\}$ then the corresponding classical limit process in the relation $T_1T_2 = qT_2T_1$ when $q \to 1$, $h \to 0$, leads us to the Lie algebra of the linear operators $[A_1, A_2] = iA_1$ ($T_1 \stackrel{def}{=} A_1$) which represents the Lie algebra of affine line transformations [7]. Thus, the quantum algebra $\{T_1, T_2\}$, $T_1T_2 = qT_2T_1$ is nothing else than the quantification of the operator Lie algebra $\{A_1, A_2\}$, $[A_1, A_2] = iA_1$ [11]. In this work following the constructions [8], the construction of the isometric expansion $\left\{V_s, \overset{+}{V}_s\right\}_{s=1}^2$ for the quantum algebra of linear bounded nonunitary operators $\{T_1, T_2\}$ given by the relation $T_1T_2 = qT_2T_1$ is offered. In Part 3, the totality of the invariants of the given quantum operator algebra is presented.

1. The Preliminary Information

I. An arbitrary linear bounded operator T acting in the Hilbert space H has the isometric (in the indefinite, generally speaking, metric) expansion [3, 4, 8]. This fact means that there exist the Hilbert spaces E and $\tilde{E}$ and such operators $\Phi : E \to H$, $\Psi : H \to \tilde{E}$, $K : E \to \tilde{E}$ that the expansion operator

$$V_T = \begin{bmatrix} T & \Phi \\ \Psi & K \end{bmatrix} : H \oplus E \to H \oplus \tilde{E} \tag{1}$$

has properties

$$V_T^* \begin{bmatrix} I & 0 \\ 0 & \tilde{\sigma} \end{bmatrix} V_T = \begin{bmatrix} I & 0 \\ 0 & \sigma \end{bmatrix}; \quad V_T \begin{bmatrix} I & 0 \\ 0 & \sigma^{-1} \end{bmatrix} V_T^* = \begin{bmatrix} I & 0 \\ 0 & \tilde{\sigma}^{-1} \end{bmatrix}, \tag{2}$$

where σ and $\tilde{\sigma}$ are the selfadjoint boundedly invertible operators in E and $\tilde{E}$ respectively. The expansion V_T (1) is built by the operator T in an ambiguous way. Consider one of the methods of the construction of such an expansion which we will need later on [4, 8]. Let $D = T^*T - I$ and $\tilde{D} = TT^* - I$ be the deficient operators [3, 4, 8] corresponding to the operator T, and $E = \overline{\tilde{D}H}$, $\tilde{E} = \overline{DH}$ be respective deficient subspaces. Define operators: $\Phi = \tilde{D} : E \to H$; $\Psi = P_{\tilde{E}}$: $H \to \tilde{E}$ is an orthoprojector in H on $\tilde{E}$; $K = T^*$: $E \to \tilde{E}$; and, finally, $\sigma = -\tilde{D}$, $\tilde{\sigma} = -D$, [4]. It is easy to see that the operator V_T (1) constructed in such a manner will satisfy relations (2).

II. Denote by $h_n \in H$, $u_n \in E$, $v_n \in \tilde{E}$ the vector-functions of discrete argument $n \in \mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$ in the respective Hilbert spaces. Further, consider the equation system which is commonly [4] known as the open expansion system V_T (1),

$$\left\{ \begin{array}{ll} h_{n+1} = Th_n + \Phi u_n; & h_0 = h; \\ v_n = \Psi h_n + K u_n; & n \in \mathbb{Z}_+; \end{array} \right. \qquad V_T \left[\begin{array}{c} h_n \\ u_n \end{array} \right] = \left[\begin{array}{c} h_{n+1} \\ v_n \end{array} \right]. \tag{3}$$

The following conservation law results from the first relation in (2):

$$\|h_n\|^2 + \langle \sigma u_n, u_n \rangle = \|h_{n+1}\|^2 + \langle \tilde{\sigma} v_n, v_n \rangle . \tag{4}$$

Note that if $u_n \equiv 0$ then h_n is generated by the semigroup $T_n = T^n$ of the discrete argument $n \in \mathbb{Z}_+$, i.e. $h_n = T_n h$ and $v_n = \Psi T_n h$.

Now consider the vector-functions $\tilde{h}_n \in H$, $\tilde{u}_n \in E$, $\tilde{v}_n \in \tilde{E}$ of the argument $n \in \mathbb{Z}_- = \{n \in \mathbb{Z} : n < 0\}$ and specify the dual open system (regarding (3)) that is generated by the operator V_T^*,

$$\left\{ \begin{array}{ll} \tilde{h}_{n-1} = T^*\tilde{h}_n + \Psi^* \tilde{v}_n; & \tilde{h}_{-1} = \tilde{h}; \\ \tilde{u}_n = \Phi^* \tilde{h}_n + K^* \tilde{v}_n; & n \in \mathbb{Z}_-; \end{array} \right. \qquad V_T^* \left[\begin{array}{c} \tilde{h}_n \\ \tilde{v}_n \end{array} \right] = \left[\begin{array}{c} \tilde{h}_{n-1} \\ \tilde{u}_n \end{array} \right]. \tag{5}$$

Then, if $\tilde{v}_n \equiv 0$ then the vector-function $\tilde{h}_{n-1}$ has the form of $\tilde{h}_{n-1} = T^*_{|n|}\tilde{h}$ and is generated by the semigroup $T^*_{|n|} = (T^*)^{|n|}$ where $(-n) \in \mathbb{Z}_+$ and $\tilde{u}_{n-1} = \Phi^* T^*_{|n|}\tilde{h}$. The conservation law for system (5) has the former type (4) if $h_n = \tilde{h}_{-n-1}$, $v_n = \tilde{\sigma}^{-1} v_{-n-1}$, $u_n = \sigma^{-1}\tilde{u}_{-n-1}$ $(n \in \mathbb{Z}_+)$.

Let $u_n = z^n u_0$ $(z \in \mathbb{C}$, $u_0 \in E)$ and assume that h_n and v_n depends on $n \in \mathbb{Z}_+$, similarly $h_n = z^n h_0$, $v_n = z^n v_0$. Then from equations (3) for the open system, we obtain that $h_0 = (zI - T_1)^{-1} \Phi u_0$, $v_0 = S(z) u_0$, where

$$S(z) = S_\Delta(z) = K + \Psi (zI - T)^{-1} \Phi \tag{6}$$

is the characteristic function of M.S. Livšic of the expansion V_T (1) [4, 8], which is defined for all z outside the spectrum of operator T.

Study of the isometric expansions V_T (1) of the open systems (3), (5) plays the fundamental role in the construction of the unitary dilation U of the contraction T ($\|T\| \leq 1$) [3, 4], and also in the construction of the functional models U and T [3, 4]. Sensible generalization of the constructions mentioned above on the case of the commutative systems of linear nonunitary operators $\{T_k\}_1^n$, $[T_k, T_s] = 0$, $1 \leq k$, $s \leq n$, was proposed in work [8]. So, in [8] the commutative isometric expansion for the system of the commutative linear operators has been built. It turned out that the construction of the isometric expansion stated in [8] has natural generalization on the quantum algebra of linear operators (the Rieffel torus).

2. Isometric Expansions and Open Systems

I. In the Hilbert space H, consider the system of the linear bounded operators $\{T_1, T_2\}$ satisfying the relation

$$T_1 T_2 = q T_2 T_1, \quad q = e^{ih}, \quad h \in \mathbb{R}. \tag{7}$$

In the same way as for the commutative operator systems $q = 1$ [8], define the isometric expansion corresponding to the case of $q \neq 1$ that generalizes the concept of the expansion V_T (1).

