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Orthogonal Polynomials on Rays: Properties of Zeros, Related Moment Problems and Symmetries

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We establish some basic properties of zeros for orthonormal polynomials on radial rays. We introduce a moment problem related to these orthonormal polynomials and obtain necessary and sufficient conditions for its solvability. We establish some properties of orthonormal polynomials in the case when the measure of orthogonality has symmetries and show that the moment problem has solutions with some symmetric properties.

Key words: orthogonal polynomials, moment problem, symmetry.

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1. Introduction

We denote by $\mathbb{C}_{n\times n}$ the set of all $n \times n$ matrices with complex elements, and by $\mathbb{C}^{\geq}_{n\times n}$ we mean all nonnegative Hermitian matrices from $\mathbb{C}_{n\times n}$, $n \in \mathbb{N}$. By $\mathbb{P}$ we denote the set of all complex polynomials. Let

$$L_N = \{\lambda \in \mathbb{C} : \lambda^N - \overline{\lambda}^N = 0\} = \{\lambda \in \mathbb{C} : \lambda^N \in \mathbb{R}\}, \quad N \in \mathbb{N}.$$

Note that L_N is an algebraic curve. In fact, it is a set of $2N$ radial rays or a pencil of N lines with the center at the origin. It is not difficult to verify that

$$L_N = \bigcup_{k=0}^{2N-1} \{x\hat{\varepsilon}^k,\ x \geq 0\}; \tag{1}$$

$$L_N = \bigcup_{k=0}^{N-1} \{x\hat{\varepsilon}^k,\ x \in \mathbb{R}\}, \tag{2}$$

where $\hat{\varepsilon} = \cos\frac{\pi}{N} + i\sin\frac{\pi}{N}$ is a primitive root of unity of order $2N$.

Observe that in the case $N = 1$ we get $L_N = \mathbb{R}$ while in the case $N = 2$ we get $L_N = \mathbb{R} \cup i\mathbb{R}$.

Set

$$L_{N,k} := \{x\hat{\varepsilon}^k,\ x \geq 0\}, \quad k \in \mathbb{Z}. \tag{3}$$

We can see that

$$L_{N,k+2N} = L_{N,k}, \quad k \in \mathbb{Z}. \tag{4}$$

Let $M(\lambda)$ be a $\mathbb{C}_{N\times N}$-valued function on $L_N\backslash\{0\}$ which is nondecreasing on each ray $L_{N,k}\backslash\{0\}$, $k = 0, 1, \ldots, 2N-1$, in the direction from 0 to ∞. Suppose that the function $M(\lambda)$ has finite moments

$$\int_{L_N} (\lambda^n, (\lambda\varepsilon)^n, (\lambda\varepsilon^2)^n, \ldots, (\lambda\varepsilon^{N-1})^n) dM(\lambda) \overline{\begin{pmatrix} \lambda^n \\ (\lambda\varepsilon)^n \\ \vdots \\ (\lambda\varepsilon^{N-1})^n \end{pmatrix}} < \infty, \tag{5}$$

where $\varepsilon = \cos\frac{2\pi}{N} + i\sin\frac{2\pi}{N}$ is a primitive root of unity of order N. Here and in what follows the integral over L_N will be understood as a sum of integrals over each ray $L_{N,k}$, $k = 0, 1, \ldots, 2N-1$. The integral over $L_{N,k}$, $k = 0, 1, \ldots, 2N-1$, is understood to be improper at zero, i.e.,

$$\int_{L_{N,k}} \ldots = \lim_{\delta\to+0} \int_{L_{N,k}\backslash U_\delta(0)} \ldots,$$

where $U_\delta(0) = \{\lambda \in \mathbb{C} :\ |\lambda| < \delta\}$.

We will show below that there exist functions $M(\lambda)$ such that (5) holds true but each entry of $M(\lambda)$ is not integrable at zero.

Let $A \in \mathbb{C}^{\geq}_{N\times N}$. Define the following functional:

$$\sigma(u, v) = \int_{L_N} (u(\lambda), u(\lambda\varepsilon), u(\lambda\varepsilon^2), \ldots, u(\lambda\varepsilon^{N-1})) dM(\lambda) \overline{\begin{pmatrix} v(\lambda) \\ v(\lambda\varepsilon) \\ \vdots \\ v(\lambda\varepsilon^{N-1}) \end{pmatrix}}$$

$$+(u(0), u'(0), u''(0), \ldots, u^{(N-1)}(0)) A \overline{\begin{pmatrix} v(0) \\ v'(0) \\ \vdots \\ v^{(N-1)}(0) \end{pmatrix}}, \qquad u, v \in \mathbb{P}. \tag{6}$$

It is well-defined as it follows from (5). The functional σ is bilinear and it is easy to see that

$$\sigma(\lambda^N u(\lambda), v(\lambda)) = \sigma(u(\lambda), \lambda^N v(\lambda)), \quad u, v \in \mathbb{P}. \tag{7}$$

Notice also that

$$\overline{\sigma(u,v)} = \sigma(v,u), \quad u,v \in \mathbb{P}; \tag{8}$$

$$\sigma(u,u) \geq 0, \quad u \in \mathbb{P}. \tag{9}$$

We assume that the functional σ is nondegenerate in the following sense:

$$\sigma(u,u) > 0 \tag{10}$$

for all $u \in \mathbb{P}: \ u \neq 0$.

Applying the Gramm–Schmidt orthogonalization method with respect to the functional σ to the sequence $1, \lambda, \lambda^2, \ldots, \lambda^n, \ldots$, we obtain a sequence of orthonormal polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n has degree n and a positive leading coefficient):

$$\int_{L_N} (p_n(\lambda), p_n(\lambda\varepsilon), p_n(\lambda\varepsilon^2), \ldots, p_n(\lambda\varepsilon^{N-1}))dM(\lambda) \overline{\begin{pmatrix} p_m(\lambda) \\ p_m(\lambda\varepsilon) \\ \vdots \\ p_m(\lambda\varepsilon^{N-1}) \end{pmatrix}}$$

$$+(p_n(0), p_n'(0), p_n''(0), \ldots, p_n^{(N-1)}(0))A \overline{\begin{pmatrix} p_m(0) \\ p_m'(0) \\ \vdots \\ p_m^{(N-1)}(0) \end{pmatrix}} = \delta_{n,m}, \qquad n,m \in \mathbb{Z}_+. \tag{11}$$

Orthonormal polynomials on radial rays (11) have a characteristic property in terms of a recurrence relation for polynomials. Namely, a set of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n has degree n and a positive leading coefficient) satisfies relation (11) iff it satisfies a recurrence relation (see [1])

$$\sum_{j=1}^{N} (\overline{\alpha_{k-j,j}} p_{k-j}(\lambda) + \alpha_{k,j} p_{k+j}(\lambda)) + \alpha_{k,0} p_k(\lambda) = \lambda^N p_k(\lambda), \quad k \in \mathbb{Z}_+, \tag{12}$$

where $\alpha_{m,n} \in \mathbb{C}, \ m,n \in \mathbb{Z}_+: \ \alpha_{m,N} > 0, \alpha_{m,0} \in \mathbb{R}$, and $\alpha_{m,n}$, p_k which appear here with negative indices are equal to zero.

Relation (12) can be written in a matrix form

$$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} & \ldots & \alpha_{0,N} & 0 & 0 & \ldots \\ \overline{\alpha_{0,1}} & \alpha_{1,0} & \alpha_{1,1} & \ldots & \alpha_{1,N-1} & \alpha_{1,N} & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ \overline{\alpha_{0,N}} & \overline{\alpha_{1,N-1}} & \overline{\alpha_{2,N-2}} & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \overline{\alpha_{1,N}} & \overline{\alpha_{2,N-1}} & \ldots & \ldots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0(\lambda) \\ p_1(\lambda) \\ \vdots \\ p_N(\lambda) \\ p_{N+1}(\lambda) \\ \vdots \end{pmatrix}$$

$$= \lambda^N \begin{pmatrix} p_0(\lambda) \\ p_1(\lambda) \\ \vdots \\ p_N(\lambda) \\ p_{N+1}(\lambda) \\ \vdots \end{pmatrix}. \tag{13}$$

We denote the matrix on the left of (13) by J and $\vec{p}(\lambda) := (p_0(\lambda), p_1(\lambda), p_2(\lambda), \ldots)^T$. We can write

$$J\vec{p}(\lambda) = \lambda^N \vec{p}(\lambda). \tag{14}$$

For the history of polynomials which satisfy high-order difference equation (11) see [1] and References therein.

The aim of our present investigation is threefold. First, we establish some basic properties of zeros for orthonormal polynomials (11). Note that some bounds for zeros under certain conditions were obtained in [2]. Second, we introduce a moment problem related to orthonormal polynomials (11) and obtain necessary and sufficient conditions for its solvability. Finally, we establish some properties of orthonormal polynomials (11) in the case when the measure $M(\lambda)$ has symmetries and show that the moment problem has solutions with some symmetric properties.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of rely, complex, positive integer, integer, nonnegative integer numbers, respectively, and $i\mathbb{R}$ stands for an imaginary axis in the complex plane. Besides the definitions given above, we should note the following notation. If $A \in \mathbb{C}_{n\times n}$, then A^* stands for its adjoint, $n \in \mathbb{N}$. If $A \in \mathbb{C}_{n\times n}$ is nondegenerate, then A^{-1} means its inverse. By $I_{n\times n}$ we denote the matrix $(\delta_{i,j})_{i,j=1}^n$, and by $O_{n\times k}$ we denote a matrix of size $(n \times k)$ whose entries are zeros, $n, k \in \mathbb{N}$. A superscript T means a transposition of the complex numerical vector or matrix.

