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On Universality of Bulk Local Regime of the Deformed Gaussian Unitary Ensemble

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We consider the deformed Gaussian Ensemble $H_n = H_n^{(0)} + M_n$ in which $H_n^{(0)}$ is a hermitian matrix (possibly random) and M_n is the Gaussian Unitary Ensemble (GUE) random matrix (independent of $H_n^{(0)}$). Assuming that the Normalized Counting Measure of $H_n^{(0)}$ converges weakly (in probability) to a nonrandom measure $N^{(0)}$ with a bounded support, we prove the universality of the local eigenvalue statistics in the bulk of the limiting spectrum of H_n.

Key words: random matrices, universality, Gaussian Unitary Ensemble.

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1. Introduction

Universality is an important topic of the random matrix theory. It deals with statistical properties of eigenvalues of $n \times n$ random matrices on the intervals whose length tends to zero as $n \to \infty$. According to the universality hypothesis these properties do not depend on large extent on the ensemble. The hypothesis was formulated in the early 60s and since then was proved in certain cases. Best of all the universality is studied in the case of ensembles with a unitary invariant probability distribution (known also as unitary matrix models) ([1–3]).

To formulate the universality hypothesis we need some notations and definitions. Denote by $\lambda_1^{(n)}, \dots, \lambda_n^{(n)}$ the eigenvalues of random matrix. Define the Normalized Counting Measure (NCM) of eigenvalues of the matrix as

$$N_n(\triangle) = \sharp\{\lambda_j^{(n)} \in \triangle, j = 1, \dots, n\}/n, \quad N_n(\mathbb{R}) = 1, \tag{1.1}$$

where $\triangle$ is an arbitrary interval of the real axis. For many known random matrices the expectation $\overline{N}_n = \mathbf{E}\{N_n\}$ is absolutely continuous, i.e.,

$$\overline{N}_n(\triangle) = \int\limits_{\triangle} \rho_n(\lambda) d\,\lambda. \tag{1.2}$$

The nonnegative function ρ_n in (1.2) is called the Density of States.

Define also the m-point correlation function $R_m^{(n)}$ by the equality:

$$\mathbf{E}\left\{\sum_{j_1\neq\ldots\neq j_m} \varphi_m(\lambda_{j_1},\ldots,\lambda_{j_m})\right\} = \int \varphi_m(\lambda_1,\ldots,\lambda_m) R_m^{(n)}(\lambda_1,\ldots,\lambda_m) d\lambda_1,\ldots,d\lambda_m, \tag{1.3}$$

where $\varphi_m : \mathbb{R}^m \to \mathbb{C}$ is bounded, continuous and symmetric in its arguments and the summation is over all m-tuples of distinct integers $j_1,\ldots,j_m = 1,\ldots,n$. Here and below integrals without limits denote the integration over the whole real axis.

The global regime of the random matrix theory, centered around the weak convergence of the Normalized Counting Measure of eigenvalues, is well studied for many ensembles. It is shown that N_n converges weakly to a nonrandom limiting measure N known as the Integrated Density of States (IDS). The IDS is normalized to unity and is absolutely continuous in many cases

$$N(\mathbb{R}) = 1, \quad N(\triangle) = \int\limits_{\triangle} \rho(\lambda) d\,\lambda. \tag{1.4}$$

The nonnegative function ρ in (1.4) is called the limiting density of states of the ensemble.

We will call the spectrum the support of N and define the bulk of the spectrum as

$$\operatorname{bulk} N = \{\lambda | \exists (a,b) \subset \operatorname{supp} N : \lambda \in (a,b),\ \inf_{\mu\in(a,b)} \rho(\mu) > 0\}$$

Then the universality hypothesis for hermitian random matrices on the bulk of the spectrum says that we have for any $\lambda_0 \in \operatorname{bulk} N$:

(i) for any fixed m uniformly in $x_1, x_2, \ldots, x_m$ varying in any compact set in $\mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{(n\rho_n(\lambda_0))^m} R_m^{(n)}\left(\lambda_0 + \frac{x_1}{\rho_n(\lambda_0)\,n},\ldots,\lambda_0 + \frac{x_m}{\rho_n(\lambda_0)\,n}\right) = \det\{S(x_i - x_j)\}_{i,j=1}^m, \tag{1.5}$$

where

$$S(x) = \frac{\sin(\pi x)}{\pi x}, \tag{1.6}$$

and $R_m^{(n)}$, ρ_n are defined in (1.3) and (1.2), respectively;

(ii) if

$$E_n(\triangle) = \mathbf{P}\{\lambda_i^{(n)} \notin \triangle,\, i = \overline{1, n}\} \tag{1.7}$$

is the gap probability, then

$$\lim_{n\to\infty} E_n\left(\left[\lambda_0 + \frac{a}{\rho_n(\lambda_0)\, n}, \lambda_0 + \frac{b}{\rho_n(\lambda_0)\, n}\right]\right) = \det\{1 - S_{a,b}\}, \tag{1.8}$$

where the operator $S_{a,b}$ is defined on $L_2[a, b]$ by the formula

$$(S_{a,b} f)(x) = \int\limits_a^b S(x-y) f(y) d\, y,$$

and S is defined in (1.6).

In this paper we study the universality of the local bulk regime of random matrices of the deformed Gaussian Unitary Ensemble (GUE)

$$H_n = H_n^{(0)} + M_n, \tag{1.9}$$

where $H_n^{(0)}$ is a Hermitian matrix (possibly random, and in this case independent of M_n) with eigenvalues $\{h_j^{(n)}\}_{j=1}^n$ and M_n is the GUE matrix, defined as

$$M_n = n^{-1/2} W, \tag{1.10}$$

where W is a Hermitian $n \times n$ matrix whose entries $\Re W_{jk}$ and $\Im W_{jk}$ are independent Gaussian random variables such that

$$\mathbf{E}\{W_{jk}\} = \mathbf{E}\{(W_{jk})^2\} = 0, \quad \mathbf{E}\{|W_{jk}|^2\} = 1, \quad j, k = 1, \ldots, n. \tag{1.11}$$

Let

$$N_n^{(0)}(\triangle) = \sharp\{h_j^{(n)} \in \triangle, j = 1, \ldots, n\}/n. \tag{1.12}$$

be the Normalized Counting Measure of eigenvalues of $H_n^{(0)}$.

Note also that since the probability law of M_n is unitary invariant, we can assume without loss of generality that $H_n^{(0)}$ is diagonal.

The global regime for the ensemble (1.9)–(1.11) is well enough studied. In particular, it was shown in [4] that if $N_n^{(0)}$ converges weakly (with probability 1) to a nonrandom measure $N^{(0)}$ as $n \to \infty$, then N_n also converges weakly (with probability 1) to a nonrandom measure N. Moreover, the Stieltjes transforms f of N and $f^{(0)}$ of $N^{(0)}$ are related as

$$f(z) = f^{(0)}(z + f(z)). \tag{1.13}$$

It follows from definition (1.1) and the above result that any n-independent interval Δ of spectral axis such that $N(\Delta) > 0$ contains $O(n)$ eigenvalues. Thus, to deal with a finite number of eigenvalues as $n \to \infty$, in particular, with the gap probability, one has to consider spectral intervals, whose length tends to zero as $n \to \infty$. In the case of local bulk regime we are about the intervals of the length $O(n^{-1})$.

Random matrix theory possesses the powerful techniques of analysis of the local regime based on the so-called determinant formulas for the correlation functions [5]. For the GUE, more generally for the hermitian matrix models, the determinant formulas follow from the possibility to write the joint probability density of its eigenvalues as the square of the determinant, formed by certain orthogonal polynomials, and then as the determinant formed by reproducing kernel of the polynomials, that are also heavily used in the subsequent asymptotic analysis [1–3]. Unfortunately, the orthogonal polynomials have not appeared so far in the studying of the deformed Gaussian Unitary Ensemble. However, it was shown in physical papers [6–8] that correlation functions of the deformed Gaussian Unitary Ensemble can be written in the determinant form, although the corresponding kernel is not, in general, a reproducing kernel of a system of orthogonal polynomials. This was done by using as a crucial step the Harish-Chandra/Itzykson–Zuber formula for certain integrals over the unitary group.

This important result was used in [9] to prove the universality of the local bulk regime of matrices (1.9), where $H_n^{(0)} = n^{-1/2}W^{(0)}$ is a hermitian random matrix with independent (modulo symmetry) entries:

$$\begin{aligned} W^{(0)} &= \{W_{jk}^{(0)}\}_{j,k}^{n},\ W_{jk}^{(0)} = \overline{W_{kj}^{(0)}}, \\ \mathbf{E}\{W_{jk}^{(0)}\} &= \mathbf{E}\{(W_{jk}^{(0)})^2\} = 0,\ \mathbf{E}\{|W_{jk}^{(0)}|^2\} = 1,\ \sup_{j,k} \mathbf{E}\{|W_{jk}^{(0)}|^p\} < \infty. \end{aligned} \quad (1.14)$$

It was proved in [9] that if $p > 2(m+2)$, then (1.5) is valid, and if $p > 6$, then (1.8) is valid.

Later in the papers [10, 11] a special case of (1.9) was studied, where $H_n^{(0)}$ has two eigenvalues $\pm a$ of equal multiplicity. In this case the universality in the bulk and at the edge of the spectrum was proved.

In this paper we consider random matrices (1.9) for a rather general class of $H_n^{(0)}$ both random and nonrandom. The main results are the following theorems.

Theorem 1. *Let $H_n^{(0)}$ in (1.9) be nonrandom and such that its Normalized Counting Measure (1.12) converges weakly to a measure $N^{(0)}$ of bounded support. Then for any λ_0, $\rho(\lambda_0) > 0$ the universality properties (1.5) and (1.8) hold.*

Theorem 2. *Let the eigenvalues $\{h_j^{(n)}\}_{j=1}^n$ of $H_n^{(0)}$ in (1.9) be a collection of random variables independent of W of (1.10). Assume that there exists* *a nonrandom measure $N^{(0)}$ of bounded support such that for any finite interval $\Delta \subset \mathbb{R}$ and for any $\varepsilon > 0$*

$$\lim_{n\to\infty} \mathbf{P}_n^{(h)}\{|N^{(0)}(\Delta) - N_n^{(0)}(\Delta)| > \varepsilon\} = 0, \tag{1.15}$$

where $\mathbf{P}_n^{(h)}\{\dots\}$ denotes the probability law of $\{h_j^{(n)}\}_{j=1}^n$. Then for any λ_0, $\rho(\lambda_0) > 0$ the universality properties (1.5) and (1.8) hold.

The paper is organized as follows. In Section 2 we give a proof of determinant formulas for correlation functions (1.3) following essentially [6, 7]. Theorem 1 is proved in Section 3. Section 4 deals with the proof of Theorem 2.

Note that we denote by C, C_1, etc. and c, c_1, etc. various constants appearing below, which can be different in different formulas.

2. The Determinant Formulas

It is well known (see, for example, [5]) that the correlation functions (1.3) for the GUE can be written in the determinant form

$$R_m^{(n)}(\lambda_1, \dots, \lambda_m) = \det\{K_n(\lambda_i, \lambda_j)\} \tag{2.1}$$

with

$$K_n(\lambda_i, \lambda_j) = \sum_{k=0}^{n-1} \phi_k(\lambda_i)\phi_k(\lambda_j), \quad \phi_k(x) = n^{1/4} h_k(\sqrt{n}x)e^{-nx^2/4},$$

where $\{h_k\}_{k\ge 0}$ are orthonormal Hermite polynomials. We want to find the analogs of these formulas in the case of random matrices (1.9). We essentially follow [6, 7].

Proposition 1. *Let H_n be the random matrix defined in (1.9). Then for its correlation function (1.3) the determinant formula (2.1) is valid with*

$$K_n(\lambda, \mu) = n \int\limits_L \frac{dt}{2\pi} \oint\limits_C \frac{dv}{2\pi} \frac{\exp\left\{-\dfrac{n}{2}(v^2 - 2v\lambda - t^2 + 2\mu t)\right\}}{v - t} \prod_{j=1}^n \left(\frac{t - h_j^{(n)}}{v - h_j^{(n)}}\right), \tag{2.2}$$

where L is a line parallel to the imaginary axis and lying to the left of all $\{h_j^{(n)}\}_{j=1}^n$, and C is a closed contour, encircling $\{h_j^{(n)}\}_{j=1}^n$ and not intersecting L.

P r o o f. The probability distribution for ensemble (1.9) is

$$P_n(H_n) = \frac{1}{Z_n} e^{-\frac{1}{2}\mathrm{Tr}\,\mathrm{W}^2} = \frac{1}{Z_n'} e^{-\frac{n}{2}\mathrm{Tr}\left(H_n^2 - 2H_nH_n^{(0)}\right)}, \qquad (2.3)$$

where

$$Z_n' = \int e^{-\frac{n}{2}\mathrm{Tr}\left(H_n^2 - 2H_nH_n^{(0)}\right)} d\,H_n, \qquad (2.4)$$

and

$$d\,H_n = \prod_{j=1}^{n} d\,H_{jj} \prod_{1\le i<j\le n} d\,\Re H_{ij} d\,\Im H_{ij}.$$

Consider the function

$$U_m(t_1,\dots,t_m) = \mathbf{E}\{\mathrm{Tr}\,e^{int_1H_n}\dots\mathrm{Tr}\,e^{int_mH_n}\} \qquad (2.5)$$

and use (2.3) to obtain

$$U_m(t_1,\dots,t_m) = \frac{1}{Z_n'}\int e^{-\frac{n}{2}\mathrm{Tr}\left(H_n^2 - 2H_nH_n^{(0)}\right)} \mathrm{Tr}\,e^{int_1H_n}\dots\mathrm{Tr}\,e^{int_mH_n}\,d\,H_n. \quad (2.6)$$

Let us change variables to $H_n = U^*\Lambda U$, where U is a unitary matrix and the matrix Λ is

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

Then the differential $d\,H_n$ in (2.6) transforms to $\triangle^2(\Lambda)d\,\Lambda d\,\mu(U)$, where $d\,\Lambda = \prod_{j=1}^{n} d\,\lambda_j$,

$$\triangle(\Lambda) = \prod_{i<j}^{n}(\lambda_i - \lambda_j) \qquad (2.7)$$

is a Vandermonde determinant, and $\mu(U)$ is the normalized to unity Haar measure on the unitary group $U(n)$. Integral over the unitary group $U(n)$ can be easily computed using the well-known Harish-Chandra/Itsykson–Zuber formula (see [5, App. 5]).

