Journal of Mathematical Physics, Analysis, Geometry
2009, vol. 5, No. 1, pp. 82–103

Simple Periodic Boundary Data and Riemann–Hilbert Problem for Integrable Model of the Stimulated Raman Scattering

E.A. Moskovchenko

Mathematical Division, B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkiv, 61103, Ukraine

E-mail:kuznetsova@ilt.kharkov.ua

Received March 24, 2008

We consider the initial-boundary value (IBV) problem for nonlinear equations related to the integrable model of the stimulated Raman scattering in the quarter xt-plane with vanishing at infinity initial conditions and single-frequency periodic boundary data ($pe^{i\omega t}$). We propose a matrix Riemann–Hilbert problem, which provides the existence of the solution of the IBV problem for all t and allows us to obtain an explicit formula for the asymptotics of the solution, using the steepest descent method for the oscillatory matrix RH problem introduced by P. Deift and X. Zhou [6].

Key words: nonlinear equations, Riemann–Hilbert problem, the steepest descent method, asymptotics.

Mathematics Subject Classification 2000: 37K15, 35Q15, 35B40.

1. Introduction

The phenomenon of stimulated Raman scattering (SRS) is described by three coupled PDEs. In the transient limit these equations are integrable [1, 2], i.e. they admit a Lax pair formulation. Paper [1] is devoted to the Raman soliton generation from laser inputs in the SRS model. In [2], the authors studied the asymptotic behavior of the solution of the initial-boundary-value (IBV) problem in the semistrip ($x \in [0, \infty)$, $t \in [0, 1]$) by using the method [3] based on the simultaneous spectral analysis of the two parts forming the Lax pair and a matrix Riemann–Hilbert problem on the complex k-plane. This method includes more boundary values than required for a well-posed IBV problem. This over-determination of the boundary data implies the so-called global relation [3, 4] between the corresponding spectral functions. Fortunately, the initial boundary value problem for nonlinear SRS equations considered below is a nice model of PDEs, which can be solved by using the matrix Riemann–Hilbert problem without restrictions caused by global relation. In this case all spectral functions are uniquely defined by given initial and boundary data only. For the finite domain $[0,L]\mathrm{x}[0,\mathrm{T}]$ the IBV problem for the SRS equations was studied in [5], where the difficulties on the presence of two essential singularities in the matrix Riemann-Hilbert problem have been overcome. In the present paper, we consider the IBV problem for the SRS equations in the quarter xt-plane with vanishing at infinity initial function and simple periodic boundary data. In general, one can propose different matrix RH problems suitable for the given IBV problem. We propose a matrix Riemann–Hilbert problem, which provides the existence of the solution for all t and allows us to obtain an explicit formula for the asymptotics of the solution, using the steepest descent method for the oscillatory matrix RH problem introduced by P. Deift and X. Zhou [6]. To make the asymptotic analysis more transparent we restrict our attention to the special case when boundary data take the single-frequency periodic form, and the initial function is identically equal to zero. We show that in the region $x > \omega^2 t$, where ω is the frequency of the boundary data, see (3) below, the asymptotics has a quasi-linear dispersive character and is described by Zakharov–Manakov type formula. In other regions the asymptotic analysis turns to be more complicated and will be presented elsewhere.

The IBV problem under consideration is

$$2\mathrm{i}q_t = \mu, \quad \mu_x = 2\mathrm{i}\nu q, \quad \nu_x = \mathrm{i}(\bar{q}\mu - q\bar{\mu}), \qquad x \in (0,\infty), \quad t \in (0,\infty), \tag{1}$$

with the initial function

$$q(x,0) = u(x), \qquad x \in (0,\infty), \tag{2}$$

and the boundary condition

$$\mu(0,t) = p\mathrm{e}^{\mathrm{i}\omega t}, \qquad p > 0, \tag{3}$$

$$\nu(0,t) = l = const, \qquad l \in \mathbb{R}. \tag{4}$$

We suppose that the function $u(x)$ is absolutely continuous, $xu(x)$ and $u'_x(x) \in L^1(0,\infty)$:

$$\int_0^\infty [x|u(x)| + |u'_x(x)|]dx < \infty. \tag{5}$$

Let the absolutely continuous in x and t functions $q(x,t), \mu(x,t) \in \mathbb{C}$, $\nu(x,t) \in \mathbb{R}$ satisfy the SRS equations (1) on the semi-infinite domain $x,t \in ((0,\infty)\times(0,\infty))$, initial (2) and boundary (3) conditions. Since (1) implies

$$\frac{\partial}{\partial x}\Big(\nu^2(x,t) + |\mu(x,t)|^2\Big) = 0,$$

in what follows we assume that

$$\nu^2(x,t) + |\mu(x,t)|^2 \equiv 1$$

and, particularly, $p^2 + l^2 = 1$. All considerations of the paper are valid if the boundary conditions (3), (4) are replaced by

$$\mu(0,t) = p\mathrm{e}^{\mathrm{i}\omega t} + v(t), \qquad \nu(0,t) = l + w(t), \tag{6}$$

where $v(t)$ and $w(t)$ are given functions decreasing fast as $t \to \infty$. The IBV problem of this type was considered in [7], where the generation of asymptotic solitons by boundary data (6) was studied using the Marchenko integral equations.

Notice that, if $q(x,t)$ is real and $2q = v_x$, $\mu = \mathrm{i}\sin v$, $\nu = \cos v$, then the SRS equations are reduced to the sine-Gordon equation

$$v_{xt} = \sin v. \tag{7}$$

The asymptotic behavior of the rapidly decreasing (as $|x| \to \infty$) solution was studied in [8].

2. Basic Solutions of Linear Over-Determined Equations

For studying the initial boundary value problem (1)–(3), we will use the simultaneous spectral analysis [3] of the linear x-equation

$$\Phi_x + \mathrm{i}k\sigma_3\Phi = Q(x,t)\Phi, \tag{8}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -\bar{q}(x,t) & 0 \end{pmatrix} \tag{9}$$

and the linear t-equation

$$\Phi_t = \frac{\mathrm{i}}{4k}\widehat{Q}(x,t)\Phi, \tag{10}$$

$$\widehat{Q}(x,t) = \begin{pmatrix} \nu(x,t) & \mathrm{i}\mu(x,t) \\ -\mathrm{i}\bar{\mu}(x,t) & -\nu(x,t) \end{pmatrix}, \tag{11}$$

where $\Phi(x,t,k)$ is a 2×2 matrix-valued function and $k \in \mathbb{C}$ is a parameter. Let us rewrite equations (8), (10) in the equivalent form:

$$W_x = U(x,t,k)W, \qquad U(x,t,k) = Q(x,t) - \mathrm{i}k\sigma_3, \tag{12}$$

$$W_t = V(x,t,k)W, \qquad V(x,t,k) = \frac{\mathrm{i}}{4k}\widehat{Q}(x,t). \tag{13}$$

It is easy to verify that the over-determined system of differential equations (12), (13) is compatible (i.e., $\frac{\partial^2 W}{\partial x \partial t} = \frac{\partial^2 W}{\partial t \partial x}$) if and only if the matrices $U(x,t,k)$ and $V(x,t,k)$ satisfy the compatibility condition

$$U_t(x,t,k) - V_x(x,t,k) + U(x,t,k)V(x,t,k) - V(x,t,k)U(x,t,k) = 0, \quad k \in \mathbb{C}, \tag{14}$$

which is equivalent to the SRS equations (1) on the functions $q(x,t)$, $\mu(x,t)$, $\nu(x,t)$.

Below we will use the following lemma.