2. Properties of Zeros

Let $\{p_n(\lambda)\}_{n=0}^\infty$ (p_n has degree n and a positive leading coefficient) be a sequence of orthonormal polynomials which satisfies (11) and therefore satisfies (12). Substitute $\lambda = z$ in (12) and then take the complex conjugate value

$$\sum_{j=1}^N (\alpha_{k-j,j}\overline{p_{k-j}(z)} + \overline{\alpha_{k,j}}\overline{p_{k+j}(z)}) + \alpha_{k,0}\overline{p_k(z)} = \overline{z}^N \overline{p_k(z)}, \quad k \in \mathbb{Z}_+. \tag{15}$$

Multiply (12) by $\overline{p_k(z)}$, (15) by $p_k(\lambda)$ and then do subtraction to get

$$\sum_{j=1}^N ((\overline{\alpha_{k-j,j}} p_{k-j}(\lambda)\overline{p_k(z)} - \alpha_{k-j,j} p_k(\lambda)\overline{p_{k-j}(z)}) + (\alpha_{k,j} p_{k+j}(\lambda)\overline{p_k(z)}$$

$$-\overline{\alpha_{k,j}}p_k(\lambda)\overline{p_{k+j}(z)})) = (\lambda^N - \overline{z}^N)p_k(\lambda)\overline{p_k(z)}, \quad k \in \mathbb{Z}_+. \tag{16}$$

Set

$$A_{k,j}(\lambda, z) := \alpha_{k,j}p_{k+j}(\lambda)\overline{p_k(z)} - \overline{\alpha_{k,j}}p_k(\lambda)\overline{p_{k+j}(z)}, \ k \in \mathbb{Z}_+, \ j = 1, 2, \ldots, N.$$

We can write (16) in the following form:

$$\sum_{j=1}^{N}(-A_{k-j,j}(\lambda, z) + A_{k,j}(\lambda, z)) = (\lambda^N - \overline{z}^N)p_k(\lambda)\overline{p_k(z)}, \quad k \in \mathbb{Z}_+. \tag{17}$$

Here $A_{k,j}$ with negative indices are equal to zero.

Summing up relations (17) over all values of k from 0 to m, $m \in \mathbb{Z}_+$, we get

$$\sum_{k=0}^{m}(\lambda^N - \overline{z}^N)p_k(\lambda)\overline{p_k(z)} = \sum_{j=1}^{N}\sum_{k=0}^{m}(-A_{k-j,j} + A_{k,j})$$

$$= \sum_{j=1}^{N}(-\sum_{k=0}^{m}A_{k-j,j} + \sum_{k=0}^{m}A_{k,j}) = \sum_{j=1}^{N}(-\sum_{r=0}^{m-j}A_{r,j} + \sum_{k=0}^{m}A_{k,j})$$

$$= \sum_{j=1}^{N}\sum_{k=\max(m-j+1,0)}^{m}A_{k,j},$$

where we set $r := k - j$ and $\sum_s^t \ldots = 0$ if $s > t$.

Hence, we get

$$\sum_{k=0}^{m}(\lambda^N - \overline{z}^N)p_k(\lambda)\overline{p_k(z)} = \sum_{j=1}^{N}\sum_{k=\max(m-j+1,0)}^{m}A_{k,j}, \quad m \in \mathbb{Z}_+. \tag{18}$$

From (18) an analog of the Christoffel–Darbou formula [3–4] follows immediately.

Theorem 2.1. *Let $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n has degree n and a positive leading coefficient) be a sequence of polynomials which satisfies (12). Then the following relation holds true:*

$$\sum_{k=0}^{m}p_k(\lambda)\overline{p_k(z)}$$

$$= \frac{\sum_{j=1}^{N}\sum_{k=\max(m-j+1,0)}^{m}\left(\alpha_{k,j}p_{k+j}(\lambda)\overline{p_k(z)} - \overline{\alpha_{k,j}}p_k(\lambda)\overline{p_{k+j}(z)}\right)}{\lambda^N - \overline{z}^N},$$

$$m \in \mathbb{Z}_+. \tag{19}$$

Passing to the limit in (19) as $\lambda \to z$, we get

$$\sum_{k=0}^{m} |p_k(z)|^2 = \lim_{\lambda \to z} \frac{\sum_{j=1}^{N} \sum_{k=\max(m-j+1,0)}^{m} A_{k,j}(\lambda, z)}{\lambda^N - \overline{z}^N}. \tag{20}$$

Consider some cases for the position of point z in the complex plane.
1) Case $z \notin L_N$. In this case the denominator on the right of (20) has a limit $2\,\mathrm{Im}(z^N) \neq 0$. Notice that the numerator on the right of (20) is a polynomial of λ. Hence, from (20) we get

$$0 < \sum_{k=0}^{m} |p_k(z)|^2 = \frac{\sum_{j=1}^{N} \sum_{k=\max(m-j+1,0)}^{m} A_{k,j}(z, z)}{2\,\mathrm{Im}(z^N)}$$
$$= \frac{\sum_{j=1}^{N} \sum_{k=\max(m-j+1,0)}^{m} \mathrm{Im}(\alpha_{k,j} p_{k+j}(z)\overline{p_k(z)})}{\mathrm{Im}(z^N)}. \tag{21}$$

From (21) it follows that if $m \geq N-1$, then polynomials $p_m, p_{m-1}, \ldots, p_{m-N+1}$ have no common roots outside L_N. Note that in the opposite case $p_k(z)$ on the right of (21) would be always equal to zero at such a root what is impossible.
2) Case $z \in L_N \backslash \{0\}$. In this case the numerator and the denominator on the right of (20) tend to zero and we can write

$$\sum_{k=0}^{m} |p_k(z)|^2 = \lim_{\lambda \to z} \frac{\sum_{j=1}^{N} \sum_{k=\max(m-j+1,0)}^{m} A'_{k;j}(\lambda, z)}{N\lambda^{N-1}}, \tag{22}$$

where the derivatives are with respect to λ.

Hence, we get

$$0 < \sum_{k=0}^{m} |p_k(z)|^2 = \frac{\sum_{j=1}^{N} \sum_{k=\max(m-j+1,0)}^{m} A'_{k;j}(z, z)}{N z^{N-1}}$$

$$= \frac{1}{N z^{N-1}} \sum_{j=1}^{N} \sum_{k=\max(m-j+1,0)}^{m} \left(\alpha_{k,j} p'_{k+j}(z)\overline{p_k(z)} - \overline{\alpha_{k,j}} p'_k(z)\overline{p_{k+j}(z)} \right). \tag{23}$$

From (23) it follows that if $m \geq N-1$, then polynomials $p_m, p_{m-1}, \ldots, p_{m-N+1}$ and their derivatives have no common roots in $L_N \backslash \{0\}$. In the opposite case $p_k(z)$ and $p'_k(z)$ on the right of (23) would be always equal to zero at such a root what is impossible.
3) Case $z = 0$. In this case the numerator and the denominator on the right of (20) and their derivatives up to the $(N-1)$-th derivative tend to zero and we can write

$$0 < \sum_{k=0}^{m} |p_k(0)|^2 = \frac{\sum_{j=1}^{N} \sum_{k=\max(m-j+1,0)}^{m} A_{k;j}^{(N)}(0, 0)}{N!}$$

$$= \frac{1}{N!} \sum_{j=1}^{N} \sum_{k=\max(m-j+1,0)}^{m} \left(\alpha_{k,j} p_{k+j}^{(N)}(0) \overline{p_k(0)} - \overline{\alpha_{k,j}} p_k^{(N)}(0) \overline{p_{k+j}(0)} \right), \tag{24}$$

where the derivative of $A_{k,j}$ is taken with respect to the first argument.

From (24) it follows that if $m \geq N-1$, then for polynomials $p_m, p_{m-1}, \ldots,$ p_{m-N+1} and their N-th derivatives $z = 0$ is not a common root.

Theorem 2.2. *Let $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n has degree n and a positive leading coefficient) be a sequence of polynomials which satisfies (11) or (12). The following statements are true:*

1. N subsequent polynomials have no common roots outside L_N.

2. $2N$ subsequent polynomials have no common roots.

3. N subsequent polynomials and their derivatives have no common roots in $L_N \setminus \{0\}$.

4. $z = 0$ can not be a common root for N subsequent polynomials and their N-th derivatives.