Proposition 2. *Let A and B be normal $n \times n$ matrices with eigenvalues $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$, correspondingly. Then we have*

$$\int \exp\{\,TrAU^*BU\}d\,\mu(U) = \frac{\det[\exp\{a_ib_j\}]_{i,j=1}^n}{\triangle(A)\triangle(B)}, \qquad (2.8)$$

where $\triangle(A)$ and $\triangle(B)$ are Vandermonde determinants (2.7) for the eigenvalues of A and B.

Thus, we get from (2.6)

$$U_m(t_1,\dots,t_m)=\frac{1}{Z'_n}\int e^{-\frac{n}{2}\sum\limits_{j=1}^n\lambda_j^2}\prod_{k=1}^n\left(\sum_{l=1}^n e^{int_k\lambda_l}\right)\triangle(\Lambda)\frac{\det\left\{e^{n\lambda_j h_k^{(n)}}\right\}_{j,k=1}^n}{\triangle(H_n^{(0)})}d\,\Lambda.$$

The integral here is symmetric function of $\{\lambda_l\}_{l=1}^n$. Thus we can rewrite the above formula as

$$U_m(t_1,\dots,t_m)=\frac{n!}{Z'_n}\sum_{k_1,\dots,k_m=1}^n\int e^{-\frac{n}{2}\sum\limits_{j=1}^n\lambda_j^2+n\sum\limits_{j=1}^n\lambda_j h_j^{(n)}+in(t_1\lambda_{k_1}+\ldots+t_m\lambda_{k_m})}\frac{\triangle(\Lambda)}{\triangle(H_n^{(0)})}d\,\Lambda. \tag{2.9}$$

It is easy to check that we have for any $b_1,\dots,b_n\in\mathbb{C}$

$$\int e^{-\frac{n}{2}\sum\limits_{j=1}^n\lambda_j^2+n\sum\limits_{j=1}^n\lambda_j b_j}\triangle(\Lambda)d\,\Lambda=\left(\frac{2\pi}{n}\right)^{n/2}\triangle(b_1,\dots,b_n)e^{\frac{n}{2}\sum\limits_{j=1}^n b_j^2}.$$

Thus, the integration over $\{\lambda_j\}_{j=1}^n$ in (2.9) yields

$$U_m(t_1,\dots,t_m)=\exp\{-\frac{n}{2}\mathrm{Tr}\,(H_n^{(0)})^2\}\sum_{k_1,\dots,k_m=1}^n\frac{\triangle(b_1,\dots,b_n)}{\triangle(H_n^{(0)})}\exp\{\frac{n}{2}\sum_{j=1}^n b_j^2\}, \tag{2.10}$$

where $b_j=h_j^{(n)}+it_1\delta_{j,k_1}+\ldots+it_m\delta_{j,k_m}$. Let us find the inverse Fourier transform of (2.10). It is easy to see that if some of k_j's coincide (for example, $k_1=k_2=\ldots=k_l$), then the inverse Fourier transform of this term becomes $\prod\limits_{i=1}^{l-1}\delta(\lambda_i-\lambda_{i+1})$ and hence can be omitted for $\lambda_i\neq\lambda_j$. Therefore, we have for $\lambda_i\neq\lambda_j$

$$R_m(\lambda_1,\dots,\lambda_m)=n^m\widetilde{\sum}\int\frac{d\,t_1\ldots d\,t_m}{(2\pi)^m}\frac{\triangle(b_1,\dots,b_n)}{\triangle(H_n^{(0)})}\exp\{\frac{n}{2}\sum_{j=1}^n b_j^2-in\sum_{l=1}^m t_l\lambda_{k_l}\},$$

where $\widetilde{\sum}$ denotes the sum over the m-tuples $(k_1,\dots,k_m)$ of distinct k_j's. Now we transform the integrals over $\{t_j\}_{j=1}^n$ to the line L' parallel to the real axis and lying in the upper half-plane. Use the identity

$$\frac{\triangle(b_1,\dots,b_n)}{\triangle(H_n^{(0)})}$$

$$=\prod_{l=1}^m\prod_{j=\overline{1,n},j\neq k_l}\left(1+\frac{it_{k_l}}{h_{k_l}^{(n)}-h_j^{(n)}}\right)\prod_{1\le l<s\le m}\frac{(h_{k_l}^{(n)}+it_{k_l}-h_{k_s}^{(n)}-it_{k_s})(h_{k_l}^{(n)}-h_{k_s}^{(n)})}{(h_{k_l}^{(n)}+it_{k_l}-h_{k_s}^{(n)})(h_{k_l}^{(n)}-h_{k_s}^{(n)}-it_{k_s})},$$

and write its r.h.s. as the integral

$$\oint\limits_{C^m} \frac{d\,u_1 \dots d\,u_m}{(2\pi i)^m} \prod_{k=1}^{m} \prod_{j=1}^{n} \left(1 + \frac{it_k}{u_k - h_j^{(n)}} \right) \prod_{k<l} \frac{(u_k + it_k - u_l - it_l)(u_k - u_l)}{(u_k + it_k - u_l)(u_k - u_l - it_l)},$$

where C is a closed contour, encircling $\{h_j^{(n)}\}_{j=1}^n$ and not intersecting L'. This leads to the representation

$$R_m(\lambda_1, \dots, \lambda_m) = n^m \int\limits_{L'} \frac{d\,t_1 \dots d\,t_m}{(2\pi)^m \prod\limits_{k=1}^{m} t_k} e^{-\frac{n}{2} \sum\limits_{k=m}^{n} t_k^2 - in \sum\limits_{k=1}^{m} t_k \lambda_k} \oint\limits_{C} \frac{d\,u_1 \dots d\,u_m}{(2\pi i)^m}$$

$$\times e^{in \sum\limits_{k=1}^{m} t_k u_k} \prod_{k=1}^{m} \prod_{j=1}^{n} \left(1 + \frac{it_k}{u_k - h_j^{(n)}} \right) \prod_{k<l} \frac{(u_k + it_k - u_l - it_l)(u_k - u_l)}{(u_k + it_k - u_l)(u_k - u_l - it_l)}.$$

Changing variables here as $t_k \to t_k + iu_k$ and then as $it_k \to t_k$, we get

$$\begin{aligned} R_m(\lambda_1, \dots, \lambda_m) &= n^m \int\limits_{L} \frac{d\,t_1 \dots d\,t_m}{(2\pi)^m} \oint\limits_{C} \frac{d\,u_1 \dots d\,u_m}{(2\pi)^m \prod\limits_{k=1}^{m} (u_k - t_k)} \\ &\times e^{n(\frac{1}{2} \sum\limits_{k=m}^{n} t_k^2 - \sum\limits_{k=1}^{m} t_k \lambda_k + \sum\limits_{k=1}^{m} u_k \lambda_k - \frac{1}{2} \sum\limits_{k=m}^{n} u_k^2)} \prod_{k=1}^{m} \prod_{j=1}^{n} \frac{t_k - h_j^{(n)}}{u_k - h_j^{(n)}} \\ &\times (-1)^{m(m-1)/2} \prod_{k<l} \frac{(t_k - t_l)(u_k - u_l)}{(u_k - t_l)(u_l - t_k)}, \end{aligned} \tag{2.11}$$

where L is a line parallel to the imaginary axis and lying to the left of all $\{h_j^{(n)}\}_{j=1}^n$, and C is a closed contour, encircling $\{h_j^{(n)}\}_{j=1}^n$ and not intersecting L. Now the identity (see [12, Probl. 7.3])

$$(-1)^{\frac{m(m-1)}{2}} \prod_{k<l} \frac{(t_k - t_l)(u_k - u_l)}{(u_k - t_l)(u_l - t_k)} \prod_{k} \frac{1}{u_k - t_k} = \det \left[\frac{1}{u_k - t_j} \right]_{k,j=1}^{m}$$

and formula (2.11) yield (2.1) with (2.2). ■

3. Proof of Theorem 1

In this section we prove Theorem 1 using (2.1) and passing to the limit $n \to \infty$ in (2.2).

Putting in formula (2.2) $\lambda = \lambda_0 + \xi/n$ and $\mu = \lambda_0 + \eta/n$, we get

$$\begin{aligned} &K_n(\lambda_0 + \xi/n, \lambda_0 + \eta/n) \\ &\qquad = n \int\limits_{L} \frac{dt}{2\pi} \oint\limits_{C} \frac{dv}{2\pi} \exp\{v\xi - t\eta\} \frac{\exp\{n(S_n(t, \lambda_0) - S_n(v, \lambda_0))\}}{v - t}, \end{aligned} \tag{3.1}$$

where

$$S_n(z,\lambda_0)=\frac{z^2}{2}+\frac{1}{n}\sum_{i=1}^{n}\ln(z-h_j^{(n)})-\lambda_0 z-S^* \tag{3.2}$$

with some constant S^* which will be chosen later (see (3.16)), and C is a closed contour, encircling $\{h_j^{(n)}\}_{j=1}^n$, and L is a line parallel to the imaginary axis and lying to the left of C. Formula (2.1) reduces the proof of (1.5) to the proof of the relation

$$\lim_{n\to\infty}\frac{1}{n\rho_n(\lambda_0)}K_n\left(\lambda_0+\frac{\xi}{n\rho_n(\lambda_0)},\lambda_0+\frac{\eta}{n\rho_n(\lambda_0)}\right)=S(\xi-\eta),$$

where S is defined in (1.6).

Now we will choose the contour C in (3.1) as some n-dependent contour C_n. Define

$$f_n^{(0)}(z)=\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_j^{(n)}-z}, \tag{3.3}$$

and for given $\lambda\in\mathbb{R}$ consider the equation

$$z-f_n^{(0)}(z)=\lambda. \tag{3.4}$$

It can be written as a polynomial equation of degree $(n+1)$ and so it has $(n+1)$ roots. Since the l.h.s. of (3.4) tends to $+\infty$, if $z\in\mathbb{R}\to h_j^{(n)}+0$, and the l.h.s. tends to $-\infty$, if $z\in\mathbb{R}\to h_j^{(n)}-0$, the $n-1$ roots are always real and belong to the segments between adjacent $h_j^{(n)}$'s . If λ is big enough, then all $n+1$ roots are real. Let $z_n(\lambda)$ be a real root equal to $\lambda-1/\lambda+O(1/\lambda^2)$, as $\lambda\to\infty$. If λ decreases, then $z_n(\lambda)$ decreases too, and coming to some λ_{c_1} the real root disappears and there appear two complex ones – $z_n(\lambda)$ and $\overline{z_n(\lambda)}$. Then $z_n(\lambda)$ may be real again, then again complex, and so on, however as soon as λ becomes less then some λ_{c_2}, the root becomes real again. Introduce

$$C_n=\{z\in\mathbb{C}:z=z_n(\lambda),\,\Im z_n(\lambda)>0\}\cup\{z\in\mathbb{C}:z=\overline{z_n(\lambda)},\,\Im z_n(\lambda)>0\}\cup S, \tag{3.5}$$

where S is a set of points $z=z_n(\lambda)$ in which $z_n(\lambda)$ becomes real. It is clear that the set of corresponding λ's is $\bigcup_{j=1}^{k}I_k$, where $\{I_j\}_{j=1}^k$ are non intersecting segments, and that C_n is closed and encircles $\{h_j^{(n)}\}_{j=1}^n$.

Let us consider the limiting equation

$$z-f^{(0)}(z)=\lambda,\quad f^{(0)}(z)=\int\frac{N^{(0)}(d\,h)}{h-z}, \tag{3.6}$$

where $\lambda\in\mathbb{R}$ is fixed. We have

Lemma 1. *Let $H_n^{(0)}$ in (1.9) be a Hermitian matrix (possibly random and in this case independent of M_n). Assume that the NCM $N_n^{(0)}$ of $H_n^{(0)}$ converges weakly with probability* 1 *to a nonrandom measure $N^{(0)}$. Then the IDS N is absolutely continuous and its density ρ is continuous, and equation (3.6) has a unique solution in the open upper half-plane $\Im z > 0$ for any λ such that $\rho(\lambda) >$* 0*. This solution is continuous in λ in the domain where it exists and*

$$\pi^{-1}\Im z(\lambda) = \rho(\lambda). \tag{3.7}$$

P r o o f. It follows from (1.13) that

$$\Im f(z) = \int \frac{(\Im z + \Im f(z))N^{(0)}(d\,h)}{|h - z - f(z)|^2}.$$

Thus, since $\Im f(z) > 0$ for $\Im z > 0$, we have

$$\int \frac{N^{(0)}(d\,h)}{|h - z - f(z)|^2} = \frac{\Im f(z)}{\Im f(z) + \Im z} \leq 1. \tag{3.8}$$

This and (1.13) yield

$$|f(z)| \leq \int \frac{N^{(0)}(d\,h)}{|h - z - f(z)|} \leq \left(\int \frac{N^{(0)}(d\,h)}{|h - z - f(z)|^2}\right)^{1/2} \leq 1. \tag{3.9}$$

According to (3.9) we have that there exists a sequence $\{z_k\}_{k=1}^{\infty} : z_k \to \lambda_0 \in \mathbb{R}$, $\Im z_k > 0$ such that $f(z_k) \to \phi_0$ as $k \to \infty$. Let $\{\widehat{z}_k\}_{k=1}^{\infty}$ be another sequence such that $\widehat{z}_k \to \lambda_0$ and $f(\widehat{z}_k) \to \phi_1 \neq \phi_0$ as $k \to \infty$. Denote $f_k = f(z_k)$, $\widehat{f}_k = f(\widehat{z}_k)$. Then we have

$$f_k = f^{(0)}(z_k + f_k), \quad \widehat{f}_k = f^{(0)}(\widehat{z}_k + \widehat{f}_k)$$

and also

$$\overline{f}_k = f^{(0)}(\overline{z}_k + \overline{f}_k).$$

Hence, we obtain

$$(\widehat{z}_k + \widehat{f}_k - \overline{z}_k - \overline{f}_k) \int \frac{N^{(0)}(d\,h)}{(h - \widehat{z}_k - \widehat{f}_k)(h - \overline{z}_k - \overline{f}_k)} = \widehat{f}_k - \overline{f}_k, \tag{3.10}$$

and thus, since $\widehat{f}_k - \overline{f}_k \to \phi_1 - \overline{\phi}_0 \neq 0, \widehat{z}_k - \overline{z}_k \to 0$ we get

$$\lim_{k\to\infty} \int \frac{N^{(0)}(d\,h)}{(h - \widehat{z}_k - \widehat{f}_k)(h - \overline{z}_k - \overline{f}_k)} = \lim_{k\to\infty} \frac{\widehat{f}_k - \overline{f}_k}{\widehat{z}_k + \widehat{f}_k - \overline{z}_k - \overline{f}_k} = 1. \tag{3.11}$$