Definition 1. *The totality of the mappings*

$$\begin{aligned} V_s &= \begin{bmatrix} T_s & \Phi N_s \\ \Psi & K \end{bmatrix} : \quad H \oplus E \to H \oplus \tilde{E}, \quad s = 1, 2; \\ \overset{+}{V}_s &= \begin{bmatrix} T_s^* & \Psi^* \tilde{N}_s^* \\ \Phi^* & K^* \end{bmatrix} : \quad H \oplus \tilde{E} \to H \oplus E, \quad s = 1, 2, \end{aligned} \tag{8}$$

is said to be the isometric expansion of the system of linear bounded operators $\{T_1, T_2\}$ in H satisfying (7) if there are such operators σ_s, τ_s, N_s, Γ and $\tilde{\sigma}_s$, $\tilde{\tau}_s$, $\tilde{N}_s$, $\tilde{\Gamma}$ in the Hilbert spaces E and $\tilde{E}$ respectively (σ_s, τ_s and $\tilde{\sigma}_s$, $\tilde{\tau}_s$ are selfadjoint ($s = 1, 2$)) that equalities

$$\begin{aligned} &1) \quad V_s^* \begin{bmatrix} I & 0 \\ 0 & \tilde{\sigma}_s \end{bmatrix} V_s = \begin{bmatrix} I & 0 \\ 0 & \sigma_s \end{bmatrix}, \quad s = 1, 2; \\ &2) \quad \overset{+}{V}_s^* \begin{bmatrix} I & 0 \\ 0 & \tau_s \end{bmatrix} \overset{+}{V}_s = \begin{bmatrix} I & 0 \\ 0 & \tau_s \end{bmatrix}, \quad s = 1, 2; \\ &3) \quad q T_2 \Phi N_1 - T_1 \Phi N_2 = q \Phi \Gamma; \quad \tilde{N}_1 \Psi T_2 - q \tilde{N}_2 \Psi T_1 = q \tilde{\Gamma} \Psi; \\ &4) \quad q \tilde{N}_2 \Psi \Phi N_1 - \tilde{N}_1 \Psi \Phi N_2 = q K \Gamma - q \tilde{\Gamma} K; \\ &5) \quad K N_s = \tilde{N}_s K, \quad s = 1, 2, \end{aligned} \tag{9}$$

are true.

First of all, show that there exists such an expansion V_s, $\overset{+}{V_s}$ (8) for every operator system $\{T_1, T_2\}$ satisfying (7). Let $D_s = T_s^* T_s - I$ and $\tilde{D}_s = T_s T_s^* - I$ be the defect operators ($s = 1, 2$). Consider the operators

$$\begin{gathered}
N_1 = \tilde{D}_1 T_2^*; \quad N_2 = q\tilde{D}_2 T_1^*; \quad \tilde{N}_1 = qT_2^* D_1; \quad \tilde{N}_2 = T_1^* D_2; \\
\tilde{\sigma}_1 = -D_1; \quad \tilde{\sigma}_2 = -D_2; \quad \sigma_1 = -T_2 \tilde{D}_1 T_2^*; \quad \sigma_2 = -T_1 \tilde{D}_2 T_1^*; \\
\tau_1 = -\tilde{D}_1; \quad \tau_2 = -\tilde{D}_2; \quad \tilde{\tau}_1 = -T_2^* D_1 T_2; \quad \tilde{\tau}_2 = -T_1^* D_2 T_1; \\
\Gamma = \tilde{D}_1 - \tilde{D}_2; \quad \tilde{\Gamma} = D_1 - D_2; \quad K = T_1^* T_2^*.
\end{gathered} \tag{10}$$

Further, define the Hilbert spaces

$$\begin{aligned}
E &= \operatorname{span}\left\{\tilde{D}_k H + N_s^* H : k, s = 1, 2\right\}; \\
\tilde{E} &= \operatorname{span}\left\{D_k H + \tilde{N}_s H : k, s = 1, 2\right\}.
\end{aligned} \tag{11}$$

It is easy to see that the operators

$$V_s = \begin{bmatrix} T_s & P_E N_s \\ P_{\tilde{E}} & K \end{bmatrix}; \quad \overset{+}{V_S} = \begin{bmatrix} T_s^* & P_{\tilde{E}} \tilde{N}_s^* \\ P_E & K^* \end{bmatrix}, \quad s = 1, 2, \tag{12}$$

satisfy the conditions 1), 2) (9). Relations 3) (9) follow from the identities

$$T_2 \tilde{D}_1 T_2^* - T_1 \tilde{D}_2 T_1^* = \tilde{D}_1 - \tilde{D}_2; \quad T_2^* D_1 T_2 - T_1^* D_2 T_1 = D_1 - D_2, \tag{13}$$

that are easily verified if one takes into account (7) and the fact that $|q| = 1$. Relations 4), 5) in (9) are verified directly in view of equality (7). The inclusion $KE \subseteq \tilde{E}$ is an obvious corollary of the relations

$$T_1^* T_2^* \tilde{D}_1 H = qT_2^* D_1 T_1^* H \subset \tilde{N}_1 T_1^* H \subseteq \tilde{E};$$

$$T_1^* T_2^* N_1^* H = D_1 T_1^* H + \tilde{N}_2 \tilde{D}_1 H \subseteq E$$

and of relations similar to them.

O b s e r v a t i o n 1. The external parameters $\{\sigma_s, \tau_s, N_s, \Gamma\}$ in E and $\left\{\tilde{\sigma}_s, \tilde{\tau}_s, \tilde{N}_s, \tilde{\Gamma}\right\}$ in $\tilde{E}$ of the expansion V_s, $\overset{+}{V_s}$ (8) are not independent. So from (10) and (13), it follows that

$$\sigma_2 - \sigma_1 = \tau_2 - \tau_1 = \Gamma; \quad \tilde{\sigma}_2 - \tilde{\sigma}_1 = \tilde{\tau}_2 - \tilde{\tau}_1 = \tilde{\Gamma}.$$

From relation 1) (9) and from the presentation of V_s, $\overset{+}{V_s}$ (8), it follows that, in the case of invertibility of σ_s and $\tilde{\sigma}_s$, operators τ_s and $\tilde{\tau}_s$ have the form of

$\tau_s = N_s \sigma_s^{-1} N_s^*$ and $\tilde{\tau}_s = \tilde{N}_s \tilde{\sigma}_s^{-1} \tilde{N}_s^*$, $s = 1, 2$, but, as can be seen from (10), the invertibility of the operators σ_s and $\tilde{\sigma}_s$, generally speaking, is not taking place.

O b s e r v a t i o n 2. It is obvious that the following relation is true:

$$\begin{bmatrix} I & 0 \\ 0 & \tilde{N}_s \end{bmatrix} V_s = \overset{+}{V_s^*} \begin{bmatrix} I & 0 \\ 0 & N_s \end{bmatrix}, \quad s = 1,\, 2, \tag{14}$$

for the operators V_s and $\overset{+}{V_s}$ (8).

