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Asymptotic Properties of Hilbert Geometry

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It is shown that the spheres in Hilbert geometry have the same volume growth entropy as those in the Lobachevsky space. Asymptotic estimates for the ratio of the volume of metric ball to the area of the metric sphere in Hilbert geometry are given. Derived estimates agree with the well-known fact in the Lobachevsky space.

Key words: Hilbert geometry, Finsler geometry, balls, spheres, volume, area, entropy.

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1. Introduction

Hilbert geometry is a generalization of the Klein model of the Lobachevsky space. The absolute there is an arbitrary convex hypersurface unlike an ellipsoid in the Lobachevsky space. Hilbert geometries are simply connected, projectively flat, complete reversible Finsler spaces of the constant negative flag curvature -1.

In [12], B. Colbois and P. Verovic proved that the balls in an $(n+1)$-dimensional Hilbert geometry have the same volume growth entropy as those in $\mathbb{H}^{n+1}$, namely n. We obtain an analogous result for the spheres in Hilbert geometry.

Theorem 1. *Consider an $(n+1)$-dimensional Hilbert geometry associated with a bounded open convex domain $U \subset \mathbb{R}^{n+1}$ whose boundary is a C^3 hypersurface with positive normal curvatures. Then we have*

$$\lim_{t\to\infty} \frac{\ln(\mathbf{Vol}(S_t^n))}{t} = n.$$

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It is known [4–7] that in the Lobachevsky space $\mathbb{H}^{n+1}$ of constant curvature -1 for a family of metric balls $\{B_t^{n+1}\}_{t\in\mathbb{R}^+}$ the following equality holds:

$$\lim_{\rho\to\infty}\frac{\mathbf{Vol}(B_\rho^{n+1})}{\mathbf{Vol}(S_\rho^n)}=\frac{1}{n}.$$

The same ratio in a more general case for λ- and h-convex hypersurfaces in Hadamard manifolds was considered in [4, 6, 7] by A.A. Borisenko, V. Miquel, A. Reventos and E. Gallego.

The similar estimates in Finsler spaces were derived in [5] (see also [16]).

Theorem [5]. *Let (M^{n+1},F) be an $(n+1)$-dimensional Finsler–Hadamard manifold that satisfies the following conditions:*

1. the flag curvature satisfies the inequalities $-k_2^2\leqslant K\leqslant -k_1^2$, $k_1,k_2>0$;

2. the $\mathbf{S}$-curvature satisfies the inequalities $n\delta_1\leqslant S\leqslant n\delta_2$ such that $\delta_i<k_i$.

Then for a family $\{B_r^{n+1}(p)\}_{r\geqslant 0}$ we have

$$\frac{1}{n(k_2-\delta_2)}\leqslant\lim_{r\to\infty}\inf\frac{\mathbf{Vol}(B_r^{n+1}(p))}{\mathbf{Area}(S_r^n(p))}\leqslant\lim_{r\to\infty}\sup\frac{\mathbf{Vol}(B_r^{n+1}(p))}{\mathbf{Area}(S_r^n(p))}\leqslant\frac{1}{n(k_1-\delta_1)}.$$

Our goal is to prove an analogous result in Hilbert geometry for a family $\{B_t^{n+1}\}_{t\in\mathbb{R}^+}$. We can not apply the theorem from [5] because the $\mathbf{S}$-curvature in Hilbert geometry is difficult to calculate.

As a result the following theorem is obtained.

Theorem 2. *Consider an $(n+1)$-dimensional Hilbert geometry associated with a bounded open convex domain $U\in\mathbb{R}^{n+1}$ whose boundary is a C^3 hypersurface with positive normal curvatures. Fix a point $o\in U$, we assume this point to be the origin and center of all balls considered. Denote by $\omega(u):\mathbb{S}^n\to\mathbb{R}_+$ the radial function for ∂U, i.e., the mapping $\omega(u)u$, where $u\in\mathbb{S}^n$ is a parametrization of ∂U, and by $\iota:\mathbb{R}^{n+1}\to\mathbb{S}^n$ the mapping such that $\iota(p)=\frac{u_p}{||u_p||}$, where u_p is the radius-vector of a point p.*