Therefore, we obtain from (3.8) for z_k and $\widehat{z}_k$ and from (3.11)

$$\lim_{k\to\infty}\int\left|\frac{1}{h-z_k-f_k}-\frac{1}{h-\widehat{z}_k-\widehat{f}_k}\right|^2 N^{(0)}(d\,h)$$
$$=\lim_{k\to\infty}\left(\int\frac{N^{(0)}(d\,h)}{|h-z_k-f_k|^2}+\int\frac{N^{(0)}(d\,h)}{|h-\widehat{z}_k-\widehat{f}_k|^2}\right.$$
$$\left.-2\Re\int\frac{N^{(0)}(d\,h)}{(h-\widehat{z}_k-\widehat{f}_k)(h-\overline{z}_k-\overline{f}_k)}\right)=0.$$

Hence, we have also for any $M>0$

$$\lim_{k\to\infty}\int\limits_{-M}^{M}\left|\frac{1}{h-z_k-f_k}-\frac{1}{h-\widehat{z}_k-\widehat{f}_k}\right|^2 N^{(0)}(d\,h)=0. \tag{3.12}$$

Let us take the segment $\triangle_M=[-M,M]$ such that $N^{(0)}(\triangle_M)>0$. Formula (3.12) yields that for any $\varepsilon>0$ and for any $k>K$ there exists $h=h(k)\in\triangle_M$ such that

$$\left|\frac{1}{h-z_k-f_k}-\frac{1}{h-\widehat{z}_k-\widehat{f}_k}\right|=\left|\frac{z_k-\widehat{z}_k+f_k-\widehat{f}_k}{(h-z_k-f_k)(h-\widehat{z}_k-\widehat{f}_k)}\right|\le\varepsilon.$$

Since $|h-z_k-f_k|\le M+\lambda_0+|\phi_0|$, $|h-\widehat{z}_k-\widehat{f}_k|\le M+\lambda_0+|\phi_1|$, the last inequality yields

$$|z_k-\widehat{z}_k+f_k-\widehat{f}_k|\le C\varepsilon$$

for any $\varepsilon>0$ and for any $k>K=K(\varepsilon)$. This is evidently impossible for $\phi_0\neq\phi_1$. Therefore, we proved that for any $\lambda\in\mathbb{R}$ there exists $\lim\limits_{z\to\lambda}f(z)$.

Let us prove the uniqueness of the solution. Suppose that there are z_1,z_2 : $z_1\neq z_2$ in the open upper half-plane such that

$$z_1-f^{(0)}(z_1)=\lambda,\quad z_2-f^{(0)}(z_2)=\lambda.$$

Again, analogously to (3.10) and (3.11), we obtain

$$(z_1-\overline{z}_2)\left(1-\int\frac{N^{(0)}(d\,h)}{(h-z_1)(h-\overline{z}_2))}\right)=0,$$

thus

$$\int\frac{N^{(0)}(d\,h)}{(h-z_1)(h-\overline{z}_2)}=1. \tag{3.13}$$

Considering the imaginary part (3.6), we get for $i = 1, 2$

$$\int \frac{N^{(0)}(dh)}{|z_i - h|^2} = 1. \tag{3.14}$$

Therefore, (3.13) and (3.14) yield

$$\int \left| \frac{1}{h - z_1} - \frac{1}{h - z_2} \right| N^{(0)}(dh) = 0,$$

and hence $z_1 = z_2$.

Set

$$z(\lambda) = \lambda + f(\lambda + i\,0) \tag{3.15}$$

for λ such that $\Im z(\lambda) = \Im f(\lambda + i\,0) > 0$. Using (1.13), we obtain that

$$z(\lambda) - f^{(0)}(z(\lambda)) = \lambda.$$

Hence, for any λ, such that $\Im f(\lambda + i\,0) > 0$, there exists a solution of (3.6) in the open upper half-plane.

Let us prove now the continuity of $f(\lambda + i0)$. Given $\varepsilon > 0$ and $\lambda_1 \in \mathbb{R}$, there exists $\delta_1 > 0$ such that

$$|f(z) - f(\lambda_1 + i0)| < \varepsilon/2, \quad \forall z : |z - \lambda_1| < \delta_1, \ \Im z > 0.$$

Choose $\lambda_2 \in \mathbb{R}$ such that $|\lambda_1 - \lambda_2| < \delta_1$. Then there exists $\delta_2 > 0$ such that

$$|f(z) - f(\lambda_2 + i0)| < \varepsilon/2, \quad \forall z : |z - \lambda_2| < \delta_2, \ \Im z > 0.$$

Hence, there exists $z \in \mathbb{C}^+$, satisfying the both inequalities, and we can write the inequality

$$|f(\lambda_1 + i0) - f(\lambda_2 + i0)| \le |f(z) - f(\lambda_1 + i0)| + |f(z) - f(\lambda_2 + i0)| < \varepsilon,$$

implying the continuity of $f(\lambda + i0)$ and, thus, the continuity of $z(\lambda)$ of (3.15).

Therefore, we proved that for any $\lambda \in \mathbb{R}$ there exists $\lim\limits_{z \to \lambda} \Im f(z)$ and this limit is continuous in λ. According to the Stieltjes–Perron formula it means that the measure N is absolutely continuous and its density $\rho(\lambda) = 1/\pi \Im f(\lambda + i\,0)$ is continuous. Moreover, (1.13) and (3.6) imply (3.7). The lemma is proved. ■

Now let us choose the constant in (3.2) such that

$$S^* = \Re \left(z_n^2(\lambda_0)/2 + \frac{1}{n} \sum_{j=1}^{n} \ln(z_n(\lambda_0) - h_j^{(n)}) - \lambda_0 z_n(\lambda_0) \right) \tag{3.16}$$

and study the behavior of $\Re S_n(z_n(\lambda), \lambda_0)$ on the contour C_n of (3.5).

Lemma 2. *Let z belong to the upper part of C_n, i.e., $z = z_n(\lambda) = x_n(\lambda) + iy_n(\lambda)$, $y_n(\lambda) > 0$, $\lambda \in \bigcup_{j=1}^{k} I_j$, where*

$$z_n(\lambda) - f_n^{(0)}(z_n(\lambda)) = \lambda. \tag{3.17}$$

Then $\Re S_n(z_n(\lambda), \lambda_0) \geq 0$, and the equality holds only at $\lambda = \lambda_0$. The same is valid for the lower part of C_n, i.e., $z = \overline{z_n(\lambda)}$.

P r o o f. The real and the imaginary parts of (3.17) yield for $x_n = \Re z_n$ and $y_n = \Im z_n$:

$$\begin{cases} x_n(\lambda) + \dfrac{1}{n} \displaystyle\sum_{j=1}^{n} \dfrac{x_n(\lambda) - h_j^{(n)}}{(x_n(\lambda) - h_j^{(n)})^2 + y_n^2(\lambda)} & = \lambda, \\ y_n(\lambda) \left(1 - \dfrac{1}{n} \displaystyle\sum_{j=1}^{n} \dfrac{1}{(x_n(\lambda) - h_j^{(n)})^2 + y_n^2(\lambda)} \right) & = 0. \end{cases} \tag{3.18}$$

Differentiating (3.17) with respect to λ, we obtain

$$z_n'(\lambda) \left(1 - \frac{d}{d\,z} f_n^{(0)}(z_n(\lambda)) \right) = 1,$$

i.e.,

$$z_n'(\lambda) = \left(1 - \frac{d}{d\,z} f_n^{(0)}(z_n(\lambda)) \right)^{-1}, \tag{3.19}$$

where $f_n^{(0)}(z)$ is defined in (3.3).

It follows from the implicit function theorem that C_n intersects the real axis at the points where

$$1 - \frac{d}{d\,x} f_n^{(0)}(x) = 0.$$

Since

$$\frac{d}{d\,x} f_n^{(0)}(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{(x - h_j^{(n)})^2},$$

the inequality

$$1 - \frac{d}{d\,x} f_n^{(0)}(x) < 0 \tag{3.20}$$

holds in a neighborhood of every $h_j^{(n)}$, $j = 1, \ldots, n$. Thus, the function $1 - \frac{d}{d\,x} f_n^{(0)}(x)$ is always positive for real x outside C_n. On the other hand, we have $z_n(\lambda) = x_n(\lambda)$ outside C_n and in this case we get

$$x_n'(\lambda) = z_n'(\lambda) = \left(1 - \frac{d}{d\,z} f_n^{(0)}(z_n(\lambda)) \right)^{-1} > 0. \tag{3.21}$$

Now let $\lambda \in \bigcup_{j=1}^{k} I_j$, i.e., $z_n(\lambda)$ belongs to C_n. We get from (3.19)

$$\Re z_n'(\lambda) = x_n'(\lambda) = \Re\left(\left(1 - \frac{d}{d\,z} f_n^{(0)}(z_n(\lambda))\right)^{-1}\right) = \frac{a_n(\lambda)}{a_n^2(\lambda) + b_n^2(\lambda)}, \tag{3.22}$$

where

$$\begin{cases} a_n(\lambda) & = \Re\left(1 - \dfrac{d}{d\,z} f_n^{(0)}(z_n(\lambda))\right), \\ b_n(\lambda) & = \Im\left(1 - \dfrac{d}{d\,z} f_n^{(0)}(z_n(\lambda))\right), \end{cases} \tag{3.23}$$

and hence

$$a_n(\lambda) = 1 - \frac{1}{n}\sum_{j=1}^{n} \frac{(x_n(\lambda) - h_j^{(n)})^2 - y_n^2(\lambda)}{((x_n(\lambda) - h_j^{(n)})^2 + y_n^2(\lambda))^2}.$$

Taking into account that $y_n(\lambda) \neq 0$ for $\lambda \in \bigcup_{j=1}^{k} I_j$, we obtain from (3.18) that

$$\frac{1}{n}\sum_{j=1}^{n} \frac{1}{(x_n(\lambda) - h_j^{(n)})^2 + y_n^2(\lambda)} = 1. \tag{3.24}$$

This and the previous equation yield

$$a_n(\lambda) = \frac{1}{n}\sum_{j=1}^{n} \frac{2y_n^2(\lambda)}{((x_n(\lambda) - h_j^{(n)})^2 + y_n^2(\lambda))^2} > 0. \tag{3.25}$$

It follows from (3.23) and (3.25) that in this case $x_n'(\lambda) > 0$ if only $y_n(\lambda) \neq 0$. Thus, $x_n(\lambda)$ is a strictly monotone increasing function defined everywhere in $\mathbb{R}$.

Substituting the expression $z_n(\lambda) = x_n(\lambda) + iy_n(\lambda)$ into (3.2) and using (3.4), we obtain

$$\frac{d}{d\,\lambda}\Re S_n(z_n(\lambda), \lambda_0) = \Re\left(z_n'(\lambda)\left(z_n(\lambda) + \frac{1}{n}\sum_{j=1}^{n}\frac{1}{z_n(\lambda) - h_j^{(n)}} - \lambda_0\right)\right) = x_n'(\lambda)(\lambda - \lambda_0). \tag{3.26}$$

Since $x_n'(\lambda) > 0$ (see (3.22), (3.25) and (3.21)), the function $\Re S_n(z_n(\lambda), \lambda_0)$ has a minimum at $\lambda = \lambda_0$, and since $\Re S_n(z_n(\lambda_0), \lambda_0) = 0$, $\Re S_n(z_n(\lambda), \lambda_0) \geq 0$, and the equality holds only at $\lambda = \lambda_0$.

Note that the lower part of C_n differs from the upper one only by the sign of $y_n(\lambda)$, hence $\Re S_n(z, \lambda_0) \geq 0$, $z \in C_n$, and the equality holds only at $z = z(\lambda_0)$ and $z = \overline{z(\lambda_0)}$. ■

A similar fact about the behavior of $\Re S_n(z,\lambda_0)$ along the line

$$L_n = \{z \in \mathbb{C} : z = \zeta_n(y) = x_n(\lambda_0) + i\,y\} \tag{3.27}$$

is given by

Lemma 3. *Consider the part of L_n lying in the upper half-plane $y > 0$. On this part $\Re S_n(z,\lambda_0) = \Re S_n(\zeta_n(y),\lambda_0) \le 0$ and the equality holds only at $y = y_n(\lambda_0)$. The same is valid for the lower part of L_n and $y = -y_n(\lambda_0)$.*

P r o o f. The function $\Re S_n(z,\lambda_0)$ is for $z \in L_n$

$$\Re S_n(\zeta_n(y),\lambda_0) = \frac{x_n^2(\lambda_0) - y^2}{2} + \frac{1}{n}\Re\sum_{j=1}^{n} \ln(x_n(\lambda_0) + iy - h_j^{(n)}) - \lambda_0 x_n(\lambda_0) - S^*.$$

Differentiating this with respect to y, we obtain

$$\frac{d}{d\,y}\Re S_n(\zeta_n(y),\lambda_0) = y\left(-1 + \frac{1}{n}\sum_{j=1}^{n}\frac{1}{(x_n(\lambda_0) - h_j^{(n)})^2 + y^2}\right). \tag{3.28}$$

Taking into account that

$$\sum_{j=1}^{n}\frac{1}{(x_n(\lambda_0) - h_j)^2 + y^2}$$

is monotone in y, we have from (3.24) that $\frac{d}{d\,y}\Re S_n(\zeta_n(y),\lambda_0)$ has a unique zero at $y = y_n(\lambda_0)$ (for $y > 0$), hence, $y = y_n(\lambda_0)$ is a maximum point of $\Re S_n(\zeta_n(y),\lambda_0)$. Similarly, for $y < 0$ the maximum point is $y = -y_n(\lambda_0)$. Therefore, $\Re S_n(z,\lambda_0) \le 0$ on L_n and the equality holds only at $z = z(\lambda_0)$ or $z = \overline{z(\lambda_0)}$. ∎

Thus, we have for $t \in L_n$ and $v \in C_n$ (see Lemmas 2 and 3)

$$\Re(n(S_n(t,\lambda_0) - S_n(v,\lambda_0))) \le 0, \tag{3.29}$$

and the equality holds only if v and t are both equal to $z_n(\lambda_0)$ or $\overline{z_n(\lambda_0)}$.