II. Define the vector-functions of discrete argument $h_n \in H$, $u_n \in E$, $v_n \in \tilde{E}$ at the points of integer-valued grid $n = (n_1, n_2) \in \mathbb{Z}_+^2 = \mathbb{Z}_+ \times \mathbb{Z}_+$. Consider the differential linear operators ∂_1 ш ∂_2 that on the vector-function h_n act in the following way:

$$\partial_1 h_n = h_{(n_1+1, n_2)}, \quad \partial_2 h_n = q^{-n_1} h_{(n_1, n_2+1)}, \tag{15}$$

besides it is obvious that

$$q \partial_1 \partial_2 = \partial_2 \partial_1. \tag{16}$$

Further, define the analogue with two variables of the open system (3)

$$\left\{ \begin{array}{ll} \partial_1 h_n = T_1 h_n + \Phi N_1 u_n; & h_{(0,0)} = h_0; \\ \partial_2 h_n = T_2 h_n + \Phi N_2 u_n; & n \in \mathbb{Z}_+^2; \\ \quad v_n = \Psi h_n + K u_n; & \end{array} \right. \qquad V_s \begin{bmatrix} h_n \\ u_n \end{bmatrix} = \begin{bmatrix} \partial_s h_n \\ v_n \end{bmatrix}, \ s = 1, 2, \tag{17}$$

where $\{\partial_1, \partial_2\}$ have the form of (15). Using relations (7) and (16), we obtain the following statement.

Theorem 1. *The equation system (17) is consistent if the vector-function u_n is the solution of the equation*

$$\{q N_2 \partial_1 - N_1 \partial_2 + q\Gamma\} u_n = 0. \tag{18}$$

The proof of the theorem easily follows from (7), (16) taking into account the first relation 3) in (9).

Theorem 2. *Suppose that u_n is the solution of equation (18) and the vector-functions h_n and u_n are defined by relations (17), then v_n satisfies the following equation*

$$\left\{ q \tilde{N}_2 \partial_1 - \tilde{N}_1 \partial_2 + q \tilde{\Gamma} \right\} v_n = 0. \tag{19}$$

P r o o f. Really, from (17) and 3)–5) (9), it follows that

$$\left\{ \tilde{N}_1 \partial_2 - q \tilde{N}_2 \partial_1 \right\} v_n = \left(\tilde{N}_1 \Psi T_2 - q \tilde{N}_2 \Psi T_1 \right) h + \left(\tilde{N}_1 \Psi \Phi N_2 - q \tilde{N}_2 \Psi \Phi N_1 \right) u$$

$$+K\left(N_1\partial_2 - qN_2\partial_1\right)u = q\tilde{\Gamma}\Psi h + q(\tilde{\Gamma}K - K\Gamma)u + qK\Gamma u = q\tilde{\Gamma}v,$$

this fact ends the proof. ∎

Calculate the expression

$$\partial_2\partial_1 h_n = T_1T_2h_n + T_1\Phi N_2 u_n + \Phi N_1\partial_2 u_n = qT_2T_1h_n + qT_2\Phi N_1 u_n - q\Phi\Gamma u_n$$
$$+\Phi N_1\partial_2 u = q\left\{T_2T_1h_n + T_2\Phi N_1 u_n + \Phi N_2\partial_1 u_n\right\} = q\partial_1\partial_2 h_n$$

in view of relations (7), 3) (9) and equation (18), and thus $\|\partial_2\partial_1 h_n\| = \|\partial_1\partial_2 h_n\|$ since $|q| = 1$. Using the isometric property 1) (9) of the expansions V_s (8), it is easy to see that

$$\begin{array}{c}\|\partial_2\partial_1 h_n\|^2 + \langle\tilde{\sigma}_2\partial_1 v_n, \partial_1 v_n\rangle + \langle\tilde{\sigma}_1 v_n, v_n\rangle = \|h_n\|^2 + \langle\sigma_2\partial_1 u_n, \partial_1 u_n\rangle \\ + \langle\sigma_1 u_n, u_n\rangle; \\ \|\partial_1\partial_2 h_n\|^2 + \langle\tilde{\sigma}_1\partial_2 v_n, \partial_2 v_n\rangle + \langle\tilde{\sigma}_2 v_n, v_n\rangle = \|h_n\|^2 + \langle\sigma_1\partial_2 u_n, \partial_2 u_n\rangle \\ + \langle\sigma_2 u_n, u_n\rangle.\end{array} \tag{20}$$

Therefore similarly to (4), the following conservation laws are true for the open system (17):

$$\begin{array}{ll} 1) & \|\partial_s h_n\|^2 + \langle\tilde{\sigma}_s v_n, v_n\rangle = \|h_n\|^2 + \langle\sigma_s u_n, u_n\rangle, \quad s = 1,\, 2; \\ 2) & \langle(\tilde{\sigma}_1 - \tilde{\sigma}_2)\, v_n, v_n\rangle + \langle\tilde{\sigma}_2\partial_1 v_n, \partial_1 v_n\rangle - \langle\tilde{\sigma}_1\partial_2 v_n, \partial_2 v_n\rangle \\ & = \langle(\sigma_1 - \sigma_2)\, u_n, u_n\rangle + \langle\sigma_2\partial_1 u_n, \partial_1 u_n\rangle - \langle\sigma_1\partial_2 u_n, \partial_2 u_n\rangle. \end{array} \tag{21}$$

Note that equality 2) (21) follows from (20) after the subtraction and will play the important role hereinafter.

III. The equation system (17) corresponding to the operators V_s describes "the dynamics" of the outgoing waves that are defined on $\mathbb{Z}^2_+$. To study the dual situation, consider the vector-functions $\tilde{h}_n \in H$, $\tilde{u}_n \in E$, $\tilde{v}_n \in \tilde{E}$ at the points of the integer-valued grid $n = (n_1, n_2) \in \mathbb{Z}^2_- = \mathbb{Z}_- \times \mathbb{Z}_-$ Define the differential linear operators $\tilde{\partial}_1$ and $\tilde{\partial}_2$ that act on the function $\tilde{h}_n$ in the following way:

$$\tilde{\partial}_1\tilde{h}_n = \tilde{h}_{(n_1-1;n_2)}; \quad \tilde{\partial}_2\tilde{h}_n = q^{n_1}\tilde{h}_{(n_1;n_2-1)}, \tag{22}$$

besides, as is easy to see,

$$q\tilde{\partial}_1\tilde{\partial}_2 = \tilde{\partial}_2\tilde{\partial}_1. \tag{23}$$

Similarly to (5), consider the analogue with two variables of the dual open system

$$\left\{\begin{array}{l} \tilde{\partial}_1\tilde{h}_n = T_1^*\tilde{h}_n + \Psi^*\tilde{N}_1^*\tilde{v}_n; \quad h_{(-1;-1)} = \tilde{h}_{-1}; \\ \tilde{\partial}_2\tilde{h}_n = T_2^*\tilde{h}_n + \Psi^*\tilde{N}_2^*\tilde{v}_n; \qquad n \in \mathbb{Z}^2_-; \\ \quad \tilde{u}_n = \Phi^*\tilde{h}_n + K^*\tilde{v}_n; \end{array}\right. \quad \overset{+}{V}_s\begin{bmatrix}\tilde{h}_n \\ \tilde{v}_n\end{bmatrix} = \begin{bmatrix}\tilde{\partial}_s\tilde{h}_n \\ \tilde{u}_n\end{bmatrix},\ s = 1, 2, \tag{24}$$

and the operators $\tilde{\partial}_1$, $\tilde{\partial}_2$ are defined by formulas (23).

For the equation system (24), the statements similar to Theorems 1 and 2 are true.