Lemma 2.1. *Let the compatibility condition (14) be fulfilled for all $k \in \mathbb{C}$. Let $W(x,t,k)$ be a matrix satisfying the x-equation (12) for all t (the t-equation (13) for all x). Assume that $W(x_0,t,k)$ satisfies the t-equation (13) for some $x = x_0$ ($W(x,t_0,k)$ satisfies the x-equation (12) for some $t = t_0$), including the case when $x_0 = \infty$ ($t_0 = \infty$). Then $W(x,t,k)$ satisfies the t-equation (13) for all x (satisfies the x-equation (12) for all t).*

P r o o f. Let $W = W(x,t,k)$ be a solution to (12). Then, due to the compatibility condition the matrix $\hat{W}(x,t,k) = W_t - V(x,t,k)W$ is also the solution to (12). Indeed, $\hat{W}_x = U(x,t,k)\hat{W} + (U_t - V_x + UV - VU)W = U(x,t,k)\hat{W}$. Since the matrices W and $\hat{W}$ are the solutions of the same equation (12), it follows that $\hat{W}(x,t,k) = W(x,t,k)C(t,k)$ for some $C(t,k)$ independent of x. By assumption, $\hat{W}(x_0,t,k) = 0$. Hence $C(t,k) \equiv 0$ and thus $\hat{W}(x,t,k) \equiv 0$, what means that $W(x,t,k)$ satisfies the t-equation (13) for all x. The proof of the statement with x and t interchanged is similar. ■

To introduce the basic solutions of the over-determined equations we have to find the exact solution of the t-equation for $x = 0$. It takes the form

$$\Phi_t = \frac{\mathrm{i}}{4k} \begin{pmatrix} l & \mathrm{i}p\mathrm{e}^{\mathrm{i}\omega t} \\ -\mathrm{i}p\mathrm{e}^{-\mathrm{i}\omega t} & -l \end{pmatrix} \Phi. \tag{15}$$

The following matrix

$$\mathcal{E}(t,k) = \frac{1}{2}\mathrm{e}^{\mathrm{i}\omega\sigma_3 t/2} \begin{pmatrix} \frac{1}{\varkappa(k)} + \varkappa(k) & \frac{1}{\varkappa(k)} - \varkappa(k) \\ \frac{1}{\varkappa(k)} - \varkappa(k) & \frac{1}{\varkappa(k)} + \varkappa(k) \end{pmatrix} \mathrm{e}^{-\mathrm{i}\Omega(k)\sigma_3 t} \tag{16}$$

is a solution of this equation if

$$\varkappa(k) = \sqrt[4]{\frac{k-E}{k-\bar{E}}},$$

where
$$E = \frac{l}{2\omega} + \frac{\mathrm{i}p}{2|\omega|} = E_1 + \mathrm{i}E_2, \qquad \bar{E} = E_1 - \mathrm{i}E_2,$$
and
$$\Omega(k) = \frac{|\omega|}{2k}\sqrt{(k-E)(k-\bar{E})}.$$
Without loss of generality we assume here and in what follows that $\omega > 0$. Indeed, we obtain the case $\omega < 0$ if we take the complex conjugated functions $\bar{q}(x,t), \bar{\mu}(x,t), \nu(x,t)$ instead of $q(x,t), \mu(x,t), \nu(x,t)$. To fix the branches of the roots we choose a cut in the complex k-plane along the curve $\gamma \cup \bar{\gamma}$, where $\operatorname{Im}\Omega(k) = 0$, and define $\varkappa(k)$ and $\Omega(k)$ as
$$\varkappa(k) = 1 + O(k^{-1}), \qquad \Omega(k) = \frac{\omega}{2} + O(k^{-1}), \qquad k \to \infty.$$
A simple analysis shows that the set $\Sigma := \{k \in \mathbb{C} | \operatorname{Im}\Omega(k) = 0\}$ consists of the real line $\operatorname{Im} k = 0$ and the circle arc $\hat{\gamma}$, which is defined by
$$\Big(k_1 - \frac{|E|^2}{2E_1}\Big)^2 + k_2^2 = \Big(\frac{|E|^2}{2E_1}\Big)^2,$$
$$k_1^2 + k_2^2 \geq |E|^2$$
(see Fig. 1). If $\Omega_\pm(k)$, $\varkappa_\pm(k)$ are boundary values of the functions $\Omega(k)$, $\varkappa(k)$ on the cut $\gamma \cup \bar{\gamma}$ from the right $(+)$ and left (-) sides of the cut, then
$$\Omega_+(k) = -\Omega_-(k), \qquad \varkappa_-(k) = \mathrm{i}\varkappa_+(k).$$
Then the matrix-valued function $\mathcal{E}(t,k)$ is analytic when being away from the point 0, where it has an essential singularity, and the circle arc $\hat{\gamma}$.

The function $\Omega(k)$ has the following asymptotics:
$$\Omega(k) = \begin{cases} \dfrac{\omega}{2} - \dfrac{l}{4k} + O(k^{-2}), & k \to \infty, \\ \pm\dfrac{1}{4k} \mp \dfrac{l\omega}{2} + O(k), & k \to 0, \qquad \mathrm{sign}(\mathrm{l}\omega) = \mp 1. \end{cases}$$
In the present paper we consider the case $l < 0$ and $\omega > 0$. The matrix $\mathcal{E}(t,k)$ behaves as follows:
$$\mathcal{E}(t,k) = I + O(k^{-1}), \qquad k \to \infty$$
and
$$\mathcal{E}(t,k)e^{\mathrm{i}t\sigma_3/4k} = \mathcal{E}_0(t) + O(k), \qquad k \to 0,$$
where
$$\mathcal{E}_0(t) = \mathrm{e}^{\mathrm{i}\omega t\sigma_3/2}\begin{pmatrix} \cos(\frac{\arg E}{2}) & -\mathrm{i}\sin(\frac{\arg E}{2}) \\ -\mathrm{i}\sin(\frac{\arg E}{2}) & \cos(\frac{\arg E}{2}) \end{pmatrix}\mathrm{e}^{\mathrm{i}\omega lt\sigma_3/2}.$$

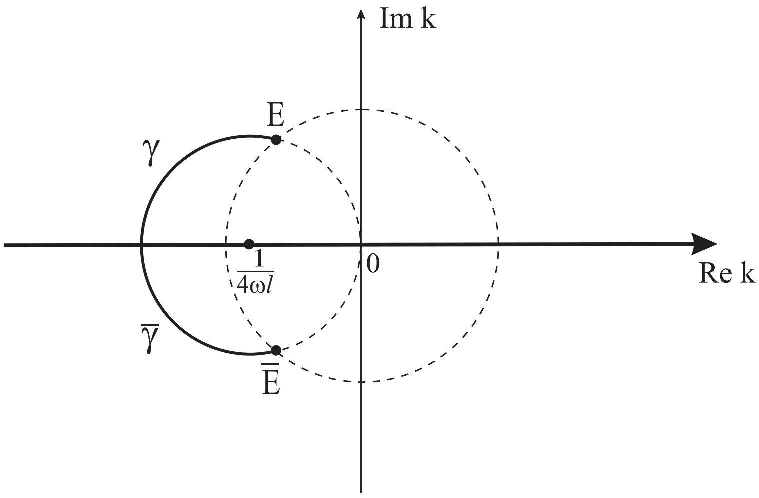

Fig. 1: Set Σ, $\omega l < 0$.

Now we introduce the basic solutions (eigenfunctions) of equations (8) and (10). The first eigenfunction has the form:

$$\Phi_1(x,t,k) = \Big(\mathrm{e}^{-\mathrm{i}kx\sigma_3} + \int_{-x}^{x} K(x,y,t)\mathrm{e}^{-\mathrm{i}ky\sigma_3}dy\Big)\mathcal{E}(t,k), \tag{17}$$

where the kernel $K(x,y,t)$ is chosen to be so that the first factor satisfies the x-equation (8) for all t, and the second factor satisfies the t-equation (10) for $x = 0$. By Lemma 2.1, $\Phi_1(x,t,k)$ satisfies both equations (8) and (10). The existence of the solution represented by the transformation operators with the kernel $K(x,y,t)$ is proved in [9].