We need below the second derivative of $\Re S_n(z,\lambda_0)$. Assume that $\lambda \in U_\delta(\lambda_0) = (\lambda_0 - \delta, \lambda_0 + \delta)$. We get from (3.26)

$$\frac{d^2}{d\,\lambda^2}\Re(-S_n(z_n(\lambda),\lambda_0)) = -x_n'(\lambda) + x_n''(\lambda)(\lambda_0 - \lambda). \tag{3.30}$$

Lemma 4. *There exists an n-independent $c > 0$ and $\delta > 0$ such that*

$$\frac{d^2}{d\,\lambda^2}\Re(-S_n(z_n(\lambda),\lambda_0)) < -c$$

for any $\lambda \in U_\delta(\lambda_0)$.

P r o o f. It follows from (3.30) that to prove the lemma it is sufficient to show that the second derivative $x_n''(\lambda)$ is bounded uniformly in n and that the first derivative $x_n'(\lambda)$ is bounded from below by a positive constant uniformly in n in some sufficiently small neighborhood $U_\delta(\lambda_0)$ of λ_0. Thus, we will show that $x_n'(\lambda) \geq C$ for all $\lambda \in U_\delta(\lambda_0)$.

Note that the inequality $2|y_n(\lambda)||x_n(\lambda) - h_j^{(n)}| \leq (x_n(\lambda) - h_j^{(n)})^2 + y_n^2(\lambda)$ and (3.24) yield for b_n of (3.23)

$$|b_n(\lambda)| = \left|\frac{1}{n}\sum_{j=1}^{n}\frac{2y_n(\lambda)(x_n(\lambda) - h_j^{(n)})}{((x_n(\lambda) - h_j^{(n)})^2 + y_n^2(\lambda))^2}\right| \leq \frac{1}{n}\sum_{j=1}^{n}\frac{2|y_n(\lambda)||(x_n(\lambda) - h_j^{(n)})|}{((x_n(\lambda) - h_j^{(n)})^2 + y_n^2(\lambda))^2}$$
$$\leq \frac{1}{n}\sum_{j=1}^{n}\frac{1}{(x_n(\lambda) - h_j^{(n)})^2 + y_n^2(\lambda)} = 1.$$

This and (3.22) imply

$$x_n'(\lambda) \geq \frac{a_n(\lambda)}{a_n^2(\lambda) + 1}. \tag{3.31}$$

Lemma 5 (see below) and (3.31) yield that $x_n'(\lambda) \geq C$ for all $\lambda \in U_\delta(\lambda_0)$. By the same lemma x_n'' is uniformly bounded, thus the second term in (3.30) is of $O(\delta)$. Lemma 4 is proved. ■

Lemma 5. *There exists n-independent C_1 and C_2 such that we have for all $\lambda \in U_\delta(\lambda_0)$*

$$|x_n(\lambda)| \leq C_1, \quad 0 < C_2 \leq |y_n(\lambda)| \leq 1, \quad |x_n''(\lambda)| \leq C_1, \tag{3.32}$$

where an n-independent δ is small enough. Moreover,

$$0 < c_1 < a_n(\lambda) < c_2,\ \lambda \in U_\delta(\lambda_0), \tag{3.33}$$

for some n-independent c_1 and c_2.

P r o o f. We use Lemma 1. Consider the solution $z(\lambda)$ of limiting equation (3.6). It follows from the lemma and the hypothesis of Theorem 1 ($\rho(\lambda_0) > 0$) that $\Im z(\lambda_0) > 0$. Taking into account the continuity of $z(\lambda)$, for any $\varepsilon_1 > 0$ we can take a sufficiently small neighborhood $U_{\delta_1}(\lambda_0)$ such that

$$|z(\lambda) - z(\lambda_0)| < \varepsilon_1/2,\ \lambda \in U_{\delta_1}(\lambda_0). \tag{3.34}$$

Note that we can choose λ_0-independent δ_1 since $z(\lambda)$ is uniformly continuous.

Consider the one-parameter family of the functions $\varphi_\lambda(z) = -f^{(0)}(z) + z - \lambda$ and the function $\phi_n(z) = -f_n^{(0)}(z) + f^{(0)}(z)$, where $f^{(0)}$, $f_n^{(0)}$ are defined in (3.6),(3.3), and the set $\omega = \{z : |z - z(\lambda_0)| \le \varepsilon_1\}$. Let us show that for any $\lambda \in U_{\delta_1}(\lambda_0)$ and $z \in \partial\omega$ we have

$$|\varphi_\lambda(z)| > c_0, \tag{3.35}$$

where c_0 does not depend on λ. Assume the opposite and choose a sequence $\{\lambda_k\}_{k\ge1}$,$\lambda_k \in U_{\delta_1}(\lambda_0)$ such that $|\varphi_{\lambda_k}(z_k)| \to 0$, as $k \to \infty$. There exists a subsequence $\{\lambda_{k_m}\}$, converging to some $\lambda \in U_{\delta_1}(\lambda_0)$ such that the subsequence $\{z_{k_m}\}$ converges to $z \in \partial\omega$. For these λ and z we have $\varphi_\lambda(z) = 0$. But the equation $\varphi_\lambda(z) = 0$ has in the upper half-plane only one root $z(\lambda)$, which is inside the circle of the radius $\varepsilon_1/2$ and with the center $z(\lambda_0)$. This contradiction proves (3.35).

Since for any $\varepsilon > 0$ there exists n_0 such that for any $n > n_0$

$$|f_n^{(0)}(z) - f^{(0)}(z)| \le \varepsilon \tag{3.36}$$

for z on any compact set of the upper half-plane (recall the weak convergence of $\{N_n^{(0)}\}$ to $N^{(0)}$), we have for $n > n_0$, where n_0 is big enough

$$|\phi_n(z)| < c_0,\ z \in \partial\omega. \tag{3.37}$$

Comparing (3.35) and (3.37), we obtain for $n > n_0$

$$|\varphi_\lambda(z)| > |\phi_n(z)|,\ z \in \partial\omega,\ \forall\lambda \in U_{\delta_1}(\lambda_0).$$

Since both functions are analytic, the Rouchet theorem implies that $\varphi_\lambda(z)$ and $\varphi_\lambda(z) + \phi_n(z) = z - f_n^{(0)}(z) - \lambda$ have the same number of zeros in ω. Since $\varphi_\lambda(z)$ has only one zero in ω, we conclude that $z_n(\lambda)$ belongs to ω, $x_n(\lambda)$ is bounded, and $y_n(\lambda) \ge C_2 > 0$ uniformly in n if $\lambda \in U_\delta(\lambda_0)$ for any $\delta < \delta_1$. Besides, (3.24) yields that $0 \le y_n(\lambda) \le 1$ for any λ. Since $z_n(\lambda)$ is real analytic, we have proved also that $x_n''(\lambda)$ is bounded uniformly in n if $\lambda \in U_\delta(\lambda_0)$ for any $\delta < \delta_1$ (since $|x''(\lambda) - x_n''(\lambda)| \le C\varepsilon_1$, $\lambda \in U_\delta(\lambda_0)$). Thus, we have proved (3.32).

Note that we have also proved that for any λ_0 such that $\rho(\lambda_0) > 0$ and for any $\varepsilon_1 > 0$ there exists δ such that for any $\lambda \in U_\delta(\lambda_0)$ and any $n > N(\delta, \varepsilon_1)$

$$|z_n(\lambda) - z(\lambda)| \le 2\varepsilon_1. \tag{3.38}$$

Since $f_n^{(0)}$ is analytic for $\Im z \ne 0$, we have

$$\left|\frac{d}{d\,z} f_n^{(0)}(z) - \frac{d}{d\,z} f^{(0)}(z)\right| \le C\varepsilon \tag{3.39}$$

uniformly on any compact set of the upper half-plane. This, (3.38) and (3.23) imply that it is sufficient to prove (3.33) for

$$\Re\left(1-\frac{d}{d\,z}f^{(0)}(z(\lambda))\right)=\int\frac{2y^2(\lambda)N^{(0)}(d\,h)}{((x(\lambda)-h)^2+y^2(\lambda))^2},\ \lambda\in U_\delta(\lambda_0). \tag{3.40}$$

Now (3.33) follows from (3.34), $\Im z(\lambda_0)>0$, and $\operatorname{supp}N^{(0)}\subset[-M,M]$, $M<\infty$. ■

It follows from Lemma 5 and the equalities

$$\Re S_n(z_n(\lambda_0),\lambda_0)=0,\quad \frac{d}{d\,\lambda}\Re S_n(z_n(\lambda),\lambda_0)\Big|_{\lambda=\lambda_0}=0$$

(see (3.2), (3.16), and (3.26)) that

$$\Re\left(-S_n(z_n(\lambda),\lambda_0)\right)<-c\frac{(\lambda-\lambda_0)^2}{2},\ \lambda\in U_\delta(\lambda_0). \tag{3.41}$$

Since $\dfrac{d}{d\,\lambda}\Re(S_n(z_n(\lambda),\lambda_0))$ has a unique zero at $\lambda=\lambda_0$ (see (3.21), (3.22), (3.25), and (3.26)), the function $\Re\left(S_n(z_n(\lambda),\lambda_0)\right)$ is monotone for $\lambda\neq\lambda_0$, and we have

$$\Re(-S_n(z_n(\lambda),\lambda_0))<-c\frac{\delta^2}{2},\ \lambda\notin U_\delta(\lambda_0). \tag{3.42}$$

We need an analogous fact in a neighborhood of $z_n(\lambda_0)$ on L_n. We get from (3.2)

$$\frac{d^2}{d\,y^2}\Re(S_n(\zeta_n(y),\lambda_0))=-\Re\left(1-\frac{d}{d\,z}f_n^{(0)}(\zeta_n(y))\right), \tag{3.43}$$

where $\zeta_n(y)$ is defined in (3.27). Since for any $\varepsilon_1>0$ there exists N such that for $n>N$ we have $|z(\lambda_0)-z_n(\lambda_0)|\leq\varepsilon_1/2$ (see (3.38)), we can choose $\delta>0$ such that $|\zeta_n(y)-z(\lambda_0)|\leq\varepsilon_1$ for $y\in U_{2\delta}(y(\lambda_0))$. We obtain for such y (see (3.38) and (3.39))

$$\left|\frac{d}{d\,z}f_n^{(0)}(\zeta_n(y))-\frac{d}{d\,z}f^{(0)}(z(\lambda_0))\right|\leq C\varepsilon_1.$$

But since expression in (3.40) for $\lambda=\lambda_0$ is bounded from below by a positive constant, the previous inequality and (3.43) yield

$$\frac{d^2}{d\,y^2}\Re(S_n(\zeta_n(y),\lambda_0))<-c,\ \ y\in U_{2\delta}(y(\lambda_0)).$$

Recall that $y_n(\lambda_0)\in U_\delta(y(\lambda_0))$ starting from some n. Hence, we obtain if n is big enough

$$|y_n(\lambda_0)-y|>\delta,\ y\notin U_{2\delta}(y(\lambda_0)).$$

Thus, since

$$\Re S_n(z_n(\lambda_0),\lambda_0)=0,\quad \frac{d}{d\,y}\Re S_n(\zeta_n(y),\lambda_0)\Big|_{y=y_n(\lambda_0)}=0$$

(see (3.2), (3.16), (3.24), and (3.28)), we get

$$\Re(S_n(\zeta_n(y),\lambda_0))<-c\frac{(y-y_n(\lambda_0))^2}{2},\ \ y\in U_{2\delta}(y(\lambda_0)). \tag{3.44}$$

Since $\frac{d}{d\,y}\Re(S_n(\zeta_n(y),\lambda_0))$ has a unique zero at $y=y_n(\lambda_0)$ (see (3.24) and (3.28)), the function $\Re(S_n(\zeta_n(y),\lambda_0))$ is monotone for $y\neq y_n(\lambda_0)$, and we have

$$\Re(S_n(\zeta_n(y),\lambda_0))<-c\frac{\delta^2}{2},\ \ y\notin U_{2\delta}(y(\lambda_0)). \tag{3.45}$$