Denote by K and k the maximum and minimum normal curvatures of ∂U, $c=\max_{u\in\mathbb{S}^n}\frac{\omega(u)}{\omega(-u)}$, $\omega_0=\min_{u\in\mathbb{S}^n}\omega(u)$, $\omega_1=\max_{u\in\mathbb{S}^n}\omega(u)$. Then we have

$$\lim_{\rho\to\infty}\sup\frac{\mathbf{Vol}(B_\rho^{n+1})}{\mathbf{Vol}(S_\rho^n)}\leqslant\frac{1}{n}c^{\frac{n}{2}}\left(\frac{K}{k}\right)^{\frac{n}{2}}\frac{1}{(k\omega_0)^{\frac{n}{2}+1}}\frac{\int_{\mathbb{S}^n}\omega(u)^{\frac{n}{2}}du}{\int_{\partial U}\omega(\iota(p))^{-\frac{n}{2}}dp},$$

$$\lim_{\rho\to\infty}\inf\frac{\mathbf{Vol}(B_\rho^{n+1})}{\mathbf{Vol}(S_\rho^n)}\geqslant\frac{1}{n}\frac{1}{c^{\frac{n}{2}}}\left(\frac{k}{K}\right)^{\frac{n}{2}}(k\omega_0)^{\frac{n}{2}}\frac{\int_{\mathbb{S}^n}\omega(u)^{\frac{n}{2}}du}{\int_{\partial U}\omega(\iota(p))^{-\frac{n}{2}}dp},$$

or, more simple expressions

$$\lim_{\rho\to\infty}\sup\frac{\mathbf{Vol}(B_\rho^{n+1})}{\mathbf{Vol}(S_\rho^n)}\leqslant\frac{1}{n}\left(\frac{K}{k}\right)^{\frac{n}{2}}\left(\frac{\omega_1}{\omega_0}\right)^{n+1}\left(\frac{\omega_1}{k}\right)^{\frac{n}{2}}\frac{1}{k\omega_1}\frac{\mathbf{Vol}_E(\mathbb{S}^n)}{\mathbf{Vol}_E(\partial U)},$$

$$\lim_{\rho\to\infty}\inf\frac{\mathbf{Vol}(B_\rho^{n+1})}{\mathbf{Vol}(S_\rho^n)}\geqslant\frac{1}{n}\left(\frac{k}{K}\right)^{\frac{n}{2}}\left(\frac{\omega_0}{\omega_1}\right)^{\frac{n}{2}}\omega_0^n(k\omega_0)^{\frac{n}{2}}\frac{\mathbf{Vol}_E(\mathbb{S}^n)}{\mathbf{Vol}_E(\partial U)}.$$

If U is a symmetric domain with respect to o, then we have

$$\lim_{\rho\to\infty}\sup\frac{\mathbf{Vol}(B_\rho^{n+1})}{\mathbf{Vol}(S_\rho^n)}\leqslant\frac{1}{n}c^{\frac{n}{2}}\left(\frac{K}{k}\right)^{\frac{n}{2}}\frac{\omega_1^n}{(k\omega_0)^{\frac{n}{2}+1}}\frac{\mathbf{Vol}_E(\mathbb{S}^n)}{\mathbf{Vol}_E(\partial U)},$$

$$\lim_{\rho\to\infty}\inf\frac{\mathbf{Vol}(B_\rho^{n+1})}{\mathbf{Vol}(S_\rho^n)}\geqslant\frac{1}{n}\frac{1}{c^{\frac{n}{2}}}\left(\frac{k}{K}\right)^{\frac{n}{2}}(k\omega_0)^{\frac{n}{2}}\omega_0^n\frac{\mathbf{Vol}_E(\mathbb{S}^n)}{\mathbf{Vol}_E(\partial U)}.$$

Notice that in this theorem the ratio of the volume of the ball to the *internal* volume of the sphere is considered, unlike in theorem [5], where the *induced* volume is used.

2. Preliminaries

2.1. Finsler geometry

In this section we recall some basic facts and theorems from Finsler geometry that we need. See [16] for details.