Theorem 3. *The consistency of the equation system (24) for $\tilde{h}_n$ takes place if the function $\tilde{v}_n$ satisfies the equation*

$$\left\{ q\tilde{N}_2^* \tilde{\partial}_1 - \tilde{N}_1^* \tilde{\partial}_2 + \tilde{\Gamma}^* \right\} \tilde{v}_n = 0. \tag{25}$$

Theorem 4. *The vector-function $\tilde{u}_n$ (24) is the solution of the difference equation*

$$\left\{ qN_2^* \tilde{\partial}_1 - N_1^* \tilde{\partial}_2 + \Gamma^* \right\} \tilde{u}_n = 0 \tag{26}$$

if $\tilde{v}_n$ satisfies equation (25) and $\tilde{h}_n$ is the solution of system (24).

As in the case of system (17), in this case $\left\| \tilde{\partial}_2 \tilde{\partial}_1 \tilde{h}_n \right\| = \left\| \tilde{\partial}_1 \tilde{\partial}_2 h_n \right\|$. Taking into account the isometric property 2) (9), it is easy to show that the conservation laws

$$\begin{aligned}
&1) \quad \left\| \tilde{\partial}_s \tilde{h}_n \right\|^2 + \langle \tau_s \tilde{u}_n, \tilde{u}_n \rangle = \left\| \tilde{h}_n \right\|^2 + \langle \tilde{\tau}_s \tilde{v}_n, \tilde{v}_n \rangle , \quad s = 1, 2; \\
&2) \quad \langle (\tau_1 - \tau_2) \tilde{u}_n, \tilde{u}_n \rangle + \left\langle \tau_2 \tilde{\partial}_1 \tilde{u}_n, \tilde{\partial}_1 \tilde{u}_n \right\rangle - \left\langle \tau_1 \tilde{\partial}_2 \tilde{u}_n, \tilde{\partial}_2 \tilde{u}_n \right\rangle \\
&\qquad = \langle (\tilde{\tau}_1 - \tilde{\tau}_2) \tilde{v}_n, \tilde{v}_n \rangle + \left\langle \tilde{\tau}_2 \tilde{\partial}_1 \tilde{v}_n, \tilde{\partial}_1 \tilde{v}_n \right\rangle - \left\langle \tilde{\tau}_1 \tilde{\partial}_2 \tilde{v}_n, \tilde{\partial}_2 \tilde{v}_n \right\rangle
\end{aligned} \tag{27}$$

take place.

As it will be shown later, the relations of isometric nature 2) (28) and 2) (27) result in the nontrivial relations for the characteristic function $S(z)$ of the operator T_1.

3. The Basic Properties of the Characteristic Function

I. Suppose the vector-functions h_n, u_n, v_n from the equations of the open system (17) have the form of $h_n = z^{n_1} h_{n_2}$, $u_n = z^{n_1} u_{n_2}$, $v_n = z^{n_1} u_{n_2}$ where $n = (n_1, n_2) \in \mathbb{Z}_+^2$ and $z \in \mathbb{C}$. Then from the first equation in (17) and the last one in (17) it follows that

$$v_{n_2} = S(z) u_{n_2},$$

where $S(z) = K + \Psi \left(zI - T_1\right)^{-1} \Phi N_1$ is the characteristic function (6) of the expansion V_1 (8) of the operator T_1.

Theorem 5. *Suppose that the operators N_1 and $\tilde{N}_1$ of the isometric expansion $\left\{ V_s, \overset{+}{V}_s \right\}_{s=1}^{2}$ (8) are such that N_1^{-1} and $\tilde{N}_1^{-1}$ exist and are bounded. Then the* *characteristic function* $S(z) = K + \Psi\,(z - T_1)^{-1}\,\Phi N_1$ *of the expansion* V_1 *satisfies the "intertwining" condition,*

$$\tilde{N}_1^{-1}\left(\tilde{N}_2 z + \tilde{\Gamma}\right) S(z) = S(zq) N_1^{-1}\,(N_2 z + \Gamma)\,. \tag{28}$$

P r o o f. First of all, note that from (7) it follows that $(T_1 - zqI)\,T_2 = qT_2\,(T_1 - zI)$ and so

$$T_2\,(T_1 - zI)^{-1} = q\,(T_1 - zqI)^{-1}\,T_2. \tag{29}$$

Rewrite the first equality in 3) (9) in the following way: $qT_2\Phi N_1 = (T_1 - zqI)\,\Phi N_2$ $+q\Phi\,(N_2 z + \Gamma)$, then

$$T_2\,(T_1 - zI)^{-1}\,\Phi N_1 = \Phi N_2 + q\,(T_1 - zqI)^{-1}\,\Phi\,(N_2 z + \Gamma) \tag{30}$$

by virtue of (29). Similarly rewriting the second equality in 3) (9) in the form of $\tilde{N}_1\Psi T_2 = q\tilde{N}_2\Psi\,(T_1 - zI) + q\left(\tilde{\Gamma} + \tilde{N}_2 z\right)\Psi$, we obtain that

$$\tilde{N}_1\Psi T_2\,(T_1 - zI)^{-1} = q\tilde{N}_2\Psi + q\left(\tilde{\Gamma} + \tilde{N}_2 z\right)\Psi\,(T_1 - zI)^{-1}\,. \tag{31}$$

Multiplying equality (30) by $\tilde{N}_1\Psi$ from the left and subtracting from it equality (31) multiplied by ΦN_1 from the right, we obtain the relation

$$\tilde{N}_2\Psi\Phi N_1 - q\tilde{N}_1\Psi\Phi N_2 + q\tilde{N}_1\Psi\,(T_1 - zqI)^{-1}\,\Phi\,(N_2 z + \Gamma)$$
$$= q\left(\tilde{N}_2 z + \tilde{\Gamma}\right)\Psi\,(T_1 - zI)^{-1}\,\Phi N_1.$$

Now using 4) (9) and the form of the characteristic function $S(z)$, we can rewrite the last equality in the following way:

$$\tilde{\Gamma}K - K\Gamma + \tilde{N}_1(K - S(zq))N_1^{-1}\,(N_2 z + \Gamma) = \left(\tilde{N}_2 z + \tilde{\Gamma}\right)(K - S(z)).$$

To complete the proof, one needs to take into account 5) (9). ∎

Using similar considerations as applied to the dual equation system (24), we obtain the characteristic function $\overset{+}{S}(z) = K^* + \Phi^*\,(zI - T_1^*)^{-1}\,\Psi^*\tilde{N}_1^*$ that also will satisfy the intertwining relation, besides the last one will coincide with (22) up to conjunction.