Besides, since it is easy to see that $\frac{d^2}{d\,y^2}\Re(S_n(\zeta_n(y),\lambda_0))\to -1$ as $y\to\infty$ uniformly in n, $\Re(S_n(\zeta_n(y),\lambda_0))$ is convex for $|y|>K$ for some sufficiently big n-independent $K>0$. Hence, we get for such K (recall that $z_n(\lambda_0)$ is in some neighborhood of $z(\lambda_0)$)

$$\Re(S_n(\zeta_n(y),\lambda_0))<-c_1|y|+c_2,\ c_1>0,\ |y|>K. \tag{3.46}$$

Denote $U_1=U_\delta(\lambda_0)$, $U_2=U_{2\delta}(y(\lambda_0))$. Using formulas (3.41), (3.42), and (3.44)–(3.46), we obtain for all sufficiently big n and K

$$\Bigg|\int\limits_{L_n}\frac{dt}{2\pi}\oint\limits_{C_n}\frac{dv}{2\pi}\exp\{v\xi-t\eta\}\frac{\exp\{n(S_n(t,\lambda_0)-S_n(v,\lambda_0))\}}{v-t}\Bigg|$$

$$\le C\left(\int\limits_{U_2}\int\limits_{U_1}+\int\limits_{U_2}\int\limits_{C_n\backslash U_1}\right)\frac{\exp\{\Re(n(S_n(\zeta_n(y),\lambda_0)-S_n(z_n(\lambda),\lambda_0)))\}|z_n'|}{|z_n(\lambda)-\zeta_n(y)|}d\,\lambda\,d\,y$$
$$+\left(\int\limits_{L_n\backslash U_2}\int\limits_{U_1}+\int\limits_{L_n\backslash U_2}\int\limits_{C_n\backslash U_1}\right)\frac{\exp\{\Re(n(S_n(\zeta_n(y),\lambda_0)-S_n(z_n(\lambda),\lambda_0)))\}|z_n'|}{|z_n(\lambda)-\zeta_n(y)|}d\,\lambda\,d\,y$$
$$\le C\int\limits_{U_2}\int\limits_{U_1}\frac{|z_n'(\lambda)|d\,\lambda dy}{|z_n(\lambda)-\zeta_n(y)|}+C_1\cdot|C_n|\cdot e^{-cn\delta^2/2}$$
$$+C_2(Ke^{-cn\delta^2/2}+e^{-n(c_1K-c_2)})+C_3\cdot|C_n|\cdot e^{-cn\delta^2/2}(Ke^{-cn\delta^2/2}+e^{-n(c_1K-c_2)}), \tag{3.47}$$

where $|C_n|$ is the length of C_n and $c_1 K - c_2 > 0$. Note that

$$\int\limits_{U_2}\int\limits_{U_1} \frac{|z_n'(\lambda)| d\,\lambda dy}{|z_n(\lambda) - \zeta_n(y)|} \le \int\limits_{U_2}\int\limits_{U_1} \frac{|z_n'(\lambda)| d\,\lambda dy}{\sqrt{(1-\cos\alpha_n + o(\delta))(|z_n(\lambda)|^2 + |\zeta_n(y)|^2)}},$$

where α_n is the angle between C_n and L_n at $z(\lambda_0)$, i.e., $\cot\alpha_n = \dfrac{y_n'(\lambda_0)}{x_n'(\lambda_0)}$, where $x_n(\lambda) = \Re z_n(\lambda)$, $y_n(\lambda) = \Im z_n(\lambda)$. Since $x_n'(\lambda_0) > c > 0$ (see (3.31) and (3.33)), $\cos\alpha_n < 1 - \varepsilon$, and we can write

$$\int\limits_{U_2}\int\limits_{U_1} \frac{|z_n'(\lambda)| d\,\lambda dy}{\sqrt{(1-\cos\alpha_n + o(\delta))(|z_n(\lambda)|^2 + |\zeta_n(y)|^2)}} \le C_0 \int\limits_{U_2}\int\limits_{U_1} \frac{|z_n'(\lambda)| d\,\lambda dy}{\sqrt{|z_n(\lambda)|^2 + |\zeta_n(y)|^2}} \le C\delta. \quad (3.48)$$

Now we need

Lemma 6. *Let $l(x)$ be the oriented length of the upper part of the contour C_n between $x_0 = x_n(\lambda_0)$ and x (we take $l(x) > 0$ for $x > x_0$ to obtain $l'(x) > 0$). Then for any collection $\{h_j^{(n)}\}_{j=1}^n$, $l(x)$ admits the bound*

$$|l(x_1) - l(x_2)| \le C|x_1 - x_2|$$

with an absolute constant C. Moreover,

$$|C_n| \le Cn,$$

where $|C_n|$ is the length of C_n.

P r o o f. We will find the bound for the length of the part of C_n between the lines $\Re z = x_1$ and $\Re z = x_2$, $x_2 - x_1 = 2$. It follows from (3.22) and (3.25) that one can express $y_n(\lambda)$ via $x_n(\lambda)$ to obtain the "graph" $y_n(x)$ of the upper part of C_n. Denote

$$\begin{gathered} y_n^2(x) = s(x), \quad x - h_j^{(n)} = \triangle_j, \\ \sigma_k = \frac{1}{n}\sum_{j=1}^n \frac{1}{(\triangle_j^2 + s)^k}, \quad \sigma_{kl} = \frac{1}{n}\sum_{j=1}^n \frac{\triangle_j^l}{(\triangle_j^2 + s)^k} \; k = \overline{1,3},\, l = 1,2. \end{gathered} \quad (3.49)$$

Differentiating (3.24) with respect to x, we obtain the equality

$$-s'\frac{1}{n}\sum_{j=1}^n \frac{1}{(\triangle_j^2 + s)^2} - \frac{2}{n}\sum_{j=1}^n \frac{\triangle_j}{(\triangle_j^2 + s)^2} = 0$$

implying that

$$|s'| = 2|\sigma_{21}|\sigma_2^{-1} \le 2\sigma_{22}^{1/2}\sigma_2^{-1/2} \le 2\sigma_2^{-1/2} \le 2\sigma_1^{-1} = 2. \tag{3.50}$$

Differentiating (3.24) with respect to x twice, we have

$$s''\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{(\triangle_j^2+s)^2}\right) - 2(s')^2\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{(\triangle_j^2+s)^3}\right)$$
$$-8s'\left(\frac{1}{n}\sum_{j=1}^{n}\frac{\triangle_j}{(\triangle_j^2+s)^3}\right) + \frac{2}{n}\sum_{j=1}^{n}\frac{(\triangle_j^2+s)^2 - 4\triangle_j^2(\triangle_j^2+s)}{(\triangle_j^2+s)^4} = 0, \tag{3.51}$$

or, in our notations (3.49),

$$s''\sigma_2 - 2(s')^2\sigma_3 - 8s'\sigma_{31} + 2(4s\sigma_3 - 3\sigma_2) = 0. \tag{3.52}$$

Note that

$$s\sigma_3 = \frac{1}{n}\sum_{j=1}^{n}\frac{s}{(\triangle_j^2+s)^3} \le \frac{1}{n}\sum_{j=1}^{n}\frac{1}{(\triangle_j^2+s)^2} = \sigma_2,$$

and also

$$\sigma_{31}^2 = \left(\frac{1}{n}\sum_{j=1}^{n}\frac{\triangle_j}{(\triangle_j^2+s)^3}\right)^2 \le \frac{1}{n}\sum_{j=1}^{n}\frac{\triangle_j^2}{(\triangle_j^2+s)^3}\cdot\frac{1}{n}\sum_{j=1}^{n}\frac{1}{(\triangle_j^2+s)^3} \le \sigma_2\sigma_3.$$

Using these inequalities, we get from (3.52)

$$s''\sigma_2 = 2(s')^2\sigma_3 + 8s'\sigma_{31} - 2(4s\sigma_3 - 3\sigma_2) = 2\sigma_3\left(s' + 2\sigma_{31}/\sigma_3\right)^2 - 8\sigma_{31}^2/\sigma_3$$
$$-8s\sigma_3 + 6\sigma_2 \ge -8\sigma_{31}^2/\sigma_3 - 2\sigma_2 \ge -10\sigma_2$$

or

$$s'' \ge -10. \tag{3.53}$$

Let $x_* \in [x_1; x_2]$ be the maximum point of $y_n(x)$ and

$$y_n'(x) = \frac{s'(x)}{2\sqrt{s(x)}} > 0, \quad x \in [x_3, x_*]$$

for some $x_3 \in [x_1; x_*]$. Then we have

$$l(x_*) - l(x_3) = \int_{x_3}^{x_*}\sqrt{1 + (y_n'(x))^2}\,d\,x = \int_{x_3}^{x_*}\sqrt{1 + \left(\frac{s'(x)}{2\sqrt{s(x)}}\right)^2}\,d\,x$$
$$\le \int_{x_3}^{x_*}\left(1 + \frac{s'(x)}{2\sqrt{s(x)}}\right)d\,x = (x_* - x_3) + \sqrt{s_*} - \sqrt{s_3} \le (x_* - x_3) + \sqrt{s_* - s_3}, \tag{3.54}$$

where $s_* = s(x_*)$, $s_3 = s(x_3)$. Taking into account that $s'(x_*) = 0$, we write

$$s_3 - s_* = \frac{s''(\xi)(x_3 - x_*)^2}{2},$$

where $\xi \in [x_3, x_*]$. This and (3.53) imply

$$0 \le s_* - s_3 \le 5(x_3 - x_*)^2.$$

Hence, we get in view of (3.54)

$$l(x_*) - l(x_3) \le (1 + \sqrt{5})(x_* - x_3). \tag{3.55}$$

We have a similar inequality for $x_3 > x_*$ and $y_n'(x) < 0$, $x \in [x_*, x_3]$. Take now an arbitrary $x_3 \in [x_1; x_2]$ and denote x_* the nearest to x_3 maximum point of $y_n(x)$ in $[x_1, x_2]$. Then, splitting $[x_1, x_*]$ in the segments of monotonicity of y_n and using (3.50), (3.55) and its analog for decreasing $y_n(x)$, we obtain

$$\begin{aligned} l(x_3) - l(x_1) = l(x_*) - l(x_1) + \int_{x_*}^{x_3} l'(x)dx \le (1+\sqrt{5})(x_* - x_1) + \int_{x_*}^{x_3} \left(1 + \frac{|s'(x)|}{2\sqrt{s(x)}}\right) dx \\ \le (1 + \sqrt{5})(x_* - x_1) + (x_3 - x_*) + \sqrt{|s_3 - s_*|} \\ \le (1 + \sqrt{5})(x_* - x_1) + (x_3 - x_*) + \sqrt{2}\sqrt{x_3 - x_*} \le C\sqrt{x_3 - x_1}, \end{aligned} \tag{3.56}$$

where the last inequality holds, because $|x_3 - x_*| \le |x_3 - x_1|$ and $|x_3 - x_1| \le 2$. Hence,

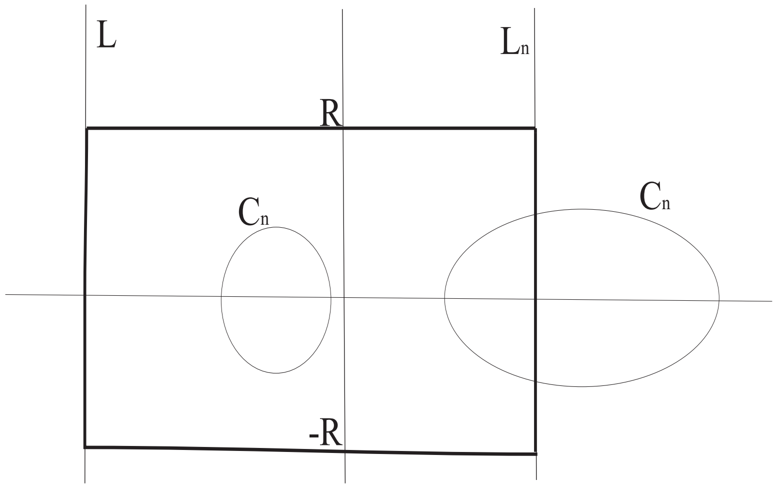
$$l(x_2) - l(x_1) \le C\sqrt{x_2 - x_1} \le C.$$

It follows from (3.24) that $\mathrm{dist}(x_n(\lambda), \{h_j^{(n)}\}_{j=1}^n) \le 1$. Thus, we can cover C_n by n strips of width 2 and obtain that $|C_n| \le Cn$. ∎

Using Lemma 6, (3.48), and (3.47), we get that

$$\lim_{n\to\infty} \int_{L_n} \frac{dt}{2\pi} \oint_{C_n} \frac{dv}{2\pi} \exp\{v\xi - ty\} \frac{\exp\{n(S_n(t, \lambda_0) - S_n(v, \lambda_0))\}}{v - t} = 0. \tag{3.57}$$

Consider the contour C_R of the figure



and the integral

$$\oint_{C_n} \frac{dv}{2\pi} I_n(v), \tag{3.58}$$

where

$$I_n(v) = -\oint_{C_R} \frac{dt}{2\pi} \exp\{v\xi - t\eta\} \frac{\exp\{n(S_n(t,\lambda_0) - S_n(v,\lambda_0))\}}{v-t} \tag{3.59}$$

and the integral is understood in the Cauchy sense for $v \in C_R$, i.e., for $v = z_n(\lambda_0)$ and $v = \overline{z_n(\lambda_0)}$. We have from the residue theorem (see [16, Sects. 1.5 and 3.3])

$$I_n(v) = \begin{cases} i\exp\{v(\xi-\eta)\}, & v \text{ is inside } C_R, \\ 0, & v \text{ is outside } C_R, \\ \dfrac{i}{2}\exp\{v(\xi-\eta)\}, & v = z_n(\lambda_0), \overline{z_n(\lambda_0)} \end{cases}$$

(according to Lemmas 2 and 3 C_n and L_n have only two points of intersection). Hence,

$$\oint_{C_n} \frac{dv}{2\pi} I_n(v) = -\frac{i}{2\pi} \int_{\overline{z_n(\lambda_0)}}^{z_n(\lambda_0)} \exp\{v(\xi-\eta)\}dv = \exp\{x_n(\lambda_0)(\xi-\eta)\} \frac{\sin(y_n(\lambda_0)(\xi-\eta))}{\pi(\xi-\eta)},$$

where $x_n(\lambda) = \Re z_n(\lambda)$, $y_n(\lambda) = \Im z_n(\lambda)$. Note that for any fixed n and any fixed set $\{h_j^{(n)}\}_{j=1}^n$, we have for $t = x \pm iR \in C_R$ in view of Lemma 2

$$\begin{aligned}\Re(S_n(t,\lambda_0) - S_n(z_n(\lambda),\lambda_0)) &\le \Re S_n(t,\lambda_0)\\ &= \frac{x^2-R^2}{2} + \frac{1}{2n}\sum_{j=1}^n \ln((x-h_j^{(n)})^2+R^2) - \lambda_0 x - S^* \le -\frac{R^2}{4}\end{aligned} \quad (3.60)$$

for sufficiently big R. Hence, the integrals over the parts of the lines $\Im z = \pm R$ in (3.59) are bounded by $C_1\, e^{-nR^2/4}$. Thus, we get after the limit $R \to \infty$

$$\begin{aligned}-\oint_{C_n}\frac{dv}{2\pi}\oint_{L\bigcup L_n}\frac{dt}{2\pi}\exp\{v\xi - t\eta\}\frac{\exp\{n(S_n(t,\lambda_0)-S_n(v,\lambda_0))\}}{v-t}\\ = \exp\{x_n(\lambda_0)(\xi-\eta)\}\frac{\sin(y_n(\lambda_0)(\xi-\eta))}{\pi(\xi-\eta)},\end{aligned} \quad (3.61)$$

where $x_n(\lambda) = \Re z_n(\lambda)$, $y_n(\lambda) = \Im z_n(\lambda)$. Therefore, adding (3.61) and (3.47), we obtain

$$\begin{aligned}\frac{1}{n}K_n\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n}\right) &= \int_L\frac{dt}{2\pi}\oint_{C_n}\frac{dv}{2\pi}\exp\{v\xi-t\eta\}\frac{\exp\{n(S_n(t,\lambda_0)-S_n(v,\lambda_0))\}}{v-t}\\ &= \left(\int_{L_n}\frac{dt}{2\pi}\oint_{C_n}\frac{dv}{2\pi} - \oint_{L\cup L_n}\frac{dt}{2\pi}\oint_{C_n}\frac{dv}{2\pi}\right)\exp\{v\xi-t\eta\}\frac{\exp\{n(S_n(t,\lambda_0)-S_n(v,\lambda_0))\}}{v-t}\\ &= \int_{L_n}\frac{dt}{2\pi}\oint_{C_n}\frac{dv}{2\pi}\exp\{v\xi-t\eta\}\frac{\exp\{n(S_n(t,\lambda_0)-S_n(v,\lambda_0))\}}{v-t}\\ &\quad + \exp\{x_n(\lambda_0)(\xi-\eta)\}\frac{\sin(y_n(\lambda_0)(\xi-\eta))}{\pi(\xi-\eta)}\\ &= \exp\{x_n(\lambda_0)(\xi-\eta)\}\frac{\sin(y_n(\lambda_0)(\xi-\eta))}{\pi(\xi-\eta)} + o(1),\ n\to\infty.\end{aligned} \quad (3.62)$$