If the functions $q(x,t)$, $\mu(x,t)$ and $\nu(x,t)$ are absolutely continuous in x and t and satisfy the initial and boundary conditions (2)–(3) and the differential equations (1) almost everywhere, then the matrix valued function (17) has the following properties:

1) $\Phi_1(x,t,k)$ satisfies the x- and t-equations (8)–(10) for $k \in \mathbb{C}\backslash(\{0\}\cup\{E\}\cup\{\bar{E}\})$;

2) $\Phi_1(x,t,k) = \Lambda\bar{\Phi}_1(x,t,\bar{k})\Lambda^{-1}$, $k \in \mathbb{C}\backslash(\{0\}\cup\{E\}\cup\{\bar{E}\})$, where $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

3) $\det\Phi_1(x,t,k) \equiv 1$, $\quad k \in \mathbb{C}$;

4) for $k \neq 0, E, \bar{E}$ the map $(x,t) \longmapsto \Phi_1(x,t,k)$ is absolutely continuous together with its partial derivatives;

5) the map $k \longmapsto \Phi_1(x,t,k)$ is analytic in $k \in \mathbb{C} \setminus (\{0\} \cup \gamma \cup \bar{\gamma})$ and it has the forth root singularities at the points E and $\bar{E}$;

6) $\Phi_1(x,t,k)\mathrm{e}^{\mathrm{i}kx\sigma_3} = I + O(k^{-1}) + O(k^{-1}\mathrm{e}^{2\mathrm{i}kx\sigma_3}), \quad k \to \infty;$
7) $\Phi_1(x,t,k)\mathrm{e}^{\frac{\mathrm{i}t\sigma_3}{4k}} = \Phi_0(x,t) + O(k), \quad k \to 0,$

$$\Phi_0(x,t) = \left(I + \int_{-x}^{x} K(x,y,t)dy \right) \mathcal{E}_0(t).$$

The eigenfunction $\Phi_2(x,t,k)$ normalized by the condition

$$\Phi_2(0,0,k) = I$$

has the form

$$\Phi_2(x,t,k) = \Big(\mathrm{e}^{-\mathrm{i}kx\sigma_3} + \int_{-x}^{x} K(x,y,t)\mathrm{e}^{-\mathrm{i}ky\sigma_3}dy\Big)\mathcal{E}(t,k)\mathcal{E}^{-1}(0,k). \tag{18}$$

It is related to $\Phi_1(x,t,k)$ by

$$\Phi_1(x,t,k) = \Phi_2(x,t,k)\mathcal{E}(0,k).$$

The eigenfunction $\Phi_2(x,t,k)$ satisfies the properties 1)–6) and 7) with $\Phi_0(x,t)\mathcal{E}_0^{-1}(0)$ instead of $\Phi_0(x,t)$.

Finally, we chose the eigenfunction $\Phi_3(x,t,k)$ in the form:

$$\Phi_3(x,t,k) = \Big(\mathrm{e}^{-\frac{\mathrm{i}t\sigma_3}{4k}} + \frac{\mathrm{i}}{4k}\int_{-t}^{t} L(x,t,s)\mathrm{e}^{-\frac{\mathrm{i}s\sigma_3}{4k}}ds\Big) \tag{19}$$

$$\times\Big(\mathrm{e}^{-\mathrm{i}kx\sigma_3} + \int_{x}^{\infty} N(x,y)\mathrm{e}^{-\mathrm{i}ky\sigma_3}dy\Big),$$

where the kernels $L(x,y,s)$ and $N(x,y)$ are such that the first factor satisfies (10) for any x, and the second factor satisfies (8) for $t=0$. Due to Lem. 2.1, $\Phi_3(x,t,k)$ satisfies both equations (8) and (10). The matrix $\Phi_3(x,t,k)$ possesses the properties 1)–4) for $k \in \mathbb{R}\setminus\{0\}$. Other important properties $\Phi_3(x,t,k)$ are as follows:
5) the matrix columns $[\Phi_3]_1(x,t,k)$ and $[\Phi_3]_2(x,t,k)$ are analytic in $k \in \mathbb{C}_{\mp}$, respectively;
6) at infinity they have the asymptotics:

$$[\Phi_3]_1(x,t,k)\mathrm{e}^{\mathrm{i}kx} = \begin{pmatrix}1\\0\end{pmatrix} + O(k^{-1}), \qquad k \to \infty, \qquad \operatorname{Im} k \le 0;$$

$$[\Phi_3]_2(x,t,k)\mathrm{e}^{-\mathrm{i}kx} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(k^{-1}), \qquad k \to \infty, \qquad \operatorname{Im} k \geq 0;$$

7) for $k \to 0$ and $\operatorname{Im} k \neq 0$ they have the following asymptotics:

$$[\Phi_3]_1(x,t,k)\mathrm{e}^{\frac{\mathrm{i}t}{4k}} = [\hat{\Phi}_3]_1(x,t) + O(k), \qquad k \to 0, \qquad \operatorname{Im} k < 0,$$

$$[\Phi_3]_2(x,t,k)\mathrm{e}^{-\frac{\mathrm{i}t}{4k}} = [\hat{\Phi}_3]_2(x,t) + O(k), \qquad k \to 0, \qquad \operatorname{Im} k > 0,$$

where $[\hat{\Phi}_3]_1(x,t)$ and $[\hat{\Phi}_3]_2(x,t)$ are some absolutely continuous vector-functions depending on the entries of matrices $L(x,t,s)$ and $N(x,y)$. The existence of the transformation operators with kernels $K(x,y,t)$, $L(x,t,s)$ and $N(x,y)$ can be proved in the same way as in [9].

Since all the introduced matrix valued functions $\Phi_j(x,t,k)$, $j = 1,2,3$, are the solutions to the x- and t-equations (8)–(10), they are linear dependent, so there exist transition matrices $S(k)$, $s(k)$ and $R(k)$ independent of x and t such that

$$\Phi_1(x,t,k) = \Phi_2(x,t,k)S(k), \quad \Phi_2(x,t,k) = \Phi_3(x,t,k)s(k), \tag{20}$$

$$\Phi_1(x,t,k) = \Phi_3(x,t,k)R(k). \tag{21}$$

They can be written as follows:

$$S(k) = \mathcal{E}(0,k), \quad s(k) = \Phi_3^{-1}(0,0,k), \quad R(k) = s(k)S(k).$$

The transition matrices have the following representations:

$$s^{-1}(k) = \begin{pmatrix} \bar{a}(\bar{k}) & b(k) \\ -\bar{b}(\bar{k}) & a(k) \end{pmatrix}, \tag{22}$$

$$S(k) = \begin{pmatrix} \bar{A}(\bar{k}) & B(k) \\ -\bar{B}(\bar{k}) & A(k) \end{pmatrix}, \tag{23}$$

where

$$a(k) = 1 + \int_0^\infty N_{22}(0,y)\mathrm{e}^{\mathrm{i}ky}dy, \qquad b(k) = \int_0^\infty N_{12}(0,y)\mathrm{e}^{\mathrm{i}ky}dy,$$

$$2A(k) = \varkappa(k) + \frac{1}{\varkappa(k)} = 2\bar{A}(\bar{k}), \quad 2B(k) = \left(\frac{1}{\varkappa(k)} - \varkappa(k)\right) = -2\bar{B}(\bar{k}).$$

The functions $N_{12}(0,y)$ and $N_{22}(0,y)$ are absolutely continuous and their derivatives belong to the space $L^1(0,\infty)$. For $R(k) = s(k)S(k)$ we have

$$R(k) = \begin{pmatrix} \bar{a}_R(\bar{k}) & b_R(k) \\ -\bar{b}_R(\bar{k}) & a_R(k) \end{pmatrix}, \tag{24}$$

where $a_R(k) = \bar{a}(\bar{k})A(k) + \bar{b}(\bar{k})B(k)$ and $b_R(k) = a(k)B(k) - b(k)A(k)$.