From the conservation laws 1) (21) and 1) (27) when $s = 1$, the following formulas result:

$$\frac{S^*(w)\tilde{\sigma}_1 S(z) - \sigma_1}{1 - z\bar{w}} = N_1^*\Phi^*\,(\bar{w}I - T_1^*)^{-1}\,(zI - T_1)^{-1}\,\Phi N_1;$$

$$\frac{\overset{+}{S}^{*}(w)\tau_1 \overset{+}{S}(z)-\tilde{\tau}_1}{1-z\bar{w}}=\tilde{N}_1\Psi\left(\bar{w}I-T_1\right)^{-1}\left(zI-T_1^*\right)^{-1}\Psi^*\tilde{N}_1^*.$$

Therefore the kernel

$$K(z,w)=\begin{bmatrix} \dfrac{S^*(w)\tilde{\sigma}_1 S(z)-\sigma_1}{1-z\bar{w}} & N_1^*\dfrac{\overset{+}{S}(z)-\overset{+}{S}(\bar{w})}{\bar{w}-z} \\ \tilde{N}_1\dfrac{S(z)-S(\bar{w})}{\bar{w}-z} & \dfrac{\overset{+}{S}^{*}(w)\tau_1 \overset{+}{S}(z)-\tilde{\tau}_1}{1-z\bar{w}} \end{bmatrix} \tag{32}$$

is the positively defined kernel [4, 8] since for all the finite sets $\{z_k, f_k\}_1^N$, $N<\infty$, where $z_k\in\mathbb{C}$ pnd $f_k\in E\oplus\tilde{E}$,

$$\sum_{k,s=1}^{N}\langle K(z_k,z_s)f_k,f_s\rangle\geq 0$$

takes place, for $K(z,w)=Y^*(w)Y(z)$ besides $Y(z)=\left[\left(zI-T_1\right)^{-1}\Phi N_1;\right.$ $\left.\left(zI-T_1^*\right)^{-1}\Psi^*\tilde{N}_1^*\right]$.

II. Consider the following subspace in H:

$$H_1=\operatorname{span}\left\{T(n)\Phi g+T^*(m)\Psi^* f: f\in\tilde{E}, g\in E, n,m\in\mathbb{Z}_+^2\right\}, \tag{33}$$

where $T(n)=T_1^{n_1}T_2^{n_2}$, $n=(n_1,n_2)\in\mathbb{Z}_+^2$.

Theorem 6. *Suppose that the operators N_1 and $\tilde{N}_1$ of the isometric expansion $\left\{V_s,\overset{+}{V}_s\right\}_{s=1}^2$ (8) are invertible. Then subspace (33) reduces both operators T_1 and T_2, besides the contractions of the operators T_1 and T_2 on the subspace $H_0=H\ominus H_1$ are the unitary operators.*

P r o o f. First, prove that the subspace H_1 reduces the operator T_1. To prove that, it is enough to show that

$$T_1T^*(m)\Psi^*\tilde{E}\subseteq H_1;\quad T_1^*T(n)\Phi E\subseteq H_1$$

for all n, $m\in\mathbb{Z}_+^2$. Prove the first inclusion (the second one is proved similarly). From 3) (9) by virtue of invertibility of $\tilde{N}_1$, it follows that $T_2^*\Psi^*=\bar{q}T_1^*\Psi\tilde{N}_2^*\tilde{N}_1^{*-1}+\bar{q}\Psi^*\tilde{\Gamma}^*\tilde{N}_1^{*-1}$, therefore using (7) we obtain that

$$T^*(m)\Psi^*\tilde{E}\subset\operatorname{span}\left\{T_1^{*n}\Psi^* g: n\in\mathbb{Z}_+, g\in\tilde{E}\right\}.$$

And since $T_1T_1^* + \Phi\tau_1\Phi^* = I$ and $T_1\Psi^* + \Phi\tau_1 K^*\tilde{N}_1^{*-1} = 0$ (in virtue of 2) (9)) then it is easy to prove by induction that $T_1 (T_1^*)^n \Psi^* \tilde{E} \subseteq H_1$ for all $n \in \mathbb{Z}_+$. Thus, the space H_1 reduces the operator T_1 and since $H_0 \subset \operatorname{Ker}(T_1^*T_1 - I)$ and $H_0 \subset \operatorname{Ker}(T_1T_1^* - I)$, for $(I - T_1^*T_1) H = \Psi^*\tilde{\sigma}_1\Psi H \subset H_1$. and $(I - T_1T_1^*) H = \Phi\tau_1\Phi^* H \subset H_1$. Then we obtain that the operator T_1 induces the unitary operator on the subspace H_0.

To prove that the subspace H_1 also reduces the operator T_2 (in virtue of the similar considerations), it is sufficient to prove that

$$T_2T_1^{*n}\Psi^*\tilde{E} \subset H_1, \quad T_2^*T_1^m\Phi E \subset H_1$$

for all n, $m \in \mathbb{Z}_+$. Prove, for example, the first inclusion. Let $L^n = \overline{T_2T_1^{*n}\Psi^*\tilde{E}}$ and let $L^n = L_1^n \oplus L_0^n$ where $L_s^n = P_sL^n$, besides P_s is the orthoprojector on H_s, $s = 0, 1$. From the reducibility of the operator T_1 by the subspaces H_1 and H_0, it follows that $T_1L_1^n \subset H_1$, $T_1L_0^n \subset H_0$. And on the other hand, $T_1T_2T_1^{*n}\Psi^*\tilde{E} = qT_2T_1T_1^{*n}\Psi^*\tilde{E}$ and so $T_2T_1\Psi^*\tilde{E} \subset H_1$ when $n = 0$, since $T_1\Psi^*\tilde{N}_1^* + \Phi\tau_1K^* = 0$, consequently $L_0^0 = \{0\}$, and using $T_1T_1^* + \Phi\tau_1\Phi^* = I$ when $n \geq 1$, we will have that $T_1L^n \subset \operatorname{span}\left\{L^{n-1} + H_1\right\}$ (by virtue of 3) (9)). Therefore $P_0T_1L_0^n \subset L_0^{n-1}$ and thus $P_0T_1^nL_0^n \subset L_0^0 = \{0\}$. Using now the unitarity of the contraction of the operator T_1 on H_0, we obtain that $L_0^n = \{0\}$. Thus $L^n \subset H_1$ for all $n \in \mathbb{Z}_+$. The unitarity of the contraction of operator T_2 on H_0 is proven in the same way as for T_1. ■

O b s e r v a t i o n 3. From the proof of Theorem 6 it follows that in the case of the reversibility of the operators N_1 and $\tilde{N}_1$ the operator T_1 generates the simple component (33) [4, 8], i.e.

$$H_1 = \operatorname{span}\left\{T_1^n\Phi g + T_1^{*m}\Psi^* f : g \in E, f \in \tilde{E}; n, m \in \mathbb{Z}_+\right\}. \tag{34}$$

The isometric expansion V_s, $\overset{+}{V}_s$ (8) is said to be the simple expansion if $H_1 = H$, where H_1 has the form of (33).

Along with the operator system $\{T_1, T_2\}$ in H satisfying relation (7), consider another system of linear bounded operators $\{T_1', T_2'\}$ in H' satisfying (7). The isometric expansion $\left\{V_s, \overset{+}{V}_s\right\}_{s=1}^{2}$ (8) of the system $\{T_1, T_2\}$ is said to be the unitarily equivalent to the expansion $\left\{V_s', \overset{+}{V}_s'\right\}_{s=1}^{2}$ (8) of the system $\{T_1', T_2'\}$ if:

1) the external spaces $E = E'$ and $\tilde{E} = \tilde{E}'$ coincide and the respective operators σ_s, τ_s, N_s, Γ and $\tilde{\sigma}_s$, $\tilde{\tau}_s$, $\tilde{N}_s$, $\tilde{\Gamma}$ acting in them coincide also;

2) there exists the unitary operator U from H into H' such that

$$UT_s = T_s'U \quad (s = 1,\, 2); \quad U\Phi = \Phi'; \quad \Psi'U = \Psi.$$

It is obvious that the characteristic functions $S(z)$ of the unitarily equivalent expansions coincide. The following theorem of unitary equivalency is true [4, 8].