In view of (3.38), we can write for $n > N$ the inequality $|y_n(\lambda_0) - y(\lambda_0)| \le \varepsilon_1$, where $y_n(\lambda_0) = \Im z_n(\lambda_0)$, $y(\lambda_0) = \Im z(\lambda_0)$ and $z_n(\lambda_0)$ and $z(\lambda_0)$ are solutions of (3.4) and (3.6), respectively, for $\lambda = \lambda_0$. Besides, it follows from (3.62) with $\xi = \eta = 0$ that for

$$\rho_n(\lambda_0) = n^{-1}K_n(\lambda_0,\lambda_0) \quad (3.63)$$

the inequality $|\rho_n(\lambda_0) - \pi^{-1}y_n(\lambda_0)| < \varepsilon_1$ is valid for any $\varepsilon_1 > 0$ and sufficiently big n. Therefore we have proved that $|\rho_n(\lambda_0) - \pi^{-1}y(\lambda_0)| \le C\varepsilon_1$ for $n > N$, i.e., that $\lim_{n\to\infty}\rho_n(\lambda_0) = \pi^{-1}\Im z(\lambda_0)$. This and (3.7) imply that $|\rho_n(\lambda_0) - \rho(\lambda_0)| \le C\varepsilon_1$ for $n > N$. Now we obtain (1.5) by using (2.1), (3.62), and the boundedness and continuity of

$$\det\left\{\frac{\sin(y(x_j-x_k))}{\pi(x_j-x_k)}\right\}_{j,k=1}^m \quad (3.64)$$

in $y \in [0,1]$ for any $m \in \mathbb{N}$ and $|x_j| \le K$, $j = 1, \dots, m$.

To prove the second statement (1.8) of the universality hypothesis we need

Lemma 7. *We have for any set* $\{h_j^{(n)}\}_{j=1}^n$ *(i.e. for any realization if* $H_n^{(0)}$ *is random) and for any* λ_0, $\rho(\lambda_0) > 0$, $|\xi|, |\eta| \le K < \infty$

$$\left| \frac{1}{n} K_n(\lambda_0 + \xi/n, \lambda_0 + \eta/n) \right| \le C, \tag{3.65}$$

where K_n *is defined in (2.2).*

P r o o f. As in the proof of (1.5), take C_n of (3.5) as a contour C in (2.2) and replace the integral over L by that over L_n of (3.27). Using (3.61), as in (3.62) we get

$$\begin{aligned} &\frac{1}{n} K_n(\lambda_0 + \xi/n, \lambda_0 + \eta/n) \\ &= \int\limits_{L_n} \frac{dt}{2\pi} \oint\limits_{C_n} \frac{dv}{2\pi} \exp\{\xi v - t\eta)\} \frac{\exp\{n(S_n(t,\lambda_0) - S_n(v,\lambda_0))\}}{v - t} \\ &\quad + \exp\{x_n(\lambda_0)(\xi - \eta)\} \frac{\sin(y_n(\lambda_0)(\xi - \eta))}{\pi(\xi - \eta)}. \end{aligned} \tag{3.66}$$

Using Lemma 8 (see below) and $|\xi|, |\eta| \le K$, we obtain

$$\exp\{x_n(\lambda_0)(\xi - \eta)\} \le C. \tag{3.67}$$

Besides,

$$\left| \frac{\sin(y_n(\lambda_0)(x_i - x_j))}{\pi(x_i - x_j)} \right| \le \pi^{-1} y_n(\lambda_0). \tag{3.68}$$

If $y_n(\lambda) \ne 0$, then (3.24) implies that $|y_n(\lambda)| \le 1$, thus $y_n(\lambda)$ is bounded uniformly in n for any λ, in particular, for $\lambda = \lambda_0$. Hence, to prove the lemma it is sufficient to find a uniform in n bound for the integral on the r.h.s. of (3.66).

Note that we have for $v = z_n(\lambda) \in C_n$ (see (3.2) and (3.16))

$$\begin{aligned} -\Re S_n(v, \lambda_0) &= -\frac{x_n^2(\lambda) - y_n^2(\lambda)}{2} + \frac{x_n^2(\lambda_0) - y_n^2(\lambda_0)}{2} \\ + \frac{1}{2n} \sum_{j=1}^n &\ln \frac{(x_n(\lambda_0) - h_j^{(n)})^2 + y_n^2(\lambda_0)}{(x_n(\lambda) - h_j^{(n)})^2 + y_n^2(\lambda)} + \lambda_0(x_n(\lambda) - x_n(\lambda_0)). \end{aligned} \tag{3.69}$$

It follows from (3.24) and (3.21) that

$$\begin{aligned}\frac{1}{2n}\sum_{j=1}^{n}\ln\frac{(x_n(\lambda_0)-h_j^{(n)})^2+y_n^2(\lambda_0)}{(x_n(\lambda)-h_j^{(n)})^2+y_n^2(\lambda)} = \frac{1}{n}\sum_{j=1}^{n}\ln\left|\frac{z_n(\lambda_0)-h_j^{(n)}}{z_n(\lambda)-h_j^{(n)}}\right| \\ = \frac{1}{n}\sum_{j=1}^{n}\ln\left|1+\frac{z_n(\lambda_0)-z_n(\lambda)}{z_n(\lambda)-h_j^{(n)}}\right| \le \frac{1}{n}\sum_{j=1}^{n}\frac{|z_n(\lambda_0)-z_n(\lambda)|}{|z_n(\lambda)-h_j^{(n)}|} \\ \le |z_n(\lambda_0)-z_n(\lambda)|\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{|z_n(\lambda)-h_j^{(n)}|^2}\right)^{1/2} \le |z_n(\lambda_0)-z_n(\lambda)|.\end{aligned} \tag{3.70}$$

Thus, (3.69), Lemma 8 (see below), (3.70), and the inequality $|y_n(\lambda)|\le 1$ yield

$$\Re(-S_n(v,\lambda_0)) \le -\frac{x_n^2(\lambda)}{2}+C_1|x_n(\lambda)|+C_2. \tag{3.71}$$

Besides, we have (see (3.2) and (3.16))

$$\Re S_n(t,\lambda_0) = \frac{x_n^2(\lambda_0)-y^2}{2}+\frac{1}{2n}\sum_{j=1}^{n}\ln\frac{(x_n(\lambda_0)-h_j^{(n)})^2+y^2}{(x_n(\lambda_0)-h_j^{(n)})^2+y_n^2(\lambda_0)}, \tag{3.72}$$

where $t = x_n(\lambda_0)+iy\in L_n$. We get analogously to (3.70):

$$\frac{1}{2n}\sum_{j=1}^{n}\ln\frac{(x_n(\lambda_0)-h_j^{(n)})^2+y^2}{(x_n(\lambda_0)-h_j^{(n)})^2+y_n^2(\lambda_0)} \le |t-z_n(\lambda_0)|. \tag{3.73}$$

Thus, (3.72), Lemma 8 (see below), (3.73), and the inequality $|y_n(\lambda_0)|\le 1$ imply

$$\Re(S_n(t,\lambda_0)) \le -\frac{y^2}{2}+C_1|y|+C_2. \tag{3.74}$$

Therefore, (3.71), (3.74), and Lemma 3 yield

$$\Re(S_n(t,\lambda_0)-S_n(v,\lambda_0)) \le \begin{cases} -\dfrac{x_n^2(\lambda)}{2}+C_1|x_n(\lambda)|-\dfrac{y^2}{2}+C_3|y|+C_4, & t\in L_n\setminus J,\\ -\dfrac{x_n^2(\lambda)}{2}+C_1|x_n(\lambda)|+C_2, & t\in J,\end{cases} \tag{3.75}$$

where $J=[x_n(\lambda_0)-iB, x_n(\lambda_0)+iB]\subset L_n$, B is big enough.

Also, it is easy to see that for $t = x_n(\lambda_0)+iy\in L_n$ and $v = z_n(\lambda)\in C_n$ we obtain (see Lemma 8 below and note that $|x|,|y|\le K$)

$$\Re(\xi v - yt) \le C_1|x_n(\lambda)|+C_2$$

and

$$\frac{1}{|v-t|} \le \min\left\{\frac{1}{|x_n(\lambda)-x_n(\lambda_0)|}, \frac{1}{|y-y_n(\lambda)|}\right\}.$$

This and (3.75) imply the bound

$$\int\limits_{L_n}\int\limits_{C_n\setminus C_n^A}\left|\exp\{\xi v-\eta t\}\frac{\exp\{n(S_n(t,\lambda_0)-S_n(v,\lambda_0))\}}{v-t}\right| d\,v\,d\,t$$
$$\le C\int\limits_A^\infty e^{-n\xi^2/2+n\xi C_1+nC_2}d\,l(\xi),$$

where $l(\xi)$ is defined in Lemma 6, and C_n^A is a part of the contour C_n where $|x_n(\lambda)|\le A$. According to Lemma 6 we have

$$\int\limits_A^\infty e^{-n\xi^2/2+n\xi C_1+nC_2}d\,l(\xi) = \sum_{k=A}^\infty \int\limits_k^{k+1} e^{-n\xi^2/2+n\xi C_1+nC_2}d\,l(\xi)$$
$$\le \sum_{k=A}^\infty e^{-nk^2/2+nkC_1+nC_2}(l(k+1)-l(k))$$
$$\le C\sum_{k=A}^\infty e^{-nk^2/2+nkC_1+nC_2} < e^{-nc},$$

if A and n are big enough. Thus we have

$$\int\limits_{L_n}\int\limits_{C_n\setminus C_n^A}\left|\exp\{\xi v-\eta t\}\frac{\exp\{n(S_n(t,\lambda_0)-S_n(v,\lambda_0))\}}{v-t}\right| d\,v\,d\,t \le e^{-nd}. \quad (3.76)$$

We have to estimate now the integral over the part of C_n, where $x_n(\lambda)\in I = [-A,A]$. Applying the same arguments as in the proof of (3.76), we obtain

$$\int\limits_{L_n\setminus J}\int\limits_{C_n^A}\left|\exp\{\xi v-\eta t\}\frac{\exp\{n(S_n(t,\lambda_0)-S_n(v,\lambda_0))\}}{v-t}\right| d\,v\,d\,t \le e^{-nc}, \quad (3.77)$$

where $J=[x_n(\lambda_0)-iB, x_n(\lambda_0)+iB]\subset L_n$ and B is big enough.

Thus, according to (3.76) and (3.77), to prove Lemma 7 it remains to estimate the integral

$$\int\limits_J \frac{dt}{2\pi}\int\limits_{C_n^A}\frac{d\,v}{2\pi}\exp\{\xi v-\eta t\}\frac{\exp\{n(S_n(t,\lambda_0)-S_n(v,\lambda_0))\}}{v-t}.$$

In view of (3.67) and the bound

$$\int\limits_J \frac{dt}{\sqrt{(x-x_0)^2+(t-y_n(x))^2}} \le \sqrt{2}\int\limits_J \frac{dt}{|x-x_0|+|t-y_n(x)|} \le 2\sqrt{2}\ln|x-x_0|^{-1}+C$$

(recall that $|y_n(x)|\le 1$), where $x_0=x_n(\lambda_0)$, we have to estimate the integral

$$\int\limits_I (\ln|x-x_0|^{-1}+C)l'(x)dx, \tag{3.78}$$

where $l(x)$ is the oriented length of the part of C_n between x_0 and x. We can find from (3.56) that

$$-\ln(x-x_0)\le -C\ln l(x),$$

and we obtain for (3.78)

$$\int\limits_I (\ln|x-x_0|^{-1}+C)l'(x)dx \le \int\limits_I (C+\ln l(x))\, l'(x)dx$$
$$= Cl(A)-l(A)\ln l(A)\le C_1.$$

∎

Prove now the statement which we use in the proof of Lemma 7.