P r o o f. Statements 1, 3, 4 were derived above. Suppose that $2N$ subsequent polynomials have a common root $z_0 \in \mathbb{C}$. From (12) it follows that z_0 is a common root for all polynomials p_n, $n \in \mathbb{Z}_+$, what is impossible. ■

For the case $N = 1$ we get the well-known properties:

1. Polynomials have no roots outside $\mathbb{R}$.

2. Two subsequent polynomials have no common roots.

3–4. A polynomial and its derivative have no common roots.

For the case $N = 2$ the restrictions on zeros will be weaker and we will illustrate them below by the examples:

1. Two subsequent polynomials have no common roots outside $\mathbb{R} \cup i\mathbb{R}$.

2. Four subsequent polynomials have no common roots.

3. Two subsequent polynomials and their derivatives have no common roots in $(\mathbb{R} \cup i\mathbb{R}) \setminus \{0\}$.

4. $z = 0$ can not be a common root for two subsequent polynomials and their second derivatives.

Our further considerations were inspired by the considerations of G. Lopez in [2, p. 128–129].

Theorem 2.3. *Let $\{p_n(\lambda)\}_{n=0}^{\infty}$ (p_n has degree n and a positive leading coefficient) be a sequence of polynomials which satisfies (11) or (12). Suppose that $p_k(\lambda)$, $k \geq N$, is a polynomial which has different roots $z_1, z_2, \ldots, z_N$ such that $z_j^N = a$, $a \in \mathbb{C}$, $j = 1, 2, \ldots, N$. Then $a \in \mathbb{R}$.*

P r o o f. For the polynomial p_k from the statement of the theorem we can write

$$p_k(z) = (z^N - a)q(z),$$

where $q(z)$ is a nonzero polynomial of degree $k - N$. Therefore $z^N q(z) = p_k(z) + aq(z)$. By virtue of (11) and (7) we get

$$\sigma(z^N q(z), q(z)) = \sigma(p_k(z) + aq(z), q(z)) = a\sigma(q(z), q(z)),$$

$$\sigma(q(z), z^N q(z)) = \sigma(q(z), p_k(z) + aq(z)) = \overline{a}\sigma(q(z), q(z)),$$

and

$$a = \overline{a}. \qquad \blacksquare$$

For the case $N = 1$ this theorem states that the roots of orthonormal polynomials are real. For the case $N = 2$ we get: if complex numbers c and $-c$ are zeros of a polynomial, then $c \in \mathbb{R} \cup i\mathbb{R}$. In other words, there are no zeros symmetric with respect to the origin outside $\mathbb{R} \cup i\mathbb{R}$.

3. Moment Problems

Consider the following problem on finding a $\mathbb{C}_{N\times N}$-valued function $M(\lambda)$ on $L_N \backslash \{0\}$ which is nondecreasing on each ray $L_{N,k}\backslash\{0\}$, $k = 0, 1, \ldots, 2N - 1$, in the direction from 0 to ∞, and a matrix $A \in \mathbb{C}^{\geq}_{N\times N}$ such that

$$\int_{L_N} (\lambda^k, (\lambda\varepsilon)^k, (\lambda\varepsilon^2)^k, \ldots, (\lambda\varepsilon^{N-1})^k) dM(\lambda) \overline{\begin{pmatrix} \lambda^l \\ (\lambda\varepsilon)^l \\ \vdots \\ (\lambda\varepsilon^{N-1})^l \end{pmatrix}}$$

$$+ (\lambda^k, (\lambda^k)', (\lambda^k)'', \ldots, (\lambda^k)^{(N-1)}) A \left. \overline{\begin{pmatrix} \lambda^l \\ (\lambda^l)' \\ \vdots \\ (\lambda^l)^{(N-1)} \end{pmatrix}} \right|_{\lambda=0} = s_{k,l},$$

$$k \in \mathbb{Z}_+,\ l = 0, 1, 2, \ldots, N - 1, \tag{25}$$

where $\{s_{k,l}\}_{k\in\mathbb{Z}_+, l=0,1,2,\ldots,N-1}$ is a given set of complex numbers.

We will call this problem the N-dimensional symmetric moment problem in a Sobolev form.

Suppose that there exists a solution $M(\lambda), A$ of moment problem (25). Notice that moments (5) are finite. If $n = mN + l$, $m \geq 0$, $0 \leq l \leq N - 1$, then the integral in (5) is equal to $s_{n+mN,l}$. Define a functional σ as in (6). The functional

σ is bilinear and it satisfies (7)–(9). From (7) it follows that σ is uniquely defined by moments $\{s_{k,l}\}_{k\in\mathbb{Z}_+,l=0,1,2,\dots,N-1}$. We will suppose that the functional σ satisfies condition (10). The solutions of moment problem (25) for which the functional σ satisfies condition (10) we will call nondegenerate. Define the following polynomials (see [3, Ch. II]):

$$\tilde{p}_n(\lambda) = \begin{vmatrix} \sigma(1,1) & \sigma(\lambda,1) & \sigma(\lambda^2,1) & \dots & \sigma(\lambda^n,1) \\ \sigma(1,\lambda) & \sigma(\lambda,\lambda) & \sigma(\lambda^2,\lambda) & \dots & \sigma(\lambda^n,\lambda) \\ \sigma(1,\lambda^2) & \sigma(\lambda,\lambda^2) & \sigma(\lambda^2,\lambda^2) & \dots & \sigma(\lambda^n,\lambda^2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma(1,\lambda^{n-1}) & \sigma(\lambda,\lambda^{n-1}) & \sigma(\lambda^2,\lambda^{n-1}) & \dots & \sigma(\lambda^n,\lambda^{n-1}) \\ 1 & \lambda & \lambda^2 & \dots & \lambda^n \end{vmatrix}$$

$$= \begin{vmatrix} s_{0,0} & s_{1,0} & s_{2,0} & \dots & s_{n,0} \\ s_{0,1} & s_{1,1} & s_{2,1} & \dots & s_{n,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{0,N-1} & s_{1,N-1} & s_{2,N-1} & \dots & s_{n,N-1} \\ s_{N,0} & s_{N+1,0} & s_{N+2,0} & \dots & s_{N+n,0} \\ s_{N,1} & s_{N+1,1} & s_{N+2,1} & \dots & s_{N+n,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda & \lambda^2 & \dots & \lambda^n \end{vmatrix}, \quad n\in\mathbb{N}, \quad \tilde{p}_0(\lambda) = 1. \tag{26}$$

Notice that

$$\sigma(\tilde{p}_n(\lambda),\lambda^k) = \begin{vmatrix} \sigma(1,1) & \sigma(\lambda,1) & \sigma(\lambda^2,1) & \dots & \sigma(\lambda^n,1) \\ \sigma(1,\lambda) & \sigma(\lambda,\lambda) & \sigma(\lambda^2,\lambda) & \dots & \sigma(\lambda^n,\lambda) \\ \sigma(1,\lambda^2) & \sigma(\lambda,\lambda^2) & \sigma(\lambda^2,\lambda^2) & \dots & \sigma(\lambda^n,\lambda^2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma(1,\lambda^{n-1}) & \sigma(\lambda,\lambda^{n-1}) & \sigma(\lambda^2,\lambda^{n-1}) & \dots & \sigma(\lambda^n,\lambda^{n-1}) \\ \sigma(1,\lambda^k) & \sigma(\lambda,\lambda^k) & \sigma(\lambda^2,\lambda^k) & \dots & \sigma(\lambda^n,\lambda^k) \end{vmatrix}$$

$$= 0, \quad k = 0,1,\dots,n-1;\ n\in\mathbb{N}. \tag{27}$$

Consequently, we get

$$\sigma(\tilde{p}_n(\lambda),\tilde{p}_k(\lambda)) = 0,\ n,k\in\mathbb{Z}_+:\ n>k.$$

By virtue of (8) we get

$$\sigma(\tilde{p}_n(\lambda),\tilde{p}_k(\lambda)) = 0,\ n,k\in\mathbb{Z}_+:\ n\neq k. \tag{28}$$

Set

$$\Delta_n := \begin{vmatrix} \sigma(1,1) & \sigma(\lambda,1) & \sigma(\lambda^2,1) & \dots & \sigma(\lambda^n,1) \\ \sigma(1,\lambda) & \sigma(\lambda,\lambda) & \sigma(\lambda^2,\lambda) & \dots & \sigma(\lambda^n,\lambda) \\ \sigma(1,\lambda^2) & \sigma(\lambda,\lambda^2) & \sigma(\lambda^2,\lambda^2) & \dots & \sigma(\lambda^n,\lambda^2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma(1,\lambda^{n-1}) & \sigma(\lambda,\lambda^{n-1}) & \sigma(\lambda^2,\lambda^{n-1}) & \dots & \sigma(\lambda^n,\lambda^{n-1}) \\ \sigma(1,\lambda^n) & \sigma(\lambda,\lambda^n) & \sigma(\lambda^2,\lambda^n) & \dots & \sigma(\lambda^n,\lambda^n) \end{vmatrix}$$

$$= \begin{vmatrix} s_{0,0} & s_{1,0} & s_{2,0} & \dots & s_{n,0} \\ s_{0,1} & s_{1,1} & s_{2,1} & \dots & s_{n,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{0,N-1} & s_{1,N-1} & s_{2,N-1} & \dots & s_{n,N-1} \\ s_{N,0} & s_{N+1,0} & s_{N+2,0} & \dots & s_{N+n,0} \\ s_{N,1} & s_{N+1,1} & s_{N+2,1} & \dots & s_{N+n,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{mN,l} & s_{mN+1,l} & s_{mN+2,l} & \dots & s_{mN+n,l} \end{vmatrix}, \quad n \in \mathbb{Z}_+,$$

$$n = mN + l,\ m \in \mathbb{Z}_+,\ 0 \le l \le N-1; \qquad \Delta_{-1} := 1. \tag{29}$$

The $(n+1) \times (n+1)$ matrix of moments in (29) we denote by $D_n,\ n \in \mathbb{Z}_+$. From (8) it follows that matrices $D_n,\ n \in \mathbb{Z}_+$, are Hermitian.