Theorem 7. *Suppose that simple (33) isometric expansion* $\left\{V_s, \overset{+}{V}_s\right\}_{s=1}^{2}$ *(8) of the operator system* $\{T_1, T_2\}$ *that satisfies equality (7) and the simple isometric expansion* V'_s, $\overset{+}{V}{}'_s$ *(8) of the system* $\{T'_1, T'_2\}$*, for which also (7) takes place, are defined. And let the external spaces of these expansions* $E = E'$, $E = \tilde{E}'$ *coincide and the corresponding operators be equal,* $\Gamma = \Gamma'$, $N_s = N'_s$, $\sigma_s = \sigma'_s$, $\tau_s = \tau'_s$ *and also* $\tilde{\Gamma} = \tilde{\Gamma}'$, $\tilde{N}_s = \tilde{N}'_s$, $\tilde{\sigma}_s = \tilde{\sigma}'_s$, $\tilde{\tau}_s = \tilde{\tau}'_s$, $s = 1, 2$*. Then if in some general region of holomorphy* $S(z) = S'(z)$ *takes place and operators* N_1 *and* $\tilde{N}_1$ *are invertible then the commutative expansions* V_s, $\overset{+}{V}_s$ *and* V'_s, $\overset{+}{V}{}'_s$ *are unitarily equivalent.*

P r o o f. Denote formulas (29) and (30) in the following way:

$$\begin{array}{c} (zqI - T_1)^{-1}\,\Phi\,(N_2 z + \Gamma) = \Phi N_2 + q\,(zqI - T_1)^{-1}\,T_2 \Phi N_1; \\ q\left(\tilde{N}_2 z + \tilde{\Gamma}\right)\Psi\,(zI - T_1)^{-1} = q\tilde{N}_2\Psi + \tilde{N}_1\Psi T_2 \Psi (zI - T_1)^{-1}. \end{array} \tag{35}$$

Therefore after the necessary number of iterations from the right $N_1^{-1}\,(N_2 z + \Gamma)$ and from the left $(N_2^* \bar{w} + \Gamma^*)\,N_1^{*-1}$ of the block $K_{(z,w)}^{1,1}$ of the kernel $K(z, w)$ (32) and also after the appropriate substitutions $z \to zq$, we obtain that the expressions

$$\Phi^* T_2^{*m}\,(\bar{w}I - T_1^*)^{-1}\,(zI - T_1)^{-1}\,T_2^n \Phi, \quad \forall n, m \in \mathbb{Z}_+,$$

for the extensions V_s, $\overset{+}{V}_s$ and V'_s, $\overset{+}{V}{}'_s$ (8) coincide since $S(z) = S'(z)$. Use of the similar considerations for other blocks of the kernel $K(z, w)$ (32) leads to the coincidence of expressions

$$\Psi T_2^n\,(\bar{w}I - T_1)^{-1}\,(zI - T_1^*)^{-1}\,T_2^{*n}\Psi^*; \quad \Psi T_2^m\,(\bar{w}I - T_1)^{-1}\,(zI - T_1)^{-1}\,T_2^n \Phi$$

for all n, $m \in \mathbb{Z}_+$. Define the operator U from H into the space H' by the formulas

$$UT(n)\Phi g = T'(n)\Phi' g \quad (g \in E);$$

$$UT^*(m)\Psi^* f = \left(T'(m)\right)^* \Psi'^* f \quad (f \in \tilde{E}),$$

where $T(n) = T^{n_1} T_2^{n_2}$ for all n, $m \in \mathbb{Z}_+^2$. The unitarity of the operator U, as well as the fulfillment of the unitary equivalence conditions, follows from the coincidence of the expressions listed above. ■

III. The characteristic function $S(z)$ of the isometric expansion $\left\{V_s, \overset{+}{V_s}\right\}_{s=1}^{2}$ (8) has not only the traditional hermitian nonnegativeness of the kernel $K(z,w)$ (32) [4, 8], but also the additional properties that are inherited by the fact that T_1 and T_2 satisfy equality (7). Suppose that the operators N_1 and $\tilde{N}_1$ of the expansion $\left\{V_s, \overset{+}{V_s}\right\}_{s=1}^{2}$ (8) are invertible, then the following relations:

$$\begin{aligned}
&1)\quad S(zq)N_1^{-1}\left(N_2 z+\Gamma\right)=\tilde{N}_1^{-1}\left(\tilde{N}_2 z+\tilde{\Gamma}\right)S(z);\\
&2)\quad K^{1,1}_{(z,w)}-\left(N_2^*\bar{w}+\Gamma^*\right)N_1^{*-1}K^{1,1}_{(zq,wq)}N_1^{-1}\left(N_2 z+\Gamma\right)\\
&\qquad =S^*(w)\tilde{\sigma}_2 S(z)-\sigma_2; \qquad\qquad (36)\\
&3)\quad K^{2,2}_{(z,w)}-\left(\tilde{N}_2\bar{w}\bar{q}+\tilde{\Gamma}\right)\tilde{N}_1^{-1}K^{2,2}_{(zq,wq)}\tilde{N}_1^{*-1}\left(\tilde{N}_2^* zq+\tilde{\Gamma}^*\right)\\
&\qquad =\overset{+}{S^*}(w)\tau_2\overset{+}{S}(z)-\tilde{\tau}_2,
\end{aligned}$$

where $\left\{K^{p,s}_{(z,w)}\right\}$ are the corresponding blocks of the kernel $K(z,w)$ (32), are taking place. While relation 1) (36) follows from Theorem 5, equalities 2) and 3) in (36) follow from the conservation laws 2) (21) and 2) (27) on condition that (18), (19) and (25), (26) are taking place. For example, prove that relation 2) (36) follows from 2) (21). Really, the fact that

$$\partial_2 v_n=q\tilde{N}_1^{-1}\left(\tilde{N}_2 z+\tilde{\Gamma}\right)v_n;\quad \partial_2 u_n=qN_1^{-1}\left(N_2 z+\Gamma\right)u_n$$

follows from (18) and from (19) by virtue of the dependence of u_n and v_n by the variable n_1 chosen in Paragraph I of this section and also from the form of ∂_1 (15). Therefore relation 2) (21) we can represent in the form

$$S^*(w)\left(\tilde{\sigma}_1-\tilde{\sigma}_2\right)S(z)+z\bar{w}S^*(w)\tilde{\sigma}_2 S(z)-S^*(w)\left(\tilde{N}_2^*\bar{w}+\tilde{\Gamma}^*\right)\tilde{N}_1^{*-1}\tilde{\sigma}_1$$

$$\times\tilde{N}_1^{-1}\left(\tilde{N}_2 z+\tilde{\Gamma}\right)S(z)=\sigma_1-\sigma_2+z\bar{w}\sigma_2-\left(N_2^*\bar{w}+\Gamma^*\right)N_1^{*-1}\sigma_1 N_1^{-1}\left(N_2 z+\Gamma\right).$$

Now using the intertwining relation 1) (36), we obtain that

$$\left(1-z\bar{w}\right)\left[S^*(w)\tilde{\sigma}_2 S(z)-\sigma_2\right]=S^*(w)\tilde{\sigma}_1 S(z)-\sigma_1$$

$$-\left(N_2^*\bar{w}+\tilde{\Gamma}\right)\tilde{N}_1^{*-1}\left[S^*(wq)\tilde{\sigma}_1 S(zq)-\sigma_1\right]N_1^{-1}\left(N_2 z+\Gamma\right),$$

this fact proves 2) (36) by virtue of the form of the kernel $K^{1,1}_{(z,w)}$ (32).