Lemma 8. *There exists some n-independent constant C such that we have for any n and any set $\{h_j^{(n)}\}_{j=1}^n$ (i.e. for any realization if $H_n^{(0)}$ is random)*

$$|x_n(\lambda_0)|\le C.$$

P r o o f. Taking the real part of (3.17) we get

$$x_n(\lambda_0)+\frac{1}{n}\sum_{j=1}^n \frac{x_n(\lambda_0)-h_j^{(n)}}{(x_n(\lambda_0)-h_j^{(n)})^2+y_n^2(\lambda_0)}=\lambda_0.$$

This implies

$$|x_n(\lambda_0)|\le|\lambda_0|+\frac{1}{n}\sum_{j=1}^n \frac{|x_n(\lambda_0)-h_j^{(n)}|}{(x_n(\lambda_0)-h_j^{(n)})^2+y_n^2(\lambda_0)}\le|\lambda_0|$$
$$+\left(\frac{1}{n}\sum_{j=1}^n \frac{\left(x_n(\lambda_0)-h_j^{(n)}\right)^2}{(x_n(\lambda_0)-h_j^{(n)})^2+y_n^2(\lambda_0)}\right)^{1/2}\left(\frac{1}{n}\sum_{j=1}^n \frac{1}{(x_n(\lambda_0)-h_j^{(n)})^2+y_n^2(\lambda_0)}\right)^{1/2}$$
$$\le|\lambda_0|+\left(\frac{1}{n}\sum_{j=1}^n \frac{1}{(x_n(\lambda_0)-h_j^{(n)})^2+y_n^2(\lambda_0)}\right)^{1/2}. \tag{3.79}$$

Note that we have from (3.24)

$$\frac{1}{n}\sum_{j=1}^{n}\frac{1}{(x_n(\lambda_0)-h_j^{(n)})^2+y_n^2(\lambda_0)}=1,$$

if $y_n(\lambda_0)\neq 0$, and

$$\frac{1}{n}\sum_{j=1}^{n}\frac{1}{(x_n(\lambda_0)-h_j^{(n)})^2}\le 1,$$

if $y_n(\lambda_0)=0$ (see (3.20)). Thus, (3.79) yields

$$|x_n(\lambda_0)|\le|\lambda_0|+1,$$

and the lemma is proved. ■

Note that according to the Hadamard inequality ([14, Sect. I.5]) we have

$$\left|\det\left\{\frac{1}{n}K_n\Big(\lambda_0+\frac{x_i}{n},\lambda_0+\frac{x_j}{n}\Big)\right\}_{i,j=1}^{m}\right|\le\prod_{i=1}^{m}\left(\sum_{j=1}^{m}\left|\frac{1}{n}K_n\left(\lambda_0+\frac{x_i}{n},\lambda_0+\frac{x_j}{n}\right)\right|^2\right)^{1/2}. \tag{3.80}$$

This and Lemma 7 imply the bound

$$\left|\det\left\{\frac{1}{n}K_n\left(\lambda_0+\frac{x_i}{n},\lambda_0+\frac{x_j}{n}\right)\right\}_{i,j=1}^{m}\right|\le m^{m/2}C^m. \tag{3.81}$$

Now we are ready to prove (1.8). Indeed, it is well known (see, e.g., [5]) that

$$\begin{aligned}&E_n\left(\left[\lambda_0+\frac{a}{\rho_n(\lambda_0)\,n},\lambda_0+\frac{b}{\rho_n(\lambda_0)\,n}\right]\right)\\&=1+\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\int\limits_a^b\det\left\{n^{-1}K_n\left(\lambda_0+\frac{x_i}{\rho_n(\lambda_0)\,n},\lambda_0+\frac{x_j}{\rho_n(\lambda_0)\,n}\right)\right\}_{i,j=1}^{l}\prod_{j=1}^{l}d\,x_j.\end{aligned} \tag{3.82}$$

Thus, according to the dominant convergence theorem, (1.5) and (3.81) yield (1.8). Therefore, Theorem 1 is proved.

R e m a r k 1. *Note that all the bounds in the proofs of results of this section do not depend on (uniform in)* $\{h_j^{(n)}\}_{j=1}^n$, $n\in\mathbb{N}$.

4. Proof of Theorem 2

In this section we prove the universality (1.5) and (1.8) of the local bulk regime of Hermitian random matrices (1.9) in the conditions of Theorem 2.

Note that if the whole sequence $\{H_n^{(0)}\}$ is defined on the same (infinity dimensional) probability space and $\{N_n^{(0)}\}$ converges weakly with probability 1 to a nonrandom measure $N^{(0)}$, then the existence of the weak nonrandom limit N of $\{N_n\}$ with probability 1 and (1.13) follows from the corresponding theorem for a nonrandom sequence $\{H_n^{(0)}\}$, which was proved in [4]. Indeed, it is easy to check that all the bounds used in the proof of theorem are independent of (uniform in) $\{H_n^{(0)}\}$, thus the theorem implies the weak convergence of $\{N_n\}$ with probability 1 with respect to the (infinity dimensional) product measures of the probability law $\mathbf{P}^{(h)}$ of $\{H_n^{(0)}\}$ (or their eigenvalues $\{\{h_l^{(n)}\}_{l=1}^n\}$) and the (infinity dimensional) Gaussian law $\mathbf{P}$ of $\{M_n\}$. Likewise, the universality of local bulk regime in this case follows from Theorem 1. Indeed, note first that now ρ_n is the density of the expectation $\overline{N}_n$ of the Normalized Counting Measure N_n of H_n of (1.9) with respect to the product $\mathbf{P} \times \mathbf{P}^{(h)}$ measures of the law $\mathbf{P}^{(h)}$ of $\{H_n^{(0)}\}$ and that of $\{M_n\}$. Then the determinant formulas (2.1), (2.2) imply that

$$\rho_n(\lambda) = \frac{1}{n}\mathbf{E}^{(h)}\{K_n(\lambda, \lambda)\}, \tag{4.1}$$

where here and below the symbol $\mathbf{E}^{(h)}\{\ldots\}$ denotes the expectation with respect to the measure generated by $\{H_n^{(0)}\}$. It follows then from (3.7) and (3.62) with $\xi = \eta = 0$ that if ρ is the density of the limiting Normalized Counting Measure N (see (1.13) and Lemma 1) and $\rho(\lambda_0) > 0$, then we have with probability 1 (with respect to $\mathbf{P} \times \mathbf{P}^{(h)}$)

$$\lim_{n\to\infty} n^{-1}K_n(\lambda_0, \lambda_0) = \pi^{-1}y(\lambda_0) = \rho(\lambda_0), \tag{4.2}$$

where $y(\lambda) = \Im z(\lambda)$ and $z(\lambda)$ is the solution of (3.6). According to Lemma 7, $n^{-1}K_n(\lambda_0, \lambda_0)$ is bounded uniformly in $H_n^{(0)}$, $n \in \mathbb{N}$, thus (4.1), (4.2), and the dominated convergence theorem imply that

$$\lim_{n\to\infty} \rho_n(\lambda_0) = \rho(\lambda_0) = \pi^{-1}y(\lambda_0). \tag{4.3}$$

Denote as before $f^{(0)}$ the Stieltjes transform of (nonrandom) $N^{(0)}$ and

$$g_n^{(0)}(z) = \int \frac{N_n^{(0)}(d\,h)}{h - z}, \ \ \Im z \neq 0 \tag{4.4}$$

the Stieltjes transform of (random) $N_n^{(0)}$, and consider the (random) equation (cf (3.4))

$$z - g_n^{(0)}(z) = \lambda, \quad \lambda \in \mathbb{R}. \tag{4.5}$$

It follows from Lemma 5 that if z_n is the (random) solution of (4.5) such that $z_n(\lambda) = \lambda - \lambda^{-1} + O(\lambda^{-2})$, $\lambda \to \infty$, and $y_n = \Im z_n$, then we have with probability 1

$$\lim_{n\to\infty} \pi^{-1} y_n(\lambda_0) = \pi^{-1} y(\lambda_0) = \rho(\lambda_0)$$

and then (4.3) implies that we obtain with probability 1

$$\lim_{n\to\infty} \pi^{-1} y_n(\lambda_0)/\rho_n(\lambda_0) = 1. \tag{4.6}$$

Thus, (3.62), (3.81), and the formula

$$\mathbf{E}^{(h)} \left\{ \det \left\{ \frac{1}{n\rho_n(\lambda_0)} K_n \left(\lambda_0 + \frac{x_i}{n\rho_n(\lambda_0)}, \lambda_0 + \frac{x_j}{n\rho_n(\lambda_0)} \right) \right\}_{i,j=1}^{m} \right\}$$

for the correlation functions of eigenvalues of (1.9) lead to the universal form of the rescaled correlation functions, i.e., the analog of the first assertion (1.5) of Theorem 1, in the case where $\{N_n^{(0)}\}$ converges with probability 1 to a nonrandom limit. The universal form of the appropriately scaled gap probability, i.e., the analog of second assertion (1.8) of Theorem 1, can be proved similarly.

We will prove now analogous result for the case when $\{N_n^{(0)}\}$ converges in probability to a nonrandom limit. We denote by $\mathbf{P}_n^{(h)}$ the probability law of $H_n^{(0)}$ and by $\mathbf{E}_n^{(h)}$ the corresponding expectation.

We start from the following

Lemma 9. *Let $g_n^{(0)}$ and $f^{(0)}$ be defined in (4.4) and (3.6). Then we have under conditions of Theorem 2*

$$\lim_{n\to\infty} \mathbf{P}_n^{(h)} \{ \max_{z\in K} |g_n^{(0)}(z) - f^{(0)}(z)| > \varepsilon \} = 0 \tag{4.7}$$

uniformly in z of a compact set K in the upper half-plane. Moreover, the converse assertion is true, i.e., if (4.7) is valid for some compact set K in the upper half-plane, then we have (1.15).

P r o o f. Let us prove the first statement of the lemma. Note that it suffices to prove (4.7) for any $z \in K$. Indeed, let $\{z_j\}_{j=1}^{l}$ be a ε-net of K. Then there exists n_0 such that for any $n > n_0$ and for any $\delta > 0$

$$\mathbf{P}_n^{(h)} \left\{ \bigcup_{j=1}^{l} \{ |g_n^{(0)}(z_j) - f^{(0)}(z_j)| > \varepsilon \} \right\} \le \sum_{j=1}^{l} \mathbf{P}_n^{(h)} \{ |g_n^{(0)}(z_j) - f^{(0)}(z_j)| > \varepsilon \} < \delta.$$

Besides, for any $z \in K$ there exists $z_k \in \{z_j\}_{j=1}^{l}$ such that $|z - z_k| < \varepsilon$, and in view of the bounds

$$\left| \frac{d}{d\,z} g_n^{(0)}(z) \right| \le 1/\Im^2 z, \quad \left| \frac{d}{d\,z} f^{(0)}(z) \right| \le 1/\Im^2 z,$$

we can write

$$|g_n^{(0)}(z) - f^{(0)}(z)| \le |g_n^{(0)}(z_k) - f^{(0)}(z_k)| + 2\varepsilon/\Im^2 z.$$

Now, taking into account that $\Im z$ is bounded from below by a positive constant for $z \in K$, we have for any $n > n_0$

$$\mathbf{P}_n^{(h)}\{\max_{z\in K} |g_n^{(0)}(z) - f^{(0)}(z)| < C\varepsilon\} > 1 - \delta.$$

We are left to prove that (4.7) is valid pointwise.

There exists A such that

$$|\lambda - z|^{-1} < \varepsilon, \quad |\lambda| \ge A. \tag{4.8}$$

Set

$$\varphi(\lambda) = \frac{1}{\lambda - z} \quad (\lambda \in \mathbb{R}), \quad \varphi_A(\lambda) = \begin{cases} \dfrac{1}{\lambda - z}, & \lambda \in [-A, A], \\ 0, & \lambda \notin [-A, A], \end{cases}$$

and let φ^ε be a piecewise constant function on the segment [-A, A] such that

$$|\varphi^\varepsilon(\lambda) - \varphi_A(\lambda)| < \varepsilon, \quad |\lambda| \le A. \tag{4.9}$$

If $\varphi^\varepsilon(\lambda) = \varphi_j$, $\lambda \in \triangle_j$, $j = \overline{1, s}$, then we have using (4.8)

$$\begin{aligned} |g_n^{(0)}(z) - f^{(0)}(z)| &\le \left| \int \varphi(\lambda) N_n^{(0)}(d\,\lambda) - \int \varphi_A(\lambda) N_n^{(0)}(d\,\lambda) \right| \\ &+ \left| \int \varphi_A(\lambda) N_n^{(0)}(d\,\lambda) - \int \varphi_A(\lambda) N^{(0)}(d\,\lambda) \right| \\ &+ \left| \int \varphi_A(\lambda) N^{(0)}(d\,\lambda) - \int \varphi(\lambda) N^{(0)}(d\,\lambda) \right| \\ &\le 2\varepsilon + \left| \int \varphi_A(\lambda) N_n^{(0)}(d\,\lambda) - \int \varphi_A(\lambda) N^{(0)}(d\,\lambda) \right|. \end{aligned} \tag{4.10}$$

Besides, it follows from (4.9) that

$$\begin{aligned} &\left| \int \varphi_A(\lambda) N_n^{(0)}(d\,\lambda) - \int \varphi_A(\lambda) N^{(0)}(d\,\lambda) \right| \le \left| \int \varphi_A(\lambda) N_n^{(0)}(d\,\lambda) \right. \\ &\left. - \int \varphi^\varepsilon(\lambda) N_n^{(0)}(d\,\lambda) \right| + \left| \int \varphi^\varepsilon(\lambda) N_n^{(0)}(d\,\lambda) - \int \varphi^\varepsilon(\lambda) N^{(0)}(d\,\lambda) \right| + \left| \int \varphi^\varepsilon(\lambda) N^{(0)}(d\,\lambda) \right. \\ &\left. - \int \varphi_A(\lambda) N^{(0)}(d\,\lambda) \right| \le 2\varepsilon + \left| \int \varphi^\varepsilon(\lambda) N_n^{(0)}(d\,\lambda) - \int \varphi^\varepsilon(\lambda) N^{(0)}(d\,\lambda) \right|. \end{aligned} \tag{4.11}$$

We have also that

$$\left| \int \varphi^\varepsilon(\lambda) N_n^{(0)}(d\,\lambda) - \int \varphi^\varepsilon(\lambda) N^{(0)}(d\,\lambda) \right| = \sum_{j=1}^{s} \varphi_j \cdot |N_n^{(0)}(\triangle_j) - N^{(0)}(\triangle_j)|, \quad (4.12)$$

and by the condition of Theorem 2, for any δ there exists N such that for any $n > N$

$$\mathbf{P}_n^{(h)}\{\bigcup_{j=1}^{l}\{|N_n^{(0)}(\triangle_j) - N^{(0)}(\triangle_j)| > \varepsilon\}\} < \delta. \quad (4.13)$$

Now the first assertion of the lemma follows from (4.10)–(4.13).