Consider an arbitrary nonzero polynomial of degree $n \in \mathbb{Z}_+$ $u(\lambda) = \sum_{j=0}^n c_j \lambda^j$, $c_j \in \mathbb{C}$, $j = 0, 1, \dots, n :\ c_n \neq 0$. By virtue of (10) we get

$$0 < \sigma(u,u) = \sum_{j,k=1}^{n} c_j \overline{c_k} \sigma(\lambda^j, \lambda^k). \tag{30}$$

The matrix and the determinant of a positive quadratic form in (30) are D_n and Δ_n, respectively. Hence, matrices $D_n,\ n \in \mathbb{Z}_+$ are positive definite and

$$\Delta_n > 0, \quad n \in \mathbb{Z}_+. \tag{31}$$

Notice that the coefficient by λ^n of $\tilde{p}_n(\lambda)$ is $\Delta_{n-1},\ n \in \mathbb{Z}_+$. Therefore $\tilde{p}_n$ is a polynomial of degree n. By virtue of (27) we get

$$\sigma(\tilde{p}_n(\lambda), \tilde{p}_n(\lambda)) = \sigma(\tilde{p}_n(\lambda), \Delta_{n-1}\lambda^n + q_{n-1}(\lambda)) = \sigma(\tilde{p}_n(\lambda), \Delta_{n-1}\lambda^n)$$

$$= \Delta_{n-1}\sigma(\tilde{p}_n(\lambda), \lambda^n) = \Delta_{n-1}\Delta_n,\ n \in \mathbb{Z}_+,$$

where $q_{n-1}(\lambda)$ is a polynomial of a degree less than n.

Hence,

$$\sigma(\tilde{p}_n, \tilde{p}_n) = \Delta_{n-1}\Delta_n,\ n \in \mathbb{Z}_+. \tag{32}$$

Set

$$\hat{p}_n(\lambda) := \frac{1}{\sqrt{\Delta_{n-1}\Delta_n}}\tilde{p}_n(\lambda), \quad n \in \mathbb{Z}_+. \tag{33}$$

By virtue of (28) and (32) we get

$$\sigma(\hat{p}_n, \hat{p}_m) = \delta_{n,m}, \quad n, m \in \mathbb{Z}_+. \tag{34}$$

Let $\{p_n(\lambda)\}_{n=0}^{\infty}$ be a sequence of orthonormal polynomials obtained by the Gramm–Schmidt orthogonalization method as before (11).

Theorem 3.1. *Let $\{r_n(\lambda)\}_{n=0}^{\infty}$ and $\{\hat{r}_n(\lambda)\}_{n=0}^{\infty}$ be two sequences of polynomials, $\deg r_n = \deg \hat{r}_n = n$, $n \in \mathbb{Z}_+$, and polynomials have positive leading coefficients. If*

$$\sigma(r_n, r_m) = \sigma(\hat{r}_n, \hat{r}_m) = \delta_{n,m}, \quad n, m \in \mathbb{Z}_+, \tag{35}$$

then $r_n = \hat{r}_n$, $n \in \mathbb{Z}_+$.

P r o o f. We will apply the induction method similarly as in the proof of Th. 1.1 from [4, p. 15–16]. By virtue of (35) we get $r_0^2 = \hat{r}_0^2 = \frac{1}{\sigma(1,1)}$ and therefore $r_0 = \hat{r}_0$.

Suppose that

$$r_j = \hat{r}_j, \ j = 0, 2, \ldots, k-1, \tag{36}$$

for a number $k \in \mathbb{N}$.

For polynomials $r_k, \hat{r}_k$ we can write

$$r_k(\lambda) = \mu\lambda^k + \sum_{j=0}^{k-1} a_j r_j(\lambda), \tag{37}$$

$$\hat{r}_k(\lambda) = \hat{\mu}\lambda^k + \sum_{j=0}^{k-1} \hat{a}_j r_j(\lambda), \tag{38}$$

where $\mu > 0, \hat{\mu} > 0$, $a_j, \hat{a}_j \in \mathbb{C}, j = 0, 1, \ldots, k-1$.

By virtue of (35)–(38) we obtain

$$0 = \sigma(r_k, r_j) = \mu\sigma(\lambda^k, r_j) + a_j, \ j \le k,$$

$$0 = \sigma(\hat{r}_k, \hat{r}_j) = \hat{\mu}\sigma(\lambda^k, r_j) + \hat{a}_j, \ j \le k.$$

Set $b_j := \sigma(\lambda^k, r_j)$, $j = 0, 1, \ldots, k-1$. From the latter relations we get

$$a_j = -\mu b_j, \ \hat{a}_j = -\hat{\mu} b_j, \ j = 1, 2, \ldots, k-1.$$

From (37),(38) we get

$$r_k(\lambda) = \mu(\lambda^k - \sum_{j=0}^{k-1} b_j r_j(\lambda)),\ \hat{r}_k(\lambda) = \hat{\mu}(\lambda^k - \sum_{j=0}^{k-1} b_j r_j(\lambda)). \tag{39}$$

Consequently, we obtain

$$\hat{r}_k = c r_k,\ \text{where } c := \frac{\hat{\mu}}{\mu} > 0. \tag{40}$$

By virtue of (35) we derive

$$1 = \sigma(r_k, r_k) = \sigma(\hat{r}_k, \hat{r}_k) = |c|^2 \sigma(r_k, r_k) = |c|^2,$$

and hence $c = 1$. ■

By virtue of Th. 3.1 we obtain that

$$\hat{p}_n = p_n, \quad n \in \mathbb{Z}_+.$$

Let us turn to the solving of the moment problem (25).

Let a symmetric moment problem (25) be given. Set

$$\sigma(\lambda^k, \lambda^n) := s_{k+mN,l},\ k,n \in \mathbb{Z}_+,\ n = mN + l,\ m \in \mathbb{Z}_+, 0 \le l \le N-1. \tag{41}$$

We extend σ linear with respect to the first argument and antilinear with respect to the second argument to obtain a bilinear functional $\sigma(u,v),\ u,v \in \mathbb{P}$. From (41) it follows that

$$\sigma(\lambda^k, \lambda^N \lambda^n) = \sigma(\lambda^N \lambda^k, \lambda^n),\ k,n \in \mathbb{Z}_+. \tag{42}$$

For arbitrary polynomials $u = \sum_{j=0}^{r_1} a_j \lambda^j$, $v = \sum_{s=0}^{r_2} b_s \lambda^s$, $r_1, r_2 \in \mathbb{Z}_+$, $a_j, b_s \in \mathbb{C}$, $j = 0, 1, \dots, r_1$, $s = 0, 1, \dots, r_2$, we get

$$\sigma(u, \lambda^N v) = \sum_{j=0}^{r_1} \sum_{s=0}^{r_2} a_j \overline{b_s} \sigma(\lambda^j, \lambda^N \lambda^s) = \sum_{j=0}^{r_1} \sum_{s=0}^{r_2} a_j \overline{b_s} \sigma(\lambda^N \lambda^j, \lambda^s) = \sigma(\lambda^N u, v).$$

Hence, for the functional σ condition (7) is true.

Define Δ_n, $n \in \mathbb{Z}_+ \cup \{-1\}$, as in (29). Suppose that matrices D_n, $n \in \mathbb{Z}_+$, defined as after (29) are Hermitian and positive.

Then the quadratic form in (30) is positive and hence condition (10) is true. Also condition (31) holds true.

Since matrices D_n, $n \in \mathbb{Z}_+$, are Hermitian we get

$$\overline{\sigma(\lambda^k, \lambda^n)} = \sigma(\lambda^n, \lambda^k),\ k,n \in \mathbb{Z}_+.$$

For arbitrary polynomials u, v as above we can write

$$\overline{\sigma(u,v)} = \sum_{j=0}^{r_1}\sum_{s=0}^{r_2} \overline{a_j} b_s \overline{\sigma(\lambda^j,\lambda^s)} = \sum_{j=0}^{r_1}\sum_{s=0}^{r_2} \overline{a_j} b_s \sigma(\lambda^s,\lambda^j) = \sigma(v,u).$$

Hence, for the functional σ condition (8) holds true.