O b s e r v a t i o n 4. Equating the coefficients of equal powers in (33), we can obtain the additional relations between the external parameters of expansion (8), σ_s, N_s, τ_s, Γ and $\tilde{\sigma}_s$, $\tilde{N}_s$, $\tilde{\tau}_s$, $\tilde{\Gamma}$, for example,

$$K^*\left(\tilde{\sigma}_2-\tilde{N}_2^*\tilde{N}_1^{*-1}\tilde{\sigma}_1\tilde{N}_1^{-1}\tilde{N}_2\right)K=\sigma_2-N_2^*N_1^{*-1}\sigma_1N_1^{-1}N_2;$$

$$K\left(\tau_2-N_2N_1^{-1}\tau_1N_1^{*-1}N_2^*\right)K^*=\tilde{\tau}_2-\tilde{N}_2\tilde{N}_1^{-1}\tilde{\tau}_1\tilde{N}_1^{*-1}\tilde{N}_2.$$

The description of the independent parameters of expansion (8) is the separate problem, and we will not touch it here.

IV. Consider now the description of the class of functions $S(z)$ that are the characteristic functions of the expansion $\left\{V_s,\overset{+}{V}_s\right\}_{s=1}^2$ (8) of the operator system T_1, T_2 satisfying (7). First of all, note several obvious properties of the expansion operators V_s and $\overset{+}{V}_s$ (8). It is easy to see that for all the vectors u from E,

$$\begin{aligned}V_1\begin{bmatrix}(zI-T_1)^{-1}\Phi N_1u\\ u\end{bmatrix}&=\begin{bmatrix}z\,(zI-T_1)^{-1}\Phi N_1u\\ S(z)u\end{bmatrix};\\ V_2\begin{bmatrix}(zI-T_1)^{-1}\Phi N_1u\\ u\end{bmatrix}&=\begin{bmatrix}q\,(zqI-T_1)^{-1}\Phi N_1N_1^{-1}(N_2z+\Gamma)\\ S(z)u\end{bmatrix}\end{aligned}\tag{37}$$

take place by virtue of (30) and 3) (9). It is important that the isometric property of the expansion V_1, 1) (9), results in the fact that the block $K^{1,1}_{(z,w)}$ of the kernel $K(z,w)$ (32) has the form of $K^{1,1}_{(z,w)}=N_1^*\Phi^*\left(\tilde{w}I-T_1^*\right)^{-1}(zI-T_1)^{-1}\Phi N_1$, and the isometric condition 1) (9) in terms of $K^{1,1}_{(z,w)}$ exactly gives equality 2) (36) obtained above. Similarly, the operators $\overset{+}{V}_s$ (8) will act in the dual situation,

$$\begin{aligned}\overset{+}{V}_1\begin{bmatrix}(zI-T_1^*)^{-1}\Psi^*\tilde{N}_1^*v\\ v\end{bmatrix}&=\begin{bmatrix}z\,(zI-T_1^*)^{-1}\Psi^*\tilde{N}_1^*v\\ \overset{+}{S}(z)v\end{bmatrix};\\ \overset{+}{V}_2\begin{bmatrix}(zI-T_1^*)^{-1}\Psi^*\tilde{N}_1^*v\\ v\end{bmatrix}&=\begin{bmatrix}(zqI-T_1^*)^{-1}\Psi^*\tilde{N}_1^*\tilde{N}_1^{*-1}\left(\tilde{N}_2^*zq+\tilde{\Gamma}^*\right)v\\ \overset{+}{S}(z)v\end{bmatrix}\end{aligned}\tag{38}$$

for all $v\in\tilde{E}$ by virtue of (31) and 3) (9). As before, the isometric property 2) (9) of the operator $\overset{+}{V}_1$ (38) results in the fact that the block $K^{2,2}_{(z,w)}$ of the kernel $K(z,w)$ (32) has the form of $K^{2,2}_{(z,w)}=\tilde{N}_1\Psi\left(\bar{w}I-T_1\right)^{-1}(zI-T_1^*)^{-1}\Psi^*\tilde{N}_1^*$, and the isometric condition 2) (9) of the operator $\overset{+}{V}_2$ results in equality 3) (36) for the block $K^{2,2}_{(z,w)}$. Finally, if we take into the consideration the connection between V_s and $\overset{+}{V}_s$ (14) in these terms, then we obtain that $\tilde{N}_1 S(z) = \overset{+}{S^*}(z) N_1$ when $s = 1$, and we obtain the intertwining condition 1) (36) again when $s = 2$. Really, from (14) and from the form of the block $K^{2,1}_{(z,w)}$, we obtain that

$$q\tilde{N}\frac{S(qz) - S(\bar{w})}{\bar{w} - zq} N_1^{-1}(N_2 z + \Gamma) + N_2 S(z) = \left(\tilde{N}_2 \overline{wq} + \tilde{\Gamma}\right)\frac{S(z) - S(\overline{wq})}{\overline{wq} - z}$$

$$+\tilde{N}_1 S(\bar{w}) N_1^{-1} N_2.$$

As a result of the elementary calculations, we obtain that

$$\tilde{N}_1^{-1}\left(\tilde{N}_2 z + \tilde{\Gamma}\right) S(z) - S(zq) N_1^{-1}(N_2 z + \Gamma) = \tilde{N}_1^{-1}\left(\tilde{N}_2 \overline{wq} + \tilde{\Gamma}\right) S(\overline{wq})$$

$$-S(\bar{w}) N_1^{-1}(N_2 \overline{wq} + \Gamma),$$

so the expression $\tilde{N}_1^{-1}\left(\tilde{N}_2 z + \tilde{\Gamma}\right) S(z) - S(zq) N_1^{-1}(N_2 z + \Gamma)$ is a constant, which obviously equals to zero if $z \to \infty$ by virtue of 4), 5) (9). Thus, all relations (36) have the natural origin that results from the isometric property and from the consistency of the expansion V_s, $\overset{+}{V}_s$ (8).

Theorem 8. *Suppose that the operator-function $S(z)$ mapping E into $\tilde{E}$ is such that there exist such operators σ_s, τ_s, N_s, Γ in the Hilbert space E, and, correspondingly, the operators $\tilde{\sigma}_s$, $\tilde{\tau}_s$, $\tilde{N}_s$, $\tilde{\Gamma}$, $s = 1, 2$, exist in the space $\tilde{E}$, besides the operators N_1 and $\tilde{N}_1$ are invertible, and, moreover, the following conditions are true:*

1) the kernel $K(z, w)$ (32) is positively defined, besides $\overset{+}{S}(z) = N_1^{*-1} S^*(z) \tilde{N}^*$;

2) for the function $S(z)$ and for the kernel blocks $K(z, w)$ (32), relations (36) are true;

3) the function $S(z)$ and the kernel $K(z, w)$ are analytical by z and by $\bar{w}$ in the region $D = \{z \in \mathbb{C} : |z| \geq R\}$ for some $R >> 1$, besides $S(\infty) \neq 0$ and $K(\infty, \infty) \neq 0$.

Then there exist the Hilbert space H and the system of linear bounded operators T_1, T_2 in H satisfying (7), such that for its isometric expansion $\left\{V_s, \overset{+}{V}_s\right\}_{s=1}^{2}$ (8), relations 1)–5) (9), with operators $\{\sigma_s, \tau_s, N_s, \Gamma\}$ and $\left\{\tilde{\sigma}_s, \tilde{\tau}_s, \tilde{N}_s, \tilde{\Gamma}\right\}$ defined above, are true, besides the characteristic function of the expansion V_1 of the operator T_1 coincides with $S(z)$.