To prove the converse assertion we indicate explicitly the fact that $g_n^{(0)}(z)$ and $N_n^{(0)}(\Delta)$ are random by writing $g_n^{(0)}(z,\omega)$ and $N_n^{(0)}(\Delta,\omega)$, $\omega \in \Omega_n^{(0)}$, where $\Omega_n^{(0)}$ is the probability space on which $N_n^{(0)}$ is defined. Assume now that (4.7) is true but (1.15) is false, i.e., that there exists an interval $\Delta \subset \mathbb{R}$, $\varepsilon > 0$, a subsequence $\{n_i\}$, and $\delta > 0$ such that

$$\mathbf{P}_{n_i}^{(h)}\{|N_{n_i}^{(0)}(\Delta) - N^{(0)}(\Delta)| > \varepsilon\} \geq \delta, \quad i \geq 1.$$

On the other hand, it follows from (4.7) that for any $r \in \mathbb{N}$ there exists $\nu \in \mathbb{N}$ such that

$$\mathbf{P}_n^{(h)}\{\max_{z\in K}|g_n^{(0)}(z,\omega) - f^{(0)}(z)| < r^{-1}\} \geq 1 - \delta/2, \ n \geq \nu.$$

This and the inequality $\mathbf{P}_n^{(h)}\{A \cap B\} \geq \mathbf{P}_n^{(h)}\{A\} + \mathbf{P}_n^{(h)}\{B\} - 1$ imply that the $\mathbf{P}_n^{(h)}$-probability to have simultaneously the inequalities

$$|N_{n_i}^{(0)}(\Delta) - N^{(0)}(\Delta)| > \varepsilon \quad (4.14)$$

and

$$\max_{z\in K}|g_n^{(0)}(z,\omega) - f^{(0)}(z)| < r^{-1} \quad (4.15)$$

is not less than $\delta/2 > 0$ if $n_i \geq \max\{n_1, \nu\}$. Denote the corresponding set of realizations of $N_n^{(0)}$ by Ω'_δ. Since the collection $\{N_{n_i}^{(0)}(\cdot,\omega)\}_{n_i \geq \max\{n_1,\nu\}, \omega\in\Omega'_\delta}$ consists of probability measures, there exists a subsequence $\{n_{i'}, \omega_{i'}\}$ such that $\{N_{n'_i}^{(0)}(\cdot,\omega_{i'})\}$ converges to a certain limit N^* and their Stieltjes transforms $\{g_{n'_i}^{(0)}(\cdot,\omega_{i'})\}$ converge uniformly on K to the Stieltjes transform f^* of N^*. In view of (4.15) $f^* = f^{(0)}$, and in view of (4.14) $|N^*(\Delta) - N^{(0)}(\Delta)| > \varepsilon$, i.e., $N^* \neq N^{(0)}$. On the other hand, in view of the one-to-one correspondence between non-negative measures and their Stieltjes transforms $f^* = f^{(0)}$ implies $N^* = N^{(0)}$. The lemma is proved. ■

Next we prove the analog of (1.13) for the condition (1.15).

Lemma 10. *In the conditions of Theorem 2 there exists a nonnegative probability measure N such that*

$$\lim_{n\to\infty} \mathbf{E}\{|N_n(\Delta) - N(\Delta)|\} = 0 \tag{4.16}$$

for any interval $\Delta \subset \mathbb{R}$, where $\mathbf{E}\{...\}$ denotes the expectation with respect to the product measure of $H_n^{(0)}$ and M_n. The measure N can be found via its Stieltjes transform f that is a unique solution of the functional equation (1.13) in the class of functions, analytic for $\Im z \neq 0$ and such that $\Im f(z) \cdot \Im z \geq 0$.

P r o o f. It suffices to prove that for every z of a compact set $K \subset \mathbb{C} \setminus \mathbb{R}$ we have

$$\lim_{n\to\infty} \mathbf{E}\{|g_n(z) - f(z)|\} = 0, \tag{4.17}$$

where g_n and f are the Stieltjes transform of N_n and N. Indeed, if yes, then by using the compactness and a ε-net for $\{g_n(z) - f(z)\}$ on K (see the beginning of the proof of Lemma 9), we obtain that

$$\lim_{n\to\infty} \mathbf{E}\{\max_{z\in K} |g_n(z) - f(z)|\} = 0, \tag{4.18}$$

and thus the Tchebyshev inequality implies the analog of (4.7) with g_n instead of $g_n^{(0)}$ and f instead of $f^{(0)}$. Next, applying Lemma 9 to the pairs (g_n, f_n) and (N_n, N) instead of $(g_n^{(0)}, f^{(0)})$ and $(N_n^{(0)}, N^{(0)})$, we obtain (1.15).

We will choose the compact set K satisfying the condition

$$\min_{z\in K} |\Im z| \geq 3 \tag{4.19}$$

and prove the relations

$$\lim_{n\to\infty} \mathbf{E}\{g_n(z)\} = f(z), \tag{4.20}$$

and

$$\lim_{n\to\infty} \mathbf{Var}\{g_n(z)\} = 0 \tag{4.21}$$

for every $z \in K$.

Denote

$$\widehat{f}_n(z) = \mathbf{E}\left\{g_n(z)|H_n^{(0)}\right\}, \quad \widehat{z}_n(z) = z + \widehat{f}_n(z). \tag{4.22}$$

Then it follows from [15] that

$$|\widehat{f}_n(z) - g_n^{(0)}(\widehat{z}_n(z))| \leq \frac{1}{n^2|\Im z|^5} \tag{4.23}$$

and

$$\mathbf{E}\{|g_n(z) - \widehat{f}_n(z)|^2\} \le \frac{1}{n^2|\Im z|^4}, \tag{4.24}$$

in particular, (4.23) is valid for every realization of $H_n^{(0)}$.

Let us prove that for every $z \in K$

$$\lim_{n\to\infty} \mathbf{E}\{|\widehat{f}_n(z) - f(z)|^2\} = 0, \tag{4.25}$$

where K is defined by (4.19) and f is the solution of (1.13).

It is easy to see that (4.20) follows immediately from (4.25), and (4.21) follows from (4.25) and (4.24).

Consider the compact set

$$K_1 = \cup_{z\in K} B_1(z),$$

where $B_1(z) \subset \mathbb{C}$ is a disk of radius 1 centered in z: $B_1(z) = \{z' : |z - z'| < 1\}$, and the set of realization

$$\Omega_\varepsilon = \{\omega : \sup_{z\in K_1} |g_n^{(0)}(z,\omega) - f^{(0)}(z)| \le \varepsilon\}, \tag{4.26}$$

where $f^{(0)}$ is defined in (3.6). Then, using (4.18) for the compact K_1, we obtain

$$\lim_{n\to\infty} \mathbf{P}\{\Omega_\varepsilon^c\} = 0. \tag{4.27}$$

Let $z \in K$, $\omega \in \Omega_\varepsilon$. Since $|\widehat{f}_n(z)| \le |\Im z|^{-1}$, then $\widehat{z} = z + \widehat{f}_n(z) \in K_1$. Hence, (4.23) and (4.26) imply

$$\widehat{f}_n(z) = f^{(0)}(z + \widehat{f}_n(z)) + r_n(z), \quad |r_n(z)| \le 2\varepsilon. \tag{4.28}$$

Now we need the following general fact

Proposition 3. *Let $\mathcal{B}$ be a Banach space with the norm $||\cdot||$, $B = \{f \in \mathcal{B}, ||f|| \le 1\}$, and the function $F : B \to B$ satisfies the condition*

$$||F(f_1) - F(f_2)|| \le q||f_1 - f_2||, \quad f_1, f_2 \in B, \quad 0 < q < 1. \tag{4.29}$$

Then for any $r : ||r|| < (1 - q)$ the equation

$$f = F(f) + r \tag{4.30}$$

has a unique solution $f(r) \in B$, and

$$||f(r) - f(0)|| \le (1 - q)^{-1}||r||. \tag{4.31}$$

To prove the proposition, it suffices to consider the sequence $\{f^{(k)}(r)\}_{k=0}^{\infty}$ such that

$$f^{(0)}(r) = r, \quad f^{(k+1)}(r) = F(f^{(k)}(r)) + r.$$

Then (4.29) yield

$$||f^{(k+1)}(r) - f^{(k)}(r)|| \le q||f^{(k)}(r) - f^{(k-1)}(r)||,$$
$$||f^{(k+1)}(r) - f^{(k+1)}(0)|| \le q||f^{(k)}(r) - f^{(k)}(0)|| + r,$$

and therefore there exists $f(r) = \lim_{k\to\infty} f^{(k)}(r)$, which satisfies (4.30) and (4.31).

We use the proposition for $\mathcal{B} = \mathbb{C}$, with $F_z(f) = f^{(0)}(z + f)$. Then (4.19) guarantees that for any $f \in B$ (4.29) is valid with $q = 1/2$. Hence, we obtain from (4.28) and (4.30) for any $z \in K$, $\omega \in \Omega_\varepsilon$

$$|\widehat{f}_n(z) - f(z)| \le 4\varepsilon.$$

Now, since $|f(z)| \le |\Im z|^{-1} \le 1$ and $|\widehat{f}_n(z)| < 1$ for $\omega \in \overline{\Omega}_\varepsilon$, then the last bound and (4.27) yield

$$\lim_{n\to\infty} \mathbf{E}\{|\widehat{f}_n(z) - f(z)|^2\} \le 16\varepsilon^2,$$

and since ε is arbitrary small, we obtain (4.25). ■

Let us take the disk $\omega = \{z : |z(\lambda_0) - z| \le \varepsilon_1\}$ as the compact set K in (4.7), where $z(\lambda_0)$ is a solution of (3.6) for $\lambda = \lambda_0$. Taking into account (4.7), we get that for any $\delta > 0$ and $\varepsilon > 0$ there exists n_0 such that for all $n > n_0$ the event (cf (3.36))

$$\Omega_{\varepsilon,n_0} = \{\max_{z\in\omega} |g_n^{(0)}(z) - f^{(0)}(z)| < \varepsilon\}, \tag{4.32}$$

satisfies the condition

$$\mathbf{P}_n^{(h)}\{\Omega_{\varepsilon,n_0}\} \ge 1 - \delta. \tag{4.33}$$

Since for any realization of $\{h_j^{(n)}\}_{j=1}^n$ the determinant formulas (2.1), (2.2) are true, we can write that for any correlation function $R_m^{(n)}$ of eigenvalues of (1.9)

$$R_m^{(n)}(\lambda_1, \dots, \lambda_m) = \mathbf{E}_n^{(h)} \left\{\det\{K_n(\lambda_j, \lambda_k)\}_{j,k=1}^m\right\}.$$

Thus, the proof of (1.5) in the case of random $H_n^{(0)}$ reduces to that of the relation

$$\lim_{n\to\infty} \mathbf{E}_n^{(h)} \left\{ \det \left\{ \frac{1}{n\rho_n(\lambda_0)} K_n \left(\lambda_0 + \frac{x_i}{n\rho_n(\lambda_0)}, \lambda_0 + \frac{x_j}{n\rho_n(\lambda_0)} \right) \right\}_{i,j=1}^m \right\}$$
$$= \det \left\{ \frac{\sin \pi(x_j - x_k)}{\pi(x_j - x_k)} \right\}_{j,k=1}^m . \tag{4.34}$$

Consider the expression

$$\mathbf{E}_n^{(h)} \left\{ A_m^{(n)}(x_1,\ldots,x_m) \right\}, \tag{4.35}$$

where

$$A_m^{(n)}(x_1,\ldots,x_m) = \det\left\{ \frac{1}{n\rho_n(\lambda_0)} K_n\left(\lambda_0+\frac{x_i}{n},\lambda_0+\frac{x_j}{n}\right)\right\}_{i,j=1}^m - \det\left\{ \frac{\sin(y_n(\lambda_0)(x_i-x_j))}{\pi(x_i-x_j)}\right\}_{i,j=1}^m. \tag{4.36}$$

Split the expectation in (4.35) in two parts, over Ω_{ε,n_0} of (4.32) and it complement. Since Theorem 1 is valid on (4.32), we can write for any n-independent and nonrandom $\varepsilon_1 > 0$ that $|A_m^{(n)}(x_1,\ldots,x_m)| \le \varepsilon_1$ uniformly for $|x_j| \le K$, $j = 1,\ldots,m$. In addition, it follows from (3.81), valid for every $H_n^{(0)}$, the bound

$$\left| \det\left\{ \frac{\sin y_n(\lambda_0)(x_j-x_k)}{\pi(x_j-x_k)}\right\}_{j,k=1}^m \right| \le m^{m/2} y_n^m(\lambda_0),$$

and the relation (3.24), also valid for any $H_n^{(0)}$ and implying that $0 \le y_n(\lambda_0) \le 1$, that for any $m \in \mathbb{N}$ $A_m^{(n)}(x_1,\ldots,x_m)$ is bounded uniformly in $|x_j| \le K$, $j = 1,\ldots,m$, and $H_n^{(0)}$. This and (4.33)imply

$$|\mathbf{E}_n^{(h)} \left\{ A_m^{(n)}(x_1,\ldots,x_m) \right\}| \le \varepsilon_1 + C\delta,$$

i.e., that (4.18) vanishes as $n \to \infty$.

In particular, the case $m = 1$, $\xi = \eta = 0$ of this assertion leads to

$$\lim_{n\to\infty} \mathbf{E}_n^{(h)} \left\{ |n^{-1}K_n(\lambda_0,\lambda_0) - \pi^{-1}y_n(\lambda_0)| \right\} = 0.$$

It follows also from Lemma 5 and (4.32)–(4.33) that

$$\lim_{n\to\infty} \mathbf{E}_n^{(h)} \left\{ |y_n(\lambda_0) - y_n(\lambda_0)| \right\} = 0.$$

Then (4.1) and (3.7) imply that

$$\lim_{n\to\infty} \mathbf{E}_n^{(h)} \left\{ |\pi - y_n(\lambda_0)/\rho_n(\lambda_0)| \right\} = 0.$$

This relation and the boundedness and continuity of (3.64) in $y \in [0,1]$ imply (1.5).

In view of the above the proof of (1.8) is essentially the same that in Theorem 1.

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