Define polynomials $\{\tilde{p}_n(\lambda)\}_{n=0}^{\infty}$ by formula (26). Repeating considerations after (26) we define polynomials $\{\hat{p}_n(\lambda)\}_{n=0}^{\infty}$ which are orthonormal with respect to σ and coincide with polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ constructed by the Gramm–Schmidt orthogonalization method. Repeating the arguments from the proof of Th. 3 in [1, p. 131] for the functional σ we obtain that polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ satisfy recurrence relation (12). By virtue of Th. 2 in [1, p. 128] we get that there exists $M(\lambda)$ and A such that orthonormality relation (11) holds true. Define a functional $\sigma_1(u,v)$ by formula (6). This functional has a property

$$\sigma_1(p_n,p_m) = \delta_{n,m}, \quad n,m \in \mathbb{Z}_+. \tag{43}$$

For arbitrary polynomials u, $v \in \mathbb{P}$ we can write $u = \sum_{j=0}^{r_1} a_j p_j(\lambda)$, $v = \sum_{s=0}^{r_2} b_s p_s(\lambda)$, r_1, $r_2 \in \mathbb{Z}_+$, $a_j, b_s \in \mathbb{C}$, $j = 0,1,\ldots,r_1$, $s = 0,1,\ldots,r_2$, therefore

$$\sigma(u,v) = \sum_{j=0}^{r_1}\sum_{s=0}^{r_2} a_j \overline{b_s} \sigma(p_j(\lambda),p_s(\lambda)) = \sum_{j=0}^{r_1}\sum_{s=0}^{r_2} a_j \overline{b_s} \delta_{j,s}$$
$$= \sum_{j=0}^{r_1}\sum_{s=0}^{r_2} a_j \overline{b_s} \sigma_1(p_j(\lambda),p_s(\lambda)) = \sigma_1(u,v).$$

Hence, the functionals σ and σ_1 coincide.

If we write relations (41) using the integral representation of $\sigma_1 = \sigma$ (6), we obtain that relations (25) hold true.

From our considerations we get the following theorem.

Theorem 3.2. *Let an N-dimensional symmetric moment problem in a Sobolev form (25) with some $\{s_{k,l}\}_{k\in\mathbb{Z}_+,l=0,1,2,\ldots,N-1}$ be given. The problem has a non-degenerate solution iff the matrices D_n, $n \in \mathbb{Z}_+$, defined as above are Hermitian and positive definite.*

Similarly to the case of the Hamburger moment problem and the Stieltjes and the Hausdorf moment problems we can set some restrictions on the measure M, A in (25) to get some "partial" moment problems. In this way we can get the mentioned above problems and also the complex moment problem on radial rays [5] and the discrete Sobolev moment problem [6]. Another version of the symmetric moment problem was presented in [7].

4. Symmetries

Let a function M, a matrix A, a functional σ and orthonormal polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ be such as in Introduction. In the scalar case ($N = 1$) it is known that if the measure of orthogonality is symmetric with respect to the origin, then an orthogonal polynomial p_n, $n \in \mathbb{Z}_+$, has only terms with degrees having the same parity as n [4, Th. 1.3]. A similar and more interesting situation with symmetries is in the case of an arbitrary $N \in \mathbb{N}$.

Suppose that the function $M(\lambda)$ possesses the following property:

$$M(\lambda\hat{\varepsilon}^l) = M(\lambda), \quad 1 \leq l \leq 2N-1,\ \lambda \in L_N \backslash \{0\}, \tag{44}$$

where $\hat{\varepsilon} = \cos\frac{\pi}{N} + i\sin\frac{\pi}{N}$.

Also we will suppose that the matrix A commutes with a diagonal $(N \times N)$ matrix $D_l := \mathrm{diag}(1, \hat{\varepsilon}^{-l}, \hat{\varepsilon}^{-2l}, \dots, \hat{\varepsilon}^{-(N-1)l})$:

$$AD_l = D_l A. \tag{45}$$

In particular, (45) is true if A is diagonal.

We can write

$$\sigma(u,v) = \sum_{k=0}^{2N-1} \int_{L_{N,k}} (u(\lambda), u(\lambda\varepsilon), \dots, u(\lambda\varepsilon^{N-1}))dM(\lambda) \overline{\begin{pmatrix} v(\lambda) \\ v(\lambda\varepsilon) \\ \vdots \\ v(\lambda\varepsilon^{N-1}) \end{pmatrix}}$$

$$+(u(0), u'(0), \dots, u^{(N-1)}(0))A \overline{\begin{pmatrix} v(0) \\ v'(0) \\ \vdots \\ v^{(N-1)}(0) \end{pmatrix}}$$

$$= \sum_{k=0}^{2N-1} \int_{L_{N,k-l}} (u(y\hat{\varepsilon}^l), u(y\hat{\varepsilon}^l\varepsilon), \dots, u(y\hat{\varepsilon}^l\varepsilon^{N-1}))dM(y\hat{\varepsilon}^l) \overline{\begin{pmatrix} v(y\hat{\varepsilon}^l) \\ v(y\hat{\varepsilon}^l\varepsilon) \\ \vdots \\ v(y\hat{\varepsilon}^l\varepsilon^{N-1}) \end{pmatrix}}$$

$$+(u(0), u'(0), \dots, u^{(N-1)}(0))A \overline{\begin{pmatrix} v(0) \\ v'(0) \\ \vdots \\ v^{(N-1)}(0) \end{pmatrix}}, \qquad u,v \in \mathbb{P}, \tag{46}$$

where we applied the change of variable $\lambda = y\hat{\varepsilon}^l;\ y = \lambda\hat{\varepsilon}^{-l}$.

Set

$$\hat{u}(y) := u(y\hat{\varepsilon}^l),\ \hat{v}(y) := v(y\hat{\varepsilon}^l),\ y \in L_N\backslash\{0\}. \tag{47}$$

It is not difficult to verify that

$$\hat{u}^{(k)}(y) = u^{(k)}(y\hat{\varepsilon}^l)\hat{\varepsilon}^{lk},\ \hat{v}^{(k)}(y) = v^{(k)}(y\hat{\varepsilon}^l)\hat{\varepsilon}^{lk},\ y \in L_N\backslash\{0\}. \tag{48}$$

By virtue of (47),(48) and (45) we can write

$$(u(0), u'(0), \ldots, u^{(N-1)}(0))A\overline{\begin{pmatrix} v(0) \\ v'(0) \\ \vdots \\ v^{(N-1)}(0) \end{pmatrix}} = (\hat{u}(0), \hat{u}'(0), \ldots, \hat{u}^{(N-1)}(0))$$

$$*D_l A D_l^* \overline{\begin{pmatrix} \hat{v}(0) \\ \hat{v}'(0) \\ \vdots \\ \hat{v}^{(N-1)}(0) \end{pmatrix}} = (\hat{u}(0), \hat{u}'(0), \ldots, \hat{u}^{(N-1)}(0))A\overline{\begin{pmatrix} \hat{v}(0) \\ \hat{v}'(0) \\ \vdots \\ \hat{v}^{(N-1)}(0) \end{pmatrix}}. \tag{49}$$

If we substitute (49) in (46) and use (44),(47), we get

$$\sigma(u,v) = \sum_{k=0}^{2N-1} \int_{L_{N,k-l}} (\hat{u}(y), \hat{u}(y\varepsilon), \ldots, \hat{u}(y\varepsilon^{N-1}))dM(y)\overline{\begin{pmatrix} \hat{v}(y) \\ \hat{v}(y\varepsilon) \\ \vdots \\ \hat{v}(y\varepsilon^{N-1}) \end{pmatrix}}$$

$$+(\hat{u}(0), \hat{u}'(0), \ldots, \hat{u}^{(N-1)}(0))A\overline{\begin{pmatrix} \hat{v}(0) \\ \hat{v}'(0) \\ \vdots \\ \hat{v}^{(N-1)}(0) \end{pmatrix}} = \sigma(\hat{u}, \hat{v}).$$

Hence,

$$\sigma(u(\lambda), v(\lambda)) = \sigma(u(\lambda\hat{\varepsilon}^l), v(\lambda\hat{\varepsilon}^l)), \quad u, v \in \mathbb{P}. \tag{50}$$

For the orthonormal polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ we can write

$$\sigma(p_n(\lambda), p_m(\lambda)) = \sigma(p_n(\lambda\hat{\varepsilon}^l), p_m(\lambda\hat{\varepsilon}^l)) = \delta_{n,m}, \quad n, m \in \mathbb{Z}_+. \tag{51}$$

Let the polynomial $p_n(\lambda)$ have the following form:

$$p_n(\lambda) = \sum_{j=0}^{n} \mu_{n,j}\lambda^j, \tag{52}$$

where $\mu_{n,j} \in \mathbb{C}$, $j = 0, 1, \ldots, n-1$; $\mu_{n,n} > 0$, $n \in \mathbb{Z}_+$.