Further we prove a one-to-one correspondence between the initial function $u(x)$ and spectral data $a(k)$ and $b(k)$. Namely, let $u(x)$ be absolutely continuous, $xu(x), u'_x(x) \in L^1(0,\infty)$. Then the vector-function $\Psi(x,k) := [\Phi_3]_2(x,0,k)$ (the second column of the matrix $\Phi_3(x,t,k)$), which satisfies the equation

$$\Psi_x + \mathrm{i}k\sigma_3\Psi = \begin{pmatrix} 0 & u(x) \\ -\bar{u}(x) & 0 \end{pmatrix} \Psi, \qquad 0 < x < \infty, \tag{25}$$

and the boundary condition

$$\lim_{x\to\infty} \Psi(x,k)\mathrm{e}^{-\mathrm{i}kx} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

defines the direct map

$$\mathbb{S} : \{u(x)\} \to \{a(k), b(k)\} \tag{26}$$

by the formula

$$\begin{pmatrix} b(k) \\ a(k) \end{pmatrix} = \Psi(0,k).$$

The spectral data $a(k)$ and $b(k)$ possess the following properties:
1) $a(k)$ and $b(k)$ are analytic in $k \in \mathbb{C}_+$ and continuous in $k \in \overline{\mathbb{C}_+}$ functions represented in the form

$$a(k) = 1 + \int_0^\infty \alpha(y)\mathrm{e}^{\mathrm{i}ky}dy, \quad b(k) = \int_0^\infty \beta(y)\mathrm{e}^{\mathrm{i}ky}dy,$$

where $\alpha(y), \beta(y)$ are absolutely continuous and $\alpha'_y(y), \beta'_y(y) \in L^1(0,\infty)$;
2) $|a(k)|^2 + |b(k)|^2 \equiv 1, \qquad k \in \mathbb{R}$;
3) $a(k) = 1 + O(k^{-1}), \qquad b(k) = O(k^{-1}), \qquad k \to \infty$.

The inverse map $\mathbb{Q}$ is given by

$$u(x) = 2\mathrm{i} \lim_{k\to\infty} kM_{12}^{(x)}(x,k), \tag{27}$$

where $M_{12}^{(x)}(x,k)$ is the entry (12) of matrix $M^{(x)}(x,k)$. This matrix is the unique solution of the following Riemann–Hilbert problem:
- $M^{(x)}(x,k)$ is a sectionally analytic matrix valued function in $k \in \mathbb{C}\backslash\Gamma$, where the oriented contour Γ is a union of the real line $\mathbb{R}$ and the circle $\mathcal{S}_\infty = \{k \in \mathbb{C} : |k| = |\mathcal{S}_\infty|\}$, where $|\mathcal{S}_\infty|$ is a sufficiently large positive number. The orientation of Γ is chosen so that k-plane is a union of the two open domains $\Omega_\pm$ and their common boundary Γ (Fig. 2).
- $M^{(x)}(x,k) = I + O(k^{-1}), \qquad k \to \infty.$
- $M_+^{(x)}(x,k) = M_-^{(x)}(x,k)J^{(x)}(x,k), \qquad k \in \Gamma,$

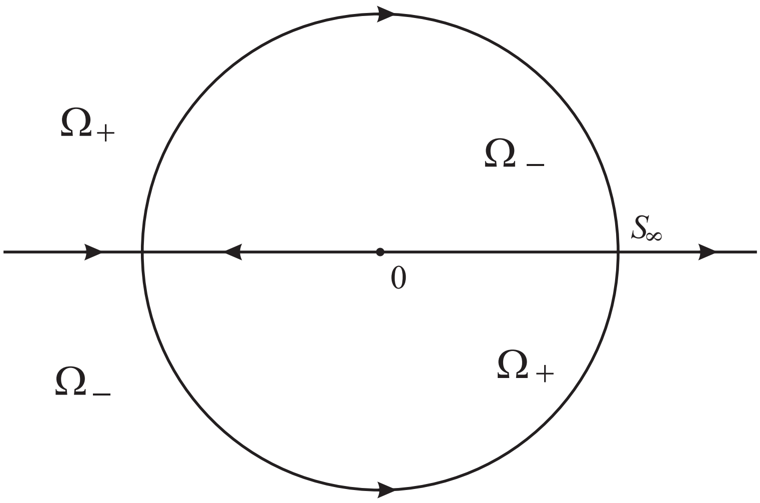

Fig. 2: The oriented contour Γ for the x-problem.

where $M_+^{(x)}(x,k)$, $M_-^{(x)}(x,k)$ are the boundary values of matrix $M^{(x)}(x,k)$ on contour Γ from domains Ω_+, Ω_-, and

$$J^{(x)}(x,k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & k \in \mathbb{R}, \quad |k| < |\mathcal{S}_\infty|, \\ \begin{pmatrix} 1 & \frac{b(k)}{a(k)} e^{-2ikx} \\ \frac{\bar{b}(\bar{k})}{\bar{a}(\bar{k})} e^{2ikx} & \frac{1}{|a(k)|^2} \end{pmatrix}, & k \in \mathbb{R}, \quad |k| > |\mathcal{S}_\infty|, \end{cases} \tag{28}$$

$$J^{(x)}(x,k) = \begin{cases} \begin{pmatrix} 1 & \frac{b(k)}{a(k)} e^{-2ikx} \\ 0 & 1 \end{pmatrix}, & |k| = |\mathcal{S}_\infty|, \quad \arg k \in (0, \pi), \\ \begin{pmatrix} 1 & 0 \\ \frac{\bar{b}(\bar{k})}{\bar{a}(\bar{k})} e^{2ikx} & 1 \end{pmatrix}, & |k| = |\mathcal{S}_\infty|, \quad \arg k \in (\pi, 2\pi). \end{cases} \tag{29}$$

This RH problem is uniquely solvable [12].

3. Main Matrix Riemann–Hilbert Problem: Reconstruction of the SRS Model

Under the assumption that x- and t-equations (8) and (10), respectively, are compatible, the relations (20) between their solutions can be written in the form of the matrix Riemann–Hilbert problem.

Let $q(x,t), \mu(x,t), \nu(x,t)$ be absolutely continuous functions with respect to $x \in [0,\infty)$ and $t \in [0,\infty)$ satisfying the SRS equations (1), the initial (2) and boundary (3) conditions. Then the relations (20) define a map

$$\mathbb{S}^R : \{q(x,t), \mu(x,t), \nu(x,t)\} \to \{a(k), b(k), A(k), B(k)\}. \tag{30}$$

In fact the spectral functions $\{a(k), b(k)\}$ are defined by initial function $u(x) = q(x,0)$, and the spectral functions $\{A(k), B(k)\}$ are defined by boundary data. In our case they take the explicit form

$$2A(k) = \varkappa(k) + \frac{1}{\varkappa(k)} = 2\bar{A}(\bar{k}), \quad 2B(k) = \Big(\frac{1}{\varkappa(k)} - \varkappa(k)\Big) = -2\bar{B}(\bar{k}).$$

To describe the map inverse to (30) we additionally use the auxiliary spectral functions $a_R(k) = \bar{a}(\bar{k})A(k) + \bar{b}(\bar{k})B(k)$ and $b_R(k) = a(k)B(k) - b(k)A(k)$ which are the entries of the transition matrix $R(k) = s(k)S(k)$.

The inverse (to (30)) map $\mathbb{Q}^R$ is defined by

$$q(x,t) = 2\mathrm{i}m_{12}^{\infty}(x,t), \tag{31}$$

$$\nu(x,t) = m_{11}(x,t), \tag{32}$$

$$\mu(x,t) = -\mathrm{i}m_{12}(x,t), \tag{33}$$

where

$$m^{\infty}(x,t) = \lim_{k\to\infty} kM(x,t,k),$$

$$m(x,t) = -m_0(x,t)\sigma_3 m_0^{-1}(x,t),$$

$$m_0(x,t) = \lim_{k\to 0} M(x,t,k),$$

and the matrix $M(x,t,k)$ is the solution of the following Riemann–Hilbert problem RH_{xt}:

- $M(x,t,k)$ is sectionally analytic for $k \in \mathbb{C} \setminus \Gamma$; the oriented contour Γ is defined as follows: $\Gamma = \mathbb{R} \cup \mathcal{S}_\infty \cup \gamma \cup \bar{\gamma}$ (Fig. 3);
- $M(x,t,k)$ has the fourth-root singularities at the points E and $\bar{E}$;
- $M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \Gamma$, where