P r o o f. On the Cartesian product $D \times (E \oplus \tilde{E})$, define the vector-functions $e_z h$, the carrier of which is amassed at the point z, and $h^T = (u, v)$ where $u \in E$,

$v \in \tilde{E}$. On the set of formal linear combinations $\sum_1^N e_{z_k} h_k$ $(N < \infty)$, define the nonnegative bilinear form with the use of the kernel $K(z,w)$ (32),

$$\langle e_z h, e_w g\rangle_K = \langle K(z,w)h, g\rangle_{E\oplus\tilde{E}}.$$

After the closing and factorization by the kernel of the given metric, we obtain the Hilbert space H. By H_E and $H_{\tilde{E}}$, define the subspaces in H generated by the elements of the type $e_z(u,0)^T = e_z u$ and $e_z(0,v)^T = e_z v$ correspondingly. First, define the expansions V_1 and V_2 on $H_E \oplus E$ in the following way:

$$V_1 \begin{bmatrix} e_z u \\ u \end{bmatrix} = \begin{bmatrix} z e_z u \\ S(z)u \end{bmatrix}; \quad V_2 \begin{bmatrix} e_z u \\ u \end{bmatrix} = \begin{bmatrix} e_{zq} q N_1^{-1}(N_2 z + \Gamma) u \\ S(z)u \end{bmatrix}, \tag{39}$$

where $u \in E$, and V_s (39), per se, have the form of (37). It is easy to see, that for V_s (37) relations 1) (9) will take place by virtue of the form of the block $K^{1,1}_{(z,w)}$ of the kernel $K(z,w)$ (32) and by virtue of equality 2) (36). Similarly, define the expansions $\overset{+}{V}_s$ on $H_{\tilde{E}} \oplus \tilde{E}$,

$$\overset{+}{V}_1 \begin{bmatrix} e_z v \\ v \end{bmatrix} = \begin{bmatrix} z e_z v \\ \overset{+}{S}(z)v \end{bmatrix}; \quad \overset{+}{V}_2 \begin{bmatrix} e_z v \\ v \end{bmatrix} = \begin{bmatrix} e_{zq} \tilde{N}_1^{*-1}\left(\tilde{N}_2^* z q + \tilde{\Gamma}^*\right) v \\ \overset{+}{S}(z)v \end{bmatrix}, \tag{40}$$

where $\overset{+}{S}(z) = N_1^{*-1} S^*(\bar{z}) \tilde{N}_1^*$ and $v \in \tilde{E}$; it is obvious that formulas (40) and (38) have the identical nature. For the operators $\overset{+}{V}_s$ (40), relations 2) (9) also take place, this fact easily follows from the structure of the block $K^{2,2}_{(z,w)}$ of the kernel $K(z,w)$ (32) and from relation 3) (36). It is easy to prove that from the intertwining relation 1) (36) it follows that

$$\left\langle \begin{bmatrix} I & 0 \\ 0 & \tilde{N}_s \end{bmatrix} V_s \begin{bmatrix} e_z u \\ u \end{bmatrix}, \begin{bmatrix} e_z v \\ v \end{bmatrix} \right\rangle_K = \left\langle \begin{bmatrix} I & 0 \\ 0 & N_s \end{bmatrix} \begin{bmatrix} e_z u \\ u \end{bmatrix}; \overset{+}{V}_s \begin{bmatrix} e_w v \\ v \end{bmatrix} \right\rangle_K. \tag{41}$$

This equality allows to continue V_s (39) (as well as $\overset{+}{V}_s$ (40)) onto the whole $H \oplus E$ (correspondingly onto the $H \oplus \tilde{E}$) correctly. Really, from (41) it follows that

$$\begin{bmatrix} I & 0 \\ 0 & \tilde{N}_s \end{bmatrix} V_s \Big|_{H\oplus E} = \overset{+}{V}{}_s^* \begin{bmatrix} I & 0 \\ 0 & N_s \end{bmatrix} \Big|_{H\oplus E}.$$

It is easy to test that 1) and 2) (9) take place as a result of such a continuation for V_s (39) and for $\overset{+}{V}_s$ (40). It is easy to prove that the operators T_s, Φ, Ψ, K (by virtue of (39)–(41)), that are the block elements of V_s (39) and of $\overset{+}{V}_s$ (40), have the form of

$$
\begin{gathered}
T_1 e_z u = z e_z u - e_0 u; \quad T_1 e_z v = \frac{1}{z}\left\{e_z v - e_0 \tau_1 \overset{+}{S}(z) v\right\};\\
T_1^* e_z v = z e_z v - e_0 v; \quad T_2^* e_z v = \frac{1}{z}\left\{e_z u - e_0 \tilde{\sigma}_1 S(z) u\right\};\\
T_2 e_z u = e_{zq} q\left[N_1^{-1}\left(N_2 z + \Gamma\right)\right] u - e_0 q N_1^{-1} N_2 u;\\
T_2 e_{zq}\left[\tilde{N}_1^{*-1}\left(\tilde{N}_2^* z q + \tilde{\Gamma}^*\right)\right] v = e_z v - \Phi \tau_2 \overset{+}{S}(z) v;\\
T_2^* e_z v = e_{zq}\left[\tilde{N}_1^{*-1}\left(\tilde{N}_2^* z q + \Gamma\right)\right] v - e_0 \tilde{N}_1^{*-1} \tilde{N}_2^* v;\\
T_2^* e_{zq}\left[q N_1^{-1}\left(N_2 z + \Gamma\right)\right] u = e_z u - e_0 \sigma_2 S(z) u;\\
K = S(\infty); \quad \Psi e_z u = [S(z) - S(\infty)] u;\\
\Psi e_z v = \frac{1}{z} \tilde{N}_1^{*-1}\left\{\tilde{\tau}_1 - K \tau_1 \overset{+}{S}(z)\right\} v; \quad \Psi^* v = e_0 \tilde{N}_1^{*-1} v;\\
\Phi u = e_0 N_1^{-1} u; \quad \Phi e_z v = [\overset{+}{S}(z) - \overset{+}{S}(\infty)] v;\\
\Phi^* e_z u = \frac{1}{z} N_1^{*-1}\left\{\sigma_1 - K^* \tilde{\sigma}_1 S(z)\right\} u;
\end{gathered}
\tag{42}
$$

besides $e_0 f = s - \lim\limits_{z\to\infty} z e_z f \in H$ where $f^T = (u, v) \in E \oplus \tilde{E}$. It is easy to test that this limit exists and belongs to the space H considering the analyticity of the kernel $K(z, w)$ (32) in $D \times D$. Finally, the trivial testing proves that relations 3)–5) (9) are true, and the characteristic function of the expansion V_1 of the operator T_1 coincides with $S(z)$ (for example, by virtue of (31)). ■

O b s e r v a t i o n 5. From Theorems 7 and 8, it follows that, in the case of the invertibility of the operators N_1 and $\tilde{N}_1$, the totality

$$\left\{S(z); \sigma_s; \tau_s; N_s; \Gamma; \tilde{\sigma}_s; \tilde{\tau}_s; \tilde{N}_s; \tilde{\Gamma}\right\}_{s=1,2}$$

is the total set of the invariants of the quantum algebra of linear operators $\{T_1, T_2\}$ defined by the commutation relation (7).

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