Consequently,

$$p_n(\lambda \hat{\varepsilon}^l) = \sum_{j=0}^{n} \mu_{n,j} \hat{\varepsilon}^{lj} \lambda^j, \quad n \in \mathbb{Z}_+. \tag{53}$$

Consider the polynomials

$$r_n(\lambda) := \hat{\varepsilon}^{-ln} p_n(\lambda \hat{\varepsilon}^l) = \mu_{n,n} \lambda^n + \sum_{j=0}^{n-1} \mu_{n,j} \hat{\varepsilon}^{l(j-n)} \lambda^j, \quad n \in \mathbb{Z}_+. \tag{54}$$

The polynomial r_n has degree n and a positive leading coefficient, $n \in \mathbb{Z}_+$. From (51), (54) it follows that

$$\sigma(r_n(\lambda), r_m(\lambda)) = \sigma(p_n(\lambda \hat{\varepsilon}^l), p_m(\lambda \hat{\varepsilon}^l)) = \delta_{n,m}, \quad n, m \in \mathbb{Z}_+. \tag{55}$$

By virtue of Th. 3.1 we obtain that

$$r_n(\lambda) = p_n(\lambda), \quad n \in \mathbb{Z}_+.$$

Consequently, we get

$$p_n(\lambda \hat{\varepsilon}^l) = \hat{\varepsilon}^{ln} p_n(\lambda), \quad n \in \mathbb{Z}_+. \tag{56}$$

Let us see what condition (56) means for the coefficients of the polynomial p_n. Subtract relation (52) multiplied by $\hat{\varepsilon}^{ln}$ from relation (53)

$$0 = \sum_{j=0}^{n-1} \mu_{n,j} (\hat{\varepsilon}^{lj} - \hat{\varepsilon}^{ln}) \lambda^j. \tag{57}$$

Consequently, if $\hat{\varepsilon}^{lj} - \hat{\varepsilon}^{ln} \neq 0$, then $\mu_{n,j} = 0$, $j = 0, 1, ..., n-1$. Therefore

$$p_n(\lambda) = \mu_{n,n} \lambda^n + \sum_{j \in [0, n-1] \cap \mathbb{Z}_+ : \hat{\varepsilon}^{l(n-j)} = 1} \mu_{n,j} \lambda^j, \ n \in \mathbb{Z}_+. \tag{58}$$

From these considerations we obtain the following theorem.

Theorem 4.1. *Let $M(\lambda), A, \sigma(\cdot, \cdot)$ and orthonormal polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ be the same as in Introduction. Suppose that conditions (44),(45) are true. Then orthonormal polynomials satisfy condition (56) and have the following form:*

$$p_n(\lambda) = \mu_{n,n} \lambda^n + \sum_{k \in [1, n] \cap \mathbb{N}: \ \hat{\varepsilon}^{lk} = 1} \mu_{n,n-k} \lambda^{n-k}, \ n \in \mathbb{Z}_+. \tag{59}$$

In particular, if $l = N$ *we get*

$$p_n(\lambda) = \mu_{n,n}\lambda^n + \sum_{k\in[1,n]\cap\mathbb{N}:\ k \text{ is even}} \mu_{n,n-k}\lambda^{n-k},\ n \in \mathbb{Z}_+, \tag{60}$$

and if $l = 1$ *we obtain*

$$p_n(\lambda) = \mu_{n,n}\lambda^n + \sum_{j=1}^{\left[\frac{n}{2N}\right]} \mu_{n,n-2Nj}\lambda^{n-2Nj},\ n \in \mathbb{Z}_+. \tag{61}$$

Let $M(\lambda), A$ be arbitrary, i.e., conditions (44),(45) are not assumed to be true. Set

$$d(u,v) := (u(0), u'(0), \dots, u^{(N-1)}(0))A\overline{\begin{pmatrix} v(0) \\ v'(0) \\ \vdots \\ v^{(N-1)}(0) \end{pmatrix}}, \qquad u,v \in \mathbb{P}. \tag{62}$$

We can write

$$\sigma(u,v) = \sum_{k=0}^{2N-1} \int_{L_{N,k}} (u(\lambda), u(\lambda\varepsilon), \dots, u(\lambda\varepsilon^{N-1}))dM(\lambda)\overline{\begin{pmatrix} v(\lambda) \\ v(\lambda\varepsilon) \\ \vdots \\ v(\lambda\varepsilon^{N-1}) \end{pmatrix}} + d(u,v)$$

$$= \sum_{k=0}^{2N-1} \int_{L_{N,k-2j}} (u(y\varepsilon^j), u(y\varepsilon^{j+1}), \dots, u(y\varepsilon^{N-1}), u(y), u(y\varepsilon), \dots, u(y\varepsilon^{j-1}))$$

$$*dM(y\varepsilon^j)\overline{\begin{pmatrix} v(y\varepsilon^j) \\ v(y\varepsilon^{j+1}) \\ \vdots \\ v(y\varepsilon^{j-1}) \end{pmatrix}} + d(u,v)$$

$$= \sum_{k=0}^{2N-1} \int_{L_{N,k-2j}} (u(y), u(y\varepsilon), \dots, u(y\varepsilon^{N-1}))B_j dM(y\varepsilon^j)B_j^*\overline{\begin{pmatrix} v(y) \\ v(y\varepsilon) \\ \vdots \\ v(y\varepsilon^{N-1}) \end{pmatrix}}$$

$$+d(u,v), \quad u,v \in \mathbb{P}, \tag{63}$$

where we applied the change of variable $\lambda = y\varepsilon^j = y\hat{\varepsilon}^{2j}$; $y = \lambda\varepsilon^{-j} = \lambda\hat{\varepsilon}^{-2j}$, $j \in [1, N-1] \cap \mathbb{N}$, and B_j is a block matrix

$$B_j := \begin{pmatrix} O_{(N-j)\times j} & I_{(N-j)\times(N-j)} \\ I_{j\times j} & O_{j\times(N-j)} \end{pmatrix}. \tag{64}$$

We change $r = k - 2j$; $k = r - 2j$ in the last sum in (63) and use (4) to get

$$\sigma(u,v) = \sum_{r=-2j}^{2N-1-2j} \int_{L_{N,r}} (u(y), u(y\varepsilon), \ldots, u(y\varepsilon^{N-1})) B_j dM(y\varepsilon^j) B_j^* \\ * \overline{\begin{pmatrix} v(y) \\ v(y\varepsilon) \\ \vdots \\ v(y\varepsilon^{N-1}) \end{pmatrix}} + d(u,v)$$

$$= \sum_{r=0}^{2N-1} \int_{L_{N,r}} (u(y), u(y\varepsilon), \ldots, u(y\varepsilon^{N-1})) B_j dM(y\varepsilon^j) B_j^* \overline{\begin{pmatrix} v(y) \\ v(y\varepsilon) \\ \vdots \\ v(y\varepsilon^{N-1}) \end{pmatrix}} \\ + d(u,v), \quad u, v \in \mathbb{P}. \tag{65}$$

Set

$$M_j(\lambda) := B_j M(\lambda\varepsilon^j) B_j^*, \quad j \in [1, N-1] \cap \mathbb{N}, \ \lambda \in L_N \backslash \{0\}. \tag{66}$$

From (65) it follows that the functions M_j and A define by (6) the same functional σ as M and A define. In particular, the functions M_j and A have the same moments of form (25) as M and A.

Set

$$\widetilde{M}(\lambda) := \frac{1}{N} \left(M(\lambda) + \sum_{j=1}^{N-1} M_j(\lambda) \right), \quad \lambda \in L_N \backslash \{0\}. \tag{67}$$

For the functions $\widetilde{M}$ and A the same can be said as for the functions M_j and A above. It turns out that the function $\widetilde{M}(\lambda)$ has some additional properties which will be obtained below.

Let the matrix $M(\lambda)$ have the following form: $M(\lambda) = (m_{n,k}(\lambda))_{n,k=0}^{N-1}$, $\lambda \in L_N \backslash \{0\}$. By direct computation we get that the functions M_j, $j = 1, 2, \ldots, N-1$ have the following structure:

$$M_j(\lambda) := \begin{pmatrix} A_j(\lambda) & C_j(\lambda) \\ C_j^*(\lambda) & F_j(\lambda) \end{pmatrix}, \quad \lambda \in L_N \backslash \{0\}, \tag{68}$$

where

$$A_j(\lambda) = \begin{pmatrix} m_{N-j,N-j}(\lambda\varepsilon^j) & m_{N-j,N-j+1}(\lambda\varepsilon^j) & \dots & m_{N-j,N-1}(\lambda\varepsilon^j) \\ m_{N-j+1,N-j}(\lambda\varepsilon^j) & m_{N-j+1,N-j+1}(\lambda\varepsilon^j) & \dots & m_{N-j+1,N-1}(\lambda\varepsilon^j) \\ \vdots & \vdots & \ddots & \vdots \\ m_{N-1,N-j}(\lambda\varepsilon^j) & m_{N-1,N-j+1}(\lambda\varepsilon^j) & \dots & m_{N-1,N-1}(\lambda\varepsilon^j) \end{pmatrix}, \tag{69}$$

$$C_j(\lambda) = \begin{pmatrix} m_{N-j,0}(\lambda\varepsilon^j) & m_{N-j,1}(\lambda\varepsilon^j) & \dots & m_{N-j,N-j-1}(\lambda\varepsilon^j) \\ m_{N-j+1,0}(\lambda\varepsilon^j) & m_{N-j+1,1}(\lambda\varepsilon^j) & \dots & m_{N-j+1,N-j-1}(\lambda\varepsilon^j) \\ \vdots & \vdots & \ddots & \vdots \\ m_{N-1,0}(\lambda\varepsilon^j) & m_{N-1,1}(\lambda\varepsilon^j) & \dots & m_{N-1,N-j-1}(\lambda\varepsilon^j) \end{pmatrix}, \tag{70}$$

$$F_j(\lambda) = \begin{pmatrix} m_{0,0}(\lambda\varepsilon^j) & m_{0,1}(\lambda\varepsilon^j) & \dots & m_{0,N-j-1}(\lambda\varepsilon^j) \\ m_{1,0}(\lambda\varepsilon^j) & m_{1,1}(\lambda\varepsilon^j) & \dots & m_{1,N-j-1}(\lambda\varepsilon^j) \\ \vdots & \vdots & \ddots & \vdots \\ m_{N-j-1,0}(\lambda\varepsilon^j) & m_{N-j-1,1}(\lambda\varepsilon^j) & \dots & m_{N-j-1,N-j-1}(\lambda\varepsilon^j) \end{pmatrix}. \tag{71}$$