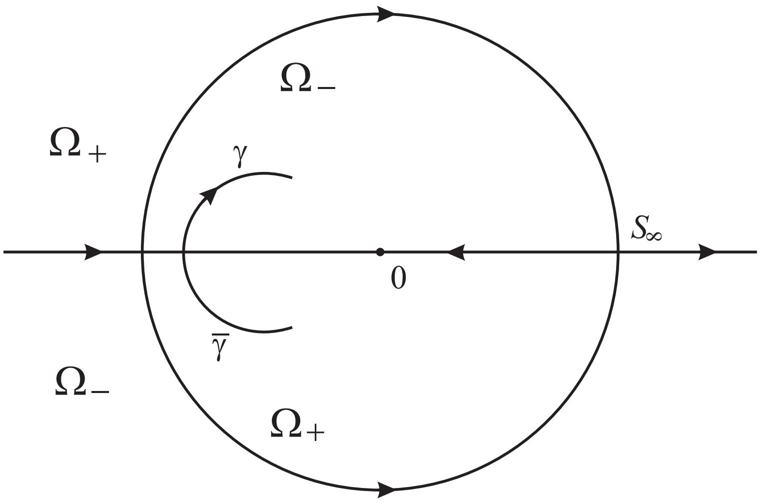

Fig. 3: The contour Γ for the xt-problem.

$$J(x,t,k)=\begin{cases}\begin{pmatrix} a_R(k) & 0 \\ \bar b_R(\bar k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & \dfrac{1}{a_R(k)}\end{pmatrix}, & |k|=|\mathcal{S}_\infty|, \quad \operatorname{Im}k<0; \\[2ex] \begin{pmatrix} 1 & \dfrac{b_R(k)}{a_R(k)}\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ \dfrac{\bar b_R(\bar k)}{\bar a_R(\bar k)}\mathrm{e}^{2\mathrm{i}t\theta(k)} & \dfrac{1}{|a_R(k)|^2}\end{pmatrix}, & |k|>|\mathcal{S}_\infty|, \quad \operatorname{Im}k=0; \\[2ex] \begin{pmatrix} \bar a_R(\bar k) & b_R(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & \dfrac{1}{\bar a_R(\bar k)}\end{pmatrix}, & |k|=|\mathcal{S}_\infty|, \quad \operatorname{Im}k>0; \end{cases}$$

$$J(x,t,k)=\begin{cases}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, & |k|<|\mathcal{S}_\infty|, \quad \operatorname{Im}k=0; \\[2ex] \begin{pmatrix} 0 & -\mathrm{i}\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ -\mathrm{i}\mathrm{e}^{2\mathrm{i}t\theta(k)} & 0\end{pmatrix}, & k\in\gamma\cup\bar\gamma; \end{cases}$$

with $\theta(k)=\dfrac{1}{4k}+k\dfrac{x}{t}$;

- $M(x,t,k)=I+O(k^{-1}), \qquad k\to\infty;$
- $M(x,t,k)=m_0(x,t)+O(k), \qquad k\to 0.$

P r o o f. To construct the Riemann–Hilbert problem RH_{xt}, we define the following matrices:

$$M(x,t,k) = \begin{cases} \left(\dfrac{[\Phi_1]_1(x,t,k)}{\bar{a}_R(\bar{k})}, \quad [\Phi_3]_2(x,t,k)\right) \mathrm{e}^{\mathrm{i}t\theta(k)\sigma_3}, & |k| > |\mathcal{S}_\infty|, \quad \operatorname{Im} k > 0, \\ \\ \Phi_1(x,t,k)\mathrm{e}^{\mathrm{i}t\theta(k)\sigma_3}, & |k| < |\mathcal{S}_\infty|, \quad \operatorname{Im} k > 0, \end{cases}$$

$$M(x,t,k) = \begin{cases} \Phi_1(x,t,k)\mathrm{e}^{\mathrm{i}t\theta(k)\sigma_3}, & |k| < |\mathcal{S}_\infty|, \quad \operatorname{Im} k < 0, \\ \\ \left([\Phi_3]_1(x,t,k), \quad \dfrac{[\Phi_1]_2(x,t,k)}{a_R(k)}\right) \mathrm{e}^{\mathrm{i}t\theta(k)\sigma_3}, & |k| > |\mathcal{S}_\infty|, \quad \operatorname{Im} k < 0, \end{cases}$$

where $[\Phi_1]_{1,2}(x,t,k)$, $[\Phi_3]_{1,2}(x,t,k)$, are the vector columns of the matrices

$$\Phi_1(x,t,k) = ([\Phi_1]_1(x,t,k), [\Phi_1]_2(x,t,k)),$$
$$\Phi_3(x,t,k) = ([\Phi_3]_1(x,t,k), [\Phi_3]_2(x,t,k)).$$

The radius $|\mathcal{S}_\infty|$ of the circle $\mathcal{S}_\infty$ is sufficiently large so that $a_R(k) \neq 0$ ($\bar{a}_R(\bar{k}) \neq 0$) for $|k| > |\mathcal{S}_\infty|$, $\operatorname{Im} k < 0$, $\operatorname{Im} k > 0$. Then the matrices $M_\pm(x,t,k)$ are analytic functions in the domains $\Omega_\pm$. They have the forth root singularities at the points E and $\bar{E}$, because the matrix $\Phi_1(x,t,k)$ as well as the matrix $\mathcal{E}(t,k)$ has the same singularities at these points. The determinants of these matrices are equal to one, which follows from the vector relations

$$[\Phi_1]_1(x,t,k) = \bar{a}_R(\bar{k})[\Phi_3]_1(x,t,k) - \bar{b}_R(\bar{k})[\Phi_3]_2(x,t,k),$$

$$[\Phi_1]_2(x,t,k) = b_R(k)[\Phi_3]_1(x,t,k) + a_R(k)[\Phi_3]_2(x,t,k)$$

arising from (20). Direct calculation gives the form of the jump matrix $J(x,t,k)$ on different parts of Γ. The asymptotic formulas for $M(x,t,k)$ as $k \to \infty$ and $k \to 0$ follow from the corresponding equations for the eigenfunctions, see Sect. 2, and from the asymptotic behavior of the spectral function $a_R(k)$. In particular, we have $M_\pm(x,t,k) = \Phi_0(x,t) + O(k), \quad k \to 0$. Therefore $m_0(x,t) = \Phi_0(x,t)$. ∎

Using general ideas of [11] and the results of [12] for contours with self-intersections, we prove the following theorem.

Theorem 3.1. *Let $u(x)$ be an absolutely continuous function satisfying (5). Let $\nu(0,t) = l$, $l < 0$, $\mu(0,t) = pe^{2\mathrm{i}\omega t}$ ($\omega, p > 0$, $l^2 + p^2 = 1$). Let $\{a(k), b(k), A(k), B(k)\}$ be the corresponding spectral functions. Then the Riemann–Hilbert problem RH_{xt} has the unique solution $M(x,t,k)$. The functions $q(x,t)$, $\mu(x,t)$ and $\nu(x,t)$, defined by the equations*

$$q(x,t) = 2\mathrm{i} \lim_{k\to\infty} kM_{12}(x,t,k), \tag{34}$$

$$\nu(x,t) = m_{11}(x,t), \qquad \mu(x,t) = -\mathrm{i}m_{12}(x,t)$$

with the matrix

$$m(x,t) = -M(x,t,0)\sigma_3 M^{-1}(x,t,0),$$

satisfy the SRS equations (1), the initial condition

$$q(x,0) = u(x), \quad x \in (0,\infty),$$

and the boundary conditions

$$\nu(0,t) = l, \quad \mu(0,t) = p\mathrm{e}^{\mathrm{i}\omega t}, \quad t \in (0,\infty).$$

The proof of this theorem is performed in the same way as in [5] and [10].

4. Asymptotic Behaviour of the Solution in the Zakharov–Manakov Region

In this section we study the asymptotic behavior of the solution to the IBV problem (1)–(3) as $t \to \infty$. To fix the ideas of asymptotic analysis and to make it more transparent we restrict our attention to a special case when the initial function is equal to zero identically. We will use the steepest descent method [6] by P. Deift and X. Zhou; many technical details of this method become much more simple in this special case. We describe the asymptotics of the solution in the sector $x > \omega^2 t$, where it has a quasilinear dispersive character. In the adjacent sector $x < \omega^2 t$ of the quarter xt-plane the asymptotics is more complicated and will be studied elsewhere.