Let $M_j(\lambda) = (m^{[j]}_{n,k}(\lambda))_{n,k=0}^{N-1}$ and $\widetilde{M}(\lambda) = (\widetilde{m}_{n,k}(\lambda))_{n,k=0}^{N-1}$, $\lambda \in L_N\backslash\{0\}$. Let us calculate $\widetilde{m}_{k,k}(\lambda)$. It is easy to see that

$$m^{[j]}_{k,k}(\lambda) = \begin{cases} m_{N-j+k,N-j+k}(\lambda\varepsilon^j), & 0 \le k \le j-1 \\ m_{k-j,k-j}(\lambda\varepsilon^j), & j \le k \le N-1 \end{cases}. \tag{72}$$

Therefore

$$\widetilde{m}_{k,k}(\lambda) = \frac{1}{N}\left(m_{k,k}(\lambda) + \sum_{j=1}^{N-1} m^{[j]}_{k,k}(\lambda)\right)$$

$$= \frac{1}{N}\left(m_{k,k}(\lambda) + \sum_{j=1}^{k} m_{k-j,k-j}(\lambda\varepsilon^j) + \sum_{j=k+1}^{N-1} m_{N-j+k,N-j+k}(\lambda\varepsilon^j)\right)$$

$$= \frac{1}{N}\left(\sum_{j=0}^{k} m_{k-j,k-j}(\lambda\varepsilon^j) + \sum_{j=k+1}^{N-1} m_{N-j+k,N-j+k}(\lambda\varepsilon^j)\right)$$

$$= \frac{1}{N}\left(\sum_{s=0}^{k} m_{s,s}(\lambda\varepsilon^{k-s}) + \sum_{t=k+1}^{N-1} m_{t,t}(\lambda\varepsilon^{N+k-t})\right)$$

$$= \frac{1}{N}\sum_{r=0}^{N-1} m_{r,r}(\lambda\varepsilon^{k-r}), \tag{73}$$

where $s := k - j,\ t := N - j + k.$

Set

$$\mathbf{d}_0(\lambda) := \frac{1}{N}\sum_{r=0}^{N-1} m_{r,r}(\lambda\varepsilon^{-r}), \quad \lambda \in L_N\backslash\{0\}. \tag{74}$$

By virtue of (73) we get

$$\widetilde{m}_{k,k}(\lambda) = \mathbf{d}_0(\lambda\varepsilon^k), \quad 0 \le k \le N-1. \tag{75}$$

Consider a number $l \in [1, N-1] \cap \mathbb{N}$. Let us calculate elements $m^{[j]}_{k,k+l}(\lambda)$, $0 \le k \le N-1-l$, i.e., the elements on the l-th upper diagonal. It is not difficult to see that

$$m^{[j]}_{k,k+l}(\lambda) = \left\{ \begin{array}{cc} m_{N-j+k,N-j+k+l}(\lambda\varepsilon^j), & 0 \le k < j-l \\ m_{N-j+k,-j+k+l}(\lambda\varepsilon^j), & j-l \le k \le j-1 \\ m_{k-j,k-j+l}(\lambda\varepsilon^j), & j \le k \le N-1-l \end{array} \right. . \tag{76}$$

Therefore

$$\begin{aligned} \widetilde{m}_{k,k+l}(\lambda) &= \frac{1}{N}\left(m_{k,k+l}(\lambda) + \sum_{j=1}^{N-1} m^{[j]}_{k,k+l}(\lambda) \right) \\ &= \frac{1}{N}\left(\sum_{j=0}^{k} m_{k-j,k-j+l}(\lambda\varepsilon^j) \right. \\ &\left. + \sum_{j=k+1}^{k+l} m_{N-j+k,-j+k+l}(\lambda\varepsilon^j) + \sum_{j=k+l+1}^{N-1} m_{N-j+k,N-j+k+l}(\lambda\varepsilon^j) \right) \\ &= \frac{1}{N}\left(\sum_{s=0}^{k} m_{s,s+l}(\lambda\varepsilon^{k-s}) + \sum_{j=k+1}^{k+l} m_{N-j+k,-j+k+l}(\lambda\varepsilon^j) \right. \\ &\left. + \sum_{t=k+1}^{N-1-l} m_{t,t+l}(\lambda\varepsilon^{N+k-t}) \right) \\ &= \frac{1}{N}\left(\sum_{r=0}^{N-1-l} m_{r,r+l}(\lambda\varepsilon^{k-r}) + \sum_{j=k+1}^{k+l} m_{N-j+k,-j+k+l}(\lambda\varepsilon^j) \right), \end{aligned} \tag{77}$$

where $s := k-j$, $t := N-j+k$.

Hence, we can write

$$\widetilde{m}_{k,k+l}(\lambda) = \frac{1}{N}\left(\sum_{r=0}^{N-1-l} m_{r,r+l}(\lambda\varepsilon^{k-r}) + \sum_{u=0}^{l-1} m_{N-l+u,u}(\lambda\varepsilon^{k+l-u}) \right), \tag{78}$$

where $u := -j + k + l$. Set

$$\mathbf{d}_l(\lambda) := \frac{1}{N}\left(\sum_{r=0}^{N-1-l} m_{r,r+l}(\lambda\varepsilon^{-r}) + \sum_{u=0}^{l-1} m_{N-l+u,u}(\lambda\varepsilon^{l-u}) \right),$$

$$1 \leq l \leq N-1,\ \lambda \in L_N \backslash \{0\}. \tag{79}$$

By virtue of (78) we obtain

$$\widetilde{m}_{k,k+l}(\lambda) = \mathbf{d}_l(\lambda\varepsilon^k), \quad 0 \leq k \leq N-1-l,\ 1 \leq l \leq N-1,\ \lambda \in L_N \backslash \{0\}. \tag{80}$$

Consequently, we get that each diagonal of matrix $\widetilde{M}(\lambda)$ is defined by a unique function on $L_N \backslash \{0\}$ by formulas (75), (80). The solutions M, A of the symmetric moment problem (25) with the matrix M having this property we will call standard. As it follows from the considerations above, standard solutions always exist.

Besides the minimization of a number of independent elements in the matrix function $M(\lambda)$, there is a possibility to minimize the support of $M(\lambda)$. It follows from the proof of Th. 1 in [1] that $M(\lambda)$ can be replaced by a function $\widehat{M}(\lambda)$ (having the same functional σ and orthonormal polynomials $\{p_n\}_{n=0}^{\infty}$) which has a support on two radial rays (corresponding to arbitrary branches of roots of 1 and -1 of order N).

5. Examples

We will illustrate our results obtained in the previous sections by several examples of orthogonal polynomials.

1. Consider the following functional:

$$\sigma(u,v) = \int_0^1 (u(\lambda), u(-\lambda)) \begin{pmatrix} 1 & \sqrt{\lambda} \\ \sqrt{\lambda} & 1 \end{pmatrix} \overline{\begin{pmatrix} v(\lambda) \\ v(-\lambda) \end{pmatrix}} d\lambda \tag{81}$$

$$= \frac{1}{2}\int_{-1}^1 (u(\lambda), u(-\lambda)) \begin{pmatrix} 1 & \sqrt{|\lambda|} \\ \sqrt{|\lambda|} & 1 \end{pmatrix} \overline{\begin{pmatrix} v(\lambda) \\ v(-\lambda) \end{pmatrix}} d\lambda, \quad u,v \in \mathbb{P}. \tag{82}$$

(Write $\int_{-1}^1 ... = \int_{-1}^0 ... + \int_0^1 ...$ in (82) and make the change of variable $\lambda = -x$ in the first addend to get (81)).

Calculate moments (25):

$$s_{2k,0} = 2\left(\frac{1}{2k+1} + \frac{1}{2k+\frac{3}{2}} \right),\ s_{2k+1,0} = 0,\ k \in \mathbb{Z}_+; \tag{83}$$

$$s_{2k,1} = 0,\ s_{2k+1,1} = 2\left(\frac{1}{2k+3} - \frac{1}{2k+3+\frac{1}{2}}\right),\ k \in \mathbb{Z}_+. \tag{84}$$

Calculate first five polynomials $\tilde{p}_n$ from (26):

$$\tilde{p}_0(\lambda) = 1,\ \tilde{p}_1(\lambda) = \frac{10}{3}\lambda,\ \tilde{p}_2(\lambda) = \frac{4}{441}(35\lambda^2 - 13),$$

$$\tilde{p}_3(\lambda) = \frac{9824}{101871}\lambda^3 - \frac{9824}{266805}\lambda,$$

$$\tilde{p}_4(\lambda) = \frac{3222272}{6471355275}\lambda^4 - \frac{2036224}{4622396625}\lambda^2 + \frac{88176896}{1779622700625}.$$

Calculate their roots:

$$\tilde{p}_1:\ \lambda_1 = 0;$$

$$\tilde{p}_2:\ \lambda_1 \approx -0.60944,\ \lambda_2 \approx 0.60944;$$

$$\tilde{p}_3:\ \lambda_1 \approx -0.61791,\ \lambda_2 = 0,\ \lambda_3 \approx 0.61791;$$

$$\tilde{p}_4:\ \lambda_1 \approx -0.86743,\ \lambda_2 \approx -0.36365,$$

$$\lambda_2 \approx 0.36365,\ \lambda_4 \approx 0.86743.$$

2. Consider the following functional:

$$\sigma(u,v) = \int_{-1}^{1} (u(\lambda), u(-\lambda)) \begin{pmatrix} 1 & \sqrt[3]{|\lambda|} \\ \sqrt[3]{|\lambda|} & 1 \end{pmatrix} \overline{\begin{pmatrix} v(\lambda) \\ v(-\lambda) \end{pmatrix}} d\lambda, \quad u,v \in \mathbb{P}. \tag{85}$$

We calculate moments (25):

$$s_{2k,0} = 4\left(\frac{1}{2k+1} + \frac{1}{2k+1+\frac{1}{3}}\right),\ s_{2k+1,0} = 0,\ k \in \mathbb{Z}_+; \tag{86}$$

$$s_{2k,1} = 0,\ s_{2k+1,1} = 4\left(\frac{1}{2k+3} - \frac{1}{2k+3+\frac{1}{3}}\right),\ k \in \mathbb{Z}_+. \tag{87}$$

Calculate first five polynomials $\tilde{p}_n$ from (26):

$$\tilde{p}_0(\lambda) = 1,\ \tilde{p}_1(\lambda) = 7\lambda,\ \tilde{p}_2(\lambda) = \frac{2}{15}(7\lambda^2 - \frac{38}{15}),$$

$$\tilde{p}_3(\lambda) = \frac{3989}{6750}\lambda^3 - \frac{3989}{18000}\lambda,$$

$$\tilde{p}_4(\lambda) = \frac{355021}{83160000}\lambda^4 - \frac{571469}{152460000}\lambda^2 + \frac{3510961}{8537760000}.$$

Calculate their roots:

$$\tilde{p}_1: \ \lambda_1 = 0;$$

$$\tilde{p}_2: \ \lambda_1 \approx -0.60158, \ \lambda_2 \approx 0.60158;$$

$$\tilde{p}_3: \ \lambda_1 \approx -0.61237, \ \lambda_2 = 0, \ \lambda_3 \approx 0.61237;$$

$$\tilde{p}_4: \ \lambda_1 \approx -0.86572, \ \lambda_2 \approx -0.35850,$$

$$\lambda_3 \approx 0.35850, \ \lambda_4 \approx 0.86572.$$

3. Consider the following functional:

$$\sigma(u,v) = \int_0^1 (u(\lambda), u(-\lambda)) \begin{pmatrix} 1+\frac{a}{\lambda} & -\frac{a}{\lambda} \\ -\frac{a}{\lambda} & 1+\frac{a}{\lambda} \end{pmatrix} \overline{\begin{pmatrix} v(\lambda) \\ v(-\lambda) \end{pmatrix}} d\lambda, \quad u,v \in \mathbb{P}, \quad (88)$$

where $a \geq 0$ is a parameter. Here the integral is understood to be improper at zero.

Notice that the elements of the (2×2) matrix in (88) are not integrable at zero if $a > 0$. However, we can calculate moments (25):

$$s_{2k,0} = \frac{2}{2k+1}, \ s_{2k+1,0} = 0, \ k \in \mathbb{Z}_+; \quad (89)$$

$$s_{2k,1} = 0, \ s_{2k+1,1} = \frac{2}{2k+3} + \frac{2a}{k+1}, \ k \in \mathbb{Z}_+. \quad (90)$$

Calculate first five polynomials $\tilde{p}_n$ from (26):

$$\tilde{p}_0(\lambda) = 1, \ \tilde{p}_1(\lambda) = 2\lambda, \ \tilde{p}_2(\lambda) = 4(a+\frac{1}{3})\lambda^2 - \frac{4}{3}a - \frac{4}{9},$$

$$\tilde{p}_3(\lambda) = \left(\frac{32}{45}a + \frac{32}{135}\right)\lambda^3 - \left(\frac{16}{45}a + \frac{32}{225}\right)\lambda,$$

$$\tilde{p}_4(\lambda) = \left(\frac{16}{135}a^2 + \frac{1088}{14175}a + \frac{256}{23625}\right)\lambda^4 - \left(\frac{32}{315}a^2 + \frac{2176}{33075}a + \frac{512}{55125}\right)\lambda^2$$
$$+ \frac{16}{1575}a^2 + \frac{1088}{165375}a + \frac{256}{275625}.$$

Calculate their roots for $a = 1$:

$$\tilde{p}_1: \ \lambda_1 = 0;$$

$$\tilde{p}_2: \ \lambda_1 \approx -0.57735, \ \lambda_2 \approx 0.57735;$$

$$\tilde{p}_3: \ \lambda_1 \approx -0.72456, \ \lambda_2 = 0, \ \lambda_3 \approx 0.72456;$$

$$\tilde{p}_4: \ \lambda_1 \approx -0.86113, \ \lambda_2 \approx -0.33998,$$

$$\lambda_3 \approx 0.33998,\ \lambda_4 \approx 0.86113.$$

4. Consider the following functional:

$$\sigma(u,v) = \int_0^1 (u(\lambda), u(-\lambda)) \begin{pmatrix} 1 & \lambda \\ \lambda & 1 \end{pmatrix} \overline{\begin{pmatrix} v(\lambda) \\ v(-\lambda) \end{pmatrix}} d\lambda$$

$$+\frac{1}{i}\int_0^i u(\lambda)\overline{v(\lambda)}d\lambda, \quad u,v \in \mathbb{P}. \tag{91}$$

We calculate moments (25):

$$s_{2k,0} = \frac{1}{k+1} + \frac{2+(-1)^k}{2k+1},\ s_{2k+1,0} = \frac{(-1)^k i}{2k+2},\ k \in \mathbb{Z}_+; \tag{92}$$

$$s_{2k,1} = \frac{(-1)^{k+1} i}{2k+2},\ s_{2k+1,1} = -\frac{1}{k+2} + \frac{2-(-1)^{k+1}}{2k+3},\ k \in \mathbb{Z}_+. \tag{93}$$

Calculate first five polynomials $\tilde{p}_n$ from (26):

$$\tilde{p}_0(\lambda) = 1,\ \tilde{p}_1(\lambda) = 4\lambda - \frac{i}{2},\ \tilde{p}_2(\lambda) = \frac{7}{4}\lambda^2 - \frac{17}{12}i\lambda - \frac{13}{24},$$

$$\tilde{p}_3(\lambda) = \frac{149}{180}\lambda^3 - \frac{343}{1440}i\lambda^2 + \frac{7}{21600}\lambda + \frac{1459}{14400}i,$$

$$\tilde{p}_4(\lambda) = \frac{1501331}{18144000}\lambda^4 - \frac{132749}{1296000}i\lambda^3 - \frac{36943}{518400}\lambda^2 + \frac{2746283}{63504000}i\lambda + \frac{3110389}{423360000}.$$

Calculate their roots:

$$\tilde{p}_1:\ \lambda_1 = \frac{i}{8};$$

$$\tilde{p}_2:\ \lambda_1 \approx -0.38169 + 0.40476i,\ \lambda_2 \approx 0.38169 + 0.40476i;$$

$$\tilde{p}_3:\ \lambda_1 \approx -0.41588 - 0.16289i,\ \lambda_2 \approx 0.61354i,\ \lambda_3 \approx 0.41588 - 0.16289i;$$

$$\tilde{p}_4:\ \lambda_1 \approx -0.67548 + 0.09831i,\ \lambda_2 \approx 0.23690i,$$

$$\lambda_3 \approx 0.80435i,\ \lambda_4 \approx 0.67548 + 0.09831i.$$

Notice that symmetric measures in (82),(85) imply that polynomial $\tilde{p}_n$ has only terms with degrees with the same parity as n, $n \in \mathbb{Z}_+$. Note that polynomials $\tilde{p}_2$ and $\tilde{p}_4$ in the last example have symmetric with respect to $i\mathbb{R}$ roots but not with respect to the origin. This agrees with Th. 2.3.

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