For the case $u(x) \equiv 0$, $\mu(0,t) = p\mathrm{e}^{\mathrm{i}\omega t}$ and $\nu(0,t) = l$ the corresponding spectral functions are as follows:

$$a(k) \equiv 1, \qquad b(k) \equiv 0, \tag{35}$$

$$a_R(k) = A(k) = \frac{1}{2}\left(\varkappa(k) + \frac{1}{\varkappa(k)}\right), \qquad b_R(k) = B(k) = \frac{1}{2}\left(\frac{1}{\varkappa(k)} - \varkappa(k)\right), \tag{36}$$

where $\varkappa(k) = \sqrt[4]{\dfrac{k-E}{k-\bar{E}}}$, $E = \dfrac{l+\mathrm{i}p}{2\omega}$ $(\omega, p > 0, l < 0, l^2 + p^2 = 1)$. These formulas show that the spectral data $A(k)$ and $B(k)$ are analytic functions everywhere with the exception of arc $\gamma \cup \bar{\gamma}$, and that $A(k) \neq 0$. We recall that the complex k-plane is cut along the contour $\gamma \cup \bar{\gamma}$. Therefore the main Riemann–Hilbert problem RH_{xt} can be reduced to the equivalent one:

- matrix valued function $M^{(1)}(x,t,k)$ is analytic in the domain $\mathbb{C}_+\setminus\gamma$ and $\mathbb{C}_-\setminus\bar\gamma$;
- $M^{(1)}(x,t,k)$ has the fourth-root singularities at the points E and $\bar E$;
- $M^{(1)}_-(x,t,k)=M^{(1)}_+(x,t,k)J^{(1)}(x,t,k),\quad k\in\mathbb{R}\cup\gamma\cup\bar\gamma$, where

$$J^{(1)}(x,t,k)=\begin{cases}\begin{pmatrix}1 & \varrho(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)}\\ -\varrho(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1-\varrho^2(k)\end{pmatrix}, & k\in\mathbb{R},\\[2ex] \begin{pmatrix}1 & 0\\ f(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1\end{pmatrix}, & k\in\gamma;\\[2ex] \begin{pmatrix}1 & f(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)}\\ 0 & 1\end{pmatrix}, & k\in\bar\gamma;\end{cases}$$

- $M^{(1)}(x,t,k)=I+O(k^{-1}),\qquad k\to\infty;$
- $M^{(1)}(x,t,k)=\tilde m_0(x,t)+O(k),\qquad k\to 0;$

where $\varrho(k):=\dfrac{B(k)}{A(k)}$ and $f(k):=\varrho_-(k)-\varrho_+(k)=-\dfrac{1}{B_+(k)A_+(k)}$.

- The functions $q(x,t)$, $\mu(x,t)$ and $\nu(x,t)$ are determined by $M^{(1)}(x,t,k)$ in the same way as in (34).

P r o o f. Since $a_R(k)\equiv A(k)\neq 0$ for all k, the $\mathrm{RH_{xt}}$ problem can be simplified. Indeed, let us transform the initial matrix $M(x,t,k)$ to the following one

$$M^{(1)}(x,t,k)=M(x,t,k)G^{(1)}(x,t,k),$$

where $G^{(1)}(x,t,k)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ for $|k|>|\mathcal{S}_\infty|$ and

$$G^{(1)}(x,t,k)=\begin{cases}\begin{pmatrix}\frac{1}{A(k)} & -B(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)}\\ 0 & A(k)\end{pmatrix}, & |k|<|\mathcal{S}_\infty|, \qquad \operatorname{Im}k>0,\\[2ex] \begin{pmatrix}A(k) & 0\\ -B(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & \frac{1}{A(k)}\end{pmatrix}, & |k|<|\mathcal{S}_\infty|, \qquad \operatorname{Im}k<0.\end{cases}$$

The transformation eliminates the circle $\mathcal{S}_\infty$, where the jump matrix $J(x,t,k)=G^{(1)}(x,t,k)$ is unbounded as $t\to\infty$. It is easy to see that the matrix valued function $M^{(1)}(x,t,k)$ is analytic in the domains $\mathbb{C}_+\setminus\gamma$ and $\mathbb{C}_-\setminus\bar\gamma$ and has the forth-root singularities at branch points. The new jump matrix coincides with the jump matrix $J^{(1)}(x,t,k)$. Furthermore, since

$$G^{(1)}(x,t,k) = \begin{cases} \begin{pmatrix} \frac{1}{A(k)} & O(\mathrm{e}^{-t\,\mathrm{Im}\,k/2|k|^2}) \\ 0 & A(k) \end{pmatrix}, & k \to 0, \qquad \mathrm{Im}\,k > 0, \\ \\ \begin{pmatrix} A(k) & 0 \\ O(\mathrm{e}^{t\,\mathrm{Im}\,k/2|k|^2}) & \frac{1}{A(k)} \end{pmatrix}, & k \to 0, \qquad \mathrm{Im}\,k < 0 \end{cases}$$

becomes diagonal in the limit as $t \to \infty$, then

$$\lim_{k\to 0} M_1(x,t,k)\sigma_3 M_1^{-1}(x,t,k) = \lim_{k\to 0} M(x,t,k)\sigma_3 M^{-1}(x,t,k)$$

and, thus, $\nu(x,t)$ and $\mu(x,t)$ given by $M_1(x,t,k)$ according to (34) are similar to the ones given by $M(x,t,k)$. Finally, since $M_1(x,t,k) = M(x,t,k)$ for $|k| > |\mathcal{S}_\infty|$, we have that the same is true for $q(x,t)$. ■

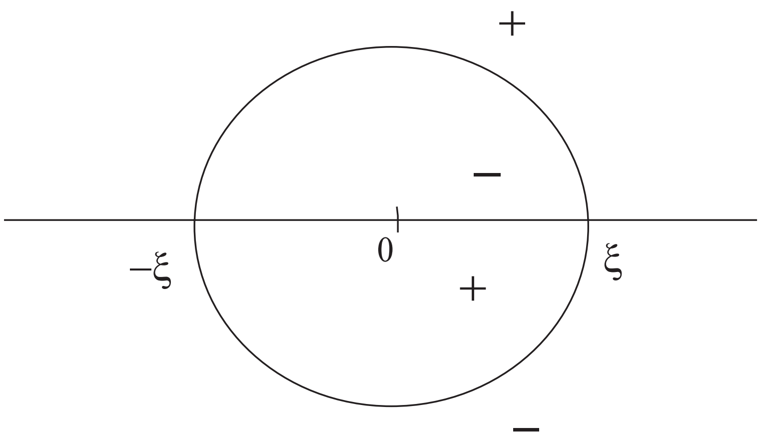

Fig. 4: The signature table of the function $\mathrm{Im}\,\theta(k)$.

To study the asymptotic behavior of the Riemann–Hilbert problem RH_{xt} in the region $x > \omega^2 t$ we use the well-known technics from [6, 13, 14]. In what follows, a significant role is played by the decomposition of the complex k-plane according to the signature table of the imaginary part of the phase function $\theta(k) = \frac{1}{4}\left(\frac{1}{k} + \frac{k}{\xi^2}\right)$, where $\xi^2 = t/4x$. The stationary points of the phase function $\theta(k)$ are real and equal to $\pm\xi$. We have

$$\mathrm{Im}\,\theta(k) = \frac{|k|^2 - \xi^2}{4|k|^2\xi^2}\,\mathrm{Im}\,k.$$

Thus $\operatorname{Im}\theta(k) > 0$ ($\operatorname{Im}\theta(k) < 0$) for k lying in the lower (upper) half-disk and out of the upper (lower) half-disk defined by the circle $|k| = \xi$ (Fig. 4). For $\xi < |E| = 1/2\omega$ (that is, for $x > \omega^2 t$), the jump matrix $J^{(1)}(x,t,k)$ approaches the identity matrix as $t \to \infty$ for $k \in \gamma \cup \bar{\gamma}$. Hence the contour $\gamma \cup \bar{\gamma}$ does not contribute to the main term of asymptotics, which is defined by the stationary points $\pm\xi$ and has the order $O(t^{-1/2})$. The contour $\gamma \cup \bar{\gamma}$ plays a crucial role in the description of asymptotics in the region $x < \omega^2 t$. We conjecture that in this region the asymptotics is of order $O(1)$ and takes the form of an elliptic modulated wave for $|E| < \xi < \xi_0$ and a plane wave for $\xi_0 < \xi < \infty$. In this paper we study the asymptotics in the region $x > \omega^2 t$ only.

To study the asymptotic behavior of the RH problem for the matrix $M^{(1)}(x,t,k)$ let us use the transform

$$M^{(2)}(x,t,k) = M^{(1)}(x,t,k)\delta^{-\sigma_3}(k),$$

where the function $\delta(k)$ is equal to (cf.[6])

$$\delta(k) = \exp\left\{\frac{1}{2\pi \mathrm{i}}\int\limits_{-\xi}^{\xi}\frac{\log(1-\varrho^2(s))ds}{s-k}\right\}, \qquad k \in C \setminus [-\xi,\xi],$$

and $\xi = \sqrt{t/4x} > 0$. Then the jump matrix $J^{(2)}(x,t,k)$ has a lower/upper factorization for $|k| < \xi$ and an upper/lower factorization for $|k| > \xi$

$$J^{(2)}(x,t,k) = \begin{cases} \begin{pmatrix} 1 & A(k)B(k)\delta_+^2(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -A(k)B(k)\delta_-^{-2}(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix}, & |k| < \xi, \\ \begin{pmatrix} 1 & 0 \\ -\varrho(k)\delta^{-2}(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix}\begin{pmatrix} 1 & \varrho(k)\delta^2(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix}, & |k| > \xi, \end{cases}$$

where we use the identity

$$\frac{\varrho(k)}{1-\varrho^2(k)} = A(k)B(k).$$

The jump matrices on the contours $\gamma \cup \bar{\gamma}$ are

$$J^{(2)}(x,t,k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ f(k)\delta^{-2}(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix}, & k \in \gamma; \\ \begin{pmatrix} 1 & -f(k)\delta^2(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix}, & k \in \bar{\gamma}. \end{cases}$$

Let us define a decomposition of the complex k-plane into six domains $D_1, \ldots, D_6$ as shown in Fig. 5. The contours L_2 and L_5 lie in the disk $|k| < \xi$; the contours L_1, L_6 (L_3, L_4) range from the point ξ ($-\xi$) to infinity along the rays $\arg k = \pm\pi/4$ ($\arg k = \pi \mp \pi/4$). Then the next transformation is

$$M^{(3)}(x,t,k) = M^{(2)}(x,t,k)G^{(2)}(k),$$

where

$$G^{(2)}(k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -\varrho(k)\delta^{-2}(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix}, & k \in D_1 \cup D_3, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & k \in D_2 \cup D_5, \\ \begin{pmatrix} 1 & -\varrho(k)\delta^{2}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix}, & k \in D_4 \cup D_6; \end{cases} \tag{37}$$

$$G^{(2)}(k) = \begin{cases} \begin{pmatrix} 1 & A(k)B(k)\delta^{2}(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix}, & k \in D_8, \\ \begin{pmatrix} 1 & 0 \\ A(k)B(k)\delta^{-2}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} & 1 \end{pmatrix}, & k \in D_7. \end{cases} \tag{38}$$

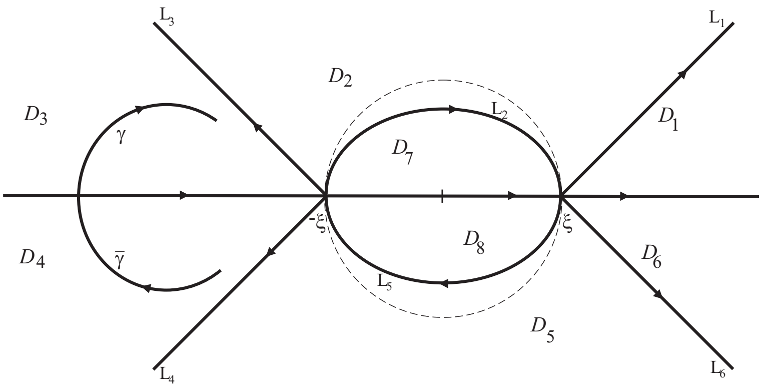

Fig. 5: The contour Γ for the $M^{(3)}(x,t,k)$-problem.

Remark 4.1. *The transformation $M^{(2)}(x,t,k) \to M^{(3)}(x,t,k)$ has the form as above due to the fact that $\varrho(k)$, corresponding to the initial and boundary data considered here, is in fact analytic outside $\gamma \cup \bar{\gamma}$ (see (35), (36)). An analogous transformation in the case of more general initial and boundary conditions (i.e. when, for example, the initial function is fast decreasing as $x \to \infty$) requires an analytic approximation of the corresponding spectral functions (cf. [6]).*

The G-transformation leads to the following RH problem

$$M_-^{(3)}(x,t,k) = M_+^{(3)}(x,t,k) J^{(3)}(x,t,k)$$

on the contour depicted in Fig. 5 with the jump matrices $J^{(3)}(x,t,k)$ which are equal to the identity matrix on real axis, they coincide with the matrices $G^{(2)}(k)$ from (37)–(38) chosen for the contours $k \in L_j$, $j = 1, 2, \ldots, 6$. Moreover, the jump matrix $J^{(3)}(x,t,k)$ is equal to the identity matrix on the arc $\gamma \cup \bar{\gamma}$ because $f(k) = \varrho_-(k) - \varrho_+(k)$. Hence, in the region $x > \omega^2 t$ ($\xi < 1/2\omega$) the jump across the arc $\gamma \cup \bar{\gamma}$ does not contribute to the asymptotics of the solution. Furthermore, it is easy to see that $J^{(3)}(x,t,k) = I + O(\mathrm{e}^{-\varepsilon t})$ as $t \to \infty$ and $k \in L_j$ with the exception of some neighborhoods of the stationary points $\pm\xi$. Since

$$M^{(3)}(x,t,k) = I + \frac{m_1^{(3)}(x,t)}{k} + O(k^{-2}), \qquad k \to \infty,$$

we have

$$q(x,t) = 2\mathrm{i}[m_1^{(3)}(x,t)]_{12} + O(e^{-\varepsilon t}), \qquad \varepsilon > 0.$$

Now we have to evaluate the main contributions from neighborhoods of the stationary points $k_0 = \pm\xi$ of the phase function $\theta(k) = 1/4k + k/4\xi^2$. To do this we use the scaling operators

$$F(k) \to [N_\pm F](z) = F(z\sqrt{\xi^3 t^{-1}} + k_0)|_{k_0 = \pm\xi},$$

which for the matrices $M^{(4)}(x,t,z)^\pm = [N_\pm M^{(3)}](z\sqrt{\xi^3 t^{-1}} + k_0)$ imply

$$[m_1^{(3)}(x,t)]_{12} = \sqrt{\frac{\xi^3}{t}}[m_1^{(4)}(x,t)]_{12}^{\pm}.$$

The scaling operators act on the product $\delta(k)e^{-\mathrm{i}t\theta(k)}$ as follows:

$$[N_\pm \delta e^{-\mathrm{i}t\theta}] = \delta_\pm^{(0)}(\xi,t)\,\delta_\pm^{(1)}(z,\xi,t),$$

where

$$\delta_\pm^{(0)}(\xi,t) = \left(\frac{4t}{\xi}\right)^{\mp\mathrm{i}\eta(k_0)/2} e^{\pm\mathrm{i}t/2\xi} e^{\chi(k_0)},$$

$$\delta_{\pm}^{(1)}(z,\xi,t) = (\pm z)^{\pm \mathrm{i}\eta(k_0)}\, e^{\mp \mathrm{i}z^2/4}\left[1 + O\left(\frac{z}{\sqrt{t}}\right)\right]$$

with the functions

$$\eta(k) = \frac{1}{2\pi}\log[1-\varrho^2(k)] = \frac{1}{2\pi}\log[1+|\varrho(k)|^2] > 0$$

and

$$\chi(k) = -\frac{1}{2\pi \mathrm{i}}\int\limits_{-\xi}^{\xi} \log|s-k_0|d\log[1-\varrho^2(s)].$$

The function $\delta_{\pm}^{(0)}(\xi,t)$ does not depend on z, but the functions $(\pm z)^{\pm \mathrm{i}\eta(k_0)}\, e^{\mp \mathrm{i}z^2/4}$ do and lead to the model RH problems

$$H_+(z) = H_-(z)e^{-\mathrm{i}z^2\sigma_3/4}J_0(k_0)e^{\mathrm{i}z^2\sigma_3/4},$$

where

$$J_0(k_0) = \begin{pmatrix} 1-\varrho^2(k_0) & -\varrho(k_0) \\ \varrho(k_0) & 1 \end{pmatrix}, \qquad k_0 = \pm\xi,$$

and

$$H(z) = \left(I + \frac{m_1^{\pm}(\xi,t)}{z} + O(z^{-2})\right) z^{\mathrm{i}\eta(k_0)\sigma_3}, \qquad z \to \infty.$$

These problems can be solved explicitly in the terms of parabolic cylinder functions [13]. Thus we have

$$q(x,t)$$

$$= \sqrt{\frac{\xi^3}{t}}\left\{\left(\delta_+^{(0)}(\xi,t)\right)^2 2\mathrm{i}[m_1^+(\xi,t)]_{12} + \left(\delta_-^{(0)}(\xi,t)\right)^2 2\mathrm{i}[m_1^-(\xi,t)]_{12}\right\} + o\left(\frac{1}{\sqrt{t}}\right),$$

where

$$[m_1^{\pm}(\xi,t)]_{12} = -\mathrm{i}\frac{\sqrt{2\pi}e^{\mathrm{i}\pi/4}e^{-\pi\eta(k_0)/2}}{\varrho(k_0)\Gamma(-\mathrm{i}\eta(k_0)))}, \qquad k_0 = \pm\xi,$$

and $\Gamma(z)$ denotes Euler's gamma-function. Finally, using the basic identity

$$|\Gamma(-\mathrm{i}\eta)|^2 = |\Gamma(\mathrm{i}\eta)|^2 = \frac{\pi}{\eta\sinh\eta},$$

we come to the following statement.

Theorem 4.1. *Let $q(x,t)$, $\mu(x,t)$ and $\nu(x,t)$ be the solution of the SRS equations (1) with the initial function satisfying (5) and the boundary conditions (3)–(4). Then in the region $x > \omega^2 t$, the function $q(x,t)$ has a quasilinear dispersive character, i.e., it is described by the Zakharov–Manakov type formulas*

$$q(x,t) = 2\sqrt{\frac{\xi^3\eta(\xi)}{t}} \exp\left\{2\mathrm{i}\sqrt{xt} - \mathrm{i}\eta(\xi)\log\sqrt{xt} + i\phi(\xi)\right\}$$
$$+2\sqrt{\frac{\xi^3\eta(-\xi)}{t}} \exp\left\{-2\mathrm{i}\sqrt{xt} + \mathrm{i}\eta(-\xi)\log\sqrt{xt} + i\phi(-\xi)\right\} + o(t^{-1/2}),\ t \to \infty,$$

where the functions $\eta(k)$ and $\phi(k)$ are given by the equations

$$\eta(k) = \frac{1}{2\pi}\log\left(1-\varrho^2(k)\right), \qquad \xi^2 = \frac{t}{4x},$$
$$\phi(k) = \frac{\pi}{4} - 3\eta(k)\log 2 - \arg\varrho(k) - \arg\Gamma(-\mathrm{i}\eta(k))$$
$$+\frac{1}{\pi}\int_{-\xi}^{\xi}\log|s-k|d\log[1-\varrho^2(s)].$$

Here $\Gamma(-\mathrm{i}\eta(k))$ is the Euler gamma-function, and $\varrho(k) = \mathrm{i}\tan[\arg\varkappa(k)]$.

The asymptotics of functions $\mu(x,t)$ and $\nu(x,t)$ can be found by formulas

$$\mu(x,t) = 2\mathrm{i}q_t(x,t), \qquad \nu(x,t) = \sqrt{1-|\mu(x,t)|^2}.$$

It is easy to find that the residual of this asymptotic solution in the SRS equations has the order $O(\log t/t^{3/2})$ as $t \to \infty$ and $x > \omega^2 t$.

Remark 4.2. *Qualitatively this result is valid for the case $u(x) \neq 0$ and the general boundary conditions (6). As we have noticed above, in this case the corresponding transformations of the RH problem include the suitable analytic approximations of spectral functions.*

Acknowledgment. The author thanks V.P. Kotlyarov and D.G. Shepelsky for helpful discussions, comments and remarks.

References

[1] *J. Leon and A.V. Mikhailov*, Raman Soliton Generation from Laser Inputs in SRS. — *Phys. Lett. A* **253** (1999), 33–40.

[2] *A.S. Fokas and C.R. Menyuk*, Integrability and Self-Similarity in Transient Stimulated Raman Scattering. — *J. Nonlinear Sci.* **9** (1999), 1–31.

[3] *A.S. Fokas*, A Unified Transform Method for Solving Linear and Certain Nonlinear PDEs. — *Proc. R. Soc. Lond. A* **453** (1997), 1411–1443.

[4] *A. Boutet de Monvel, A.S. Fokas, and D.G. Shepelsky*, The mKdV Equation on the Half-Line. — *J. Inst. Math.* **3** (2004), No. 2, 139–164.

[5] *E.A. Moscovchenko and V.P. Kotlyarov*, A new Riemann–Hilbert Problem in a Model of Stimulated Raman Scattering. — *J. Phys. A.: Math. Gen.* **39** (2006), 14591–14610.

[6] *P. Deift and X. Zhou*, A Steepest Descent Method for Oscillatory Riemann–Hilbert Problems. Asymptotics for the MKdV Equation. — *Ann. Math.* **137** (1993), 295–368.

[7] *E.Ya. Khruslov and V.P. Kotlyarov*, Generation of Asymptotic Solitons in an Integrable Model of Stimulated Raman Scattering by Periodic Boundary Data. — *Mat. fiz., anal., geom.* **10** (2003), 366–384.

[8] *A.R. Its and V.E. Petrov*, "Isomonodromic" Solutions of the Sine–Gordon Equation and the Time Asymptotics of its Rapidly Decreasing Solutions. — *Dokl. Akad. Nauk USSR* **265** (1982), No. 6, 1302–1306. (Russian)

[9] *A. Boutet de Monvel and V.P. Kotlyarov*, Scattering Problem for the Zakharov–Shabat Equations on the Semi-Axis. — *Inverse Probl.* **16** (2000), 1813–1837.

[10] *A. Boutet de Monvel and V. Kotlyarov*, Focusing Nonlinear Schrödinger Equation on the Quarter Plane with Time-Periodic Boundary Condition: a Riemann–Hilbert Approach. — *J. Inst. Math.* **6** (2007), No. 4, 579–611.

[11] *L.D. Fadeev and L. Takhtadjan*, Hamiltonian Methods in the Theory of Solitons. Springer, Berlin, 1987.

[12] *X. Zhou*, Direct and Inverse Scattering Transform with Arbitrary Spectral Singularities. — *Comm. Pure Appl. Math.* **42** (1989), 895–938.

[13] *P. Deift, A. Its, and X. Zhou*, Long-time Asymptotics for Integrable Nonlinear Wave Equations, Important Developments in Soliton Theory. Springer, Ser. Nonlinear Dynam., Berlin, 1993.

[14] *P.A. Deift, A.R. Its, and X. Zhou*, A Riemann–Hilbert Approach to Asymptotic Problems Arising in the Theory of Random Matrix Models, and also in the Theory of Integrable Statistical Mechanics. — *Ann. Math.* **146** (1997), 149–235.