Let M^n be an n-dimensional connected C^∞-manifold. Denote by $TM^n = \bigsqcup_{x\in M^n} T_xM^n$ the tangent bundle of M^n, where T_xM^n is the tangent space at x. The *Finsler metric* on M^n is a function $F : TM^n \to [0,\infty)$ with the following properties:

1. $F \in C^\infty(TM^n\backslash\{0\})$;
2. F is positively homogeneous of degree one, i.e., for any pair $(x,y)\in TM^n$ and any $\lambda>0$, $F(x,\lambda y)=\lambda F(x,y)$;
3. for any pair $(x,y)\in TM^n$ the following bilinear symmetric form $g_y : T_xM^n\times T_xM^n\to\mathbb{R}$ is positively definite

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial t\partial s}[F^2(x,y+su+tv)]|_{s=t=0}.$$

The pair (M^n, F) is called *a Finsler manifold.*

If we introduce the functions

$$\mathbf{g}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}[F^2(x,y)],$$

then one can rewrite the form $\mathbf{g}_y(u,v)$ as

$$\mathbf{g}_y(u,v) = \mathbf{g}_{ij}(x,y)u^i v^j.$$

For any fixed vector field Y defined on the subset $U \subset M^n$, $\mathbf{g}_Y(u,v)$ is a Riemannian metric on U.

Given a Finsler metric F on a manifold M^n. For a smooth curve $c : [a,b] \to M^n$ the length is defined by the integral

$$L_F(c) = \int_a^b F(c(t), \dot{c}(t))dt = \int_a^b \sqrt{\mathbf{g}_{\dot{c}(t)}(\dot{c}(t), \dot{c}(t))}dt.$$

Let $\{e_i\}_{i=1}^n$ be an arbitrary basis for T_xM^n and $\{\theta^i\}_{i=1}^n$ be a dual basis for $T_x^*M^n$. Consider the set $B^n_{F(x)} = \{(y^i) \in \mathbb{R}^n : F(x, y^i e_i) < 1\} \subset T_xM^n$. Denote by $\mathbf{Vol}_E(A)$ the Euclidean volume of A. Then define the form

$$dV_F = \sigma_F(x)\theta^1 \wedge \ldots \wedge \theta^n,$$

here

$$\sigma_F(x) := \frac{\mathbf{Vol}_E(\mathbb{B}^n)}{\mathbf{Vol}_E(B^n_{F(x)})}, \tag{1}$$

and $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$.

The volume form dV_F determines a regular measure $\mathbf{Vol}_F = \int dV_F$ and is called the *Busemann–Hausdorff volume form.*

For any Riemannian metric $g(u,v) = \mathbf{g}_{ij}(x)u^i v^j$ the Busemann–Hausdorff volume form is the standard Riemannian volume form

$$dV_g = \sqrt{\det(\mathbf{g}_{ij})}\theta^1 \wedge \ldots \wedge \theta^n.$$

In [9] it was proved that the Busemann–Hausdorff measure for reversible metric coincides with the n-dimensional outer Hausdorff measure. Recall that the *n-dimensional outer Hausdorff measure of a set A* is defined by

$$\nu_n = \lim_{r \to 0} \nu_{n,r},$$

$$\nu_{n,r} = \mathbf{Vol}_E(\mathbb{B}^n) \inf \left(\sum_i \rho_i^n : 2\rho_i < r, A \subseteq \bigcup_i B[x_i, \rho_i], x_i \in A \right).$$

It should be noticed here that if we calculate the Hausdorff measure for the submanifold in a Finsler manifold with the symmetric metric, then we will obtain the *internal* volume on submanifold in the metric induced from the ambient space. But, unfortunately, the using of this volume implies certain difficulties. In our case, when we consider the sphere as a submanifold, the following claim does not hold

$$\mathbf{Vol}(B_r^n) = \int_0^r \mathbf{Vol}(S_t^{n-1})dt$$

if we use the internal volume. For details, see [16].

2.2. Hilbert geometry

Consider a bounded open convex domain $U \subset \mathbb{R}^{n+1}$ whose boundary is a C^3 hypersurface with positive normal curvatures in $\mathbb{R}^n$ equipped with a Euclidean norm $\|\cdot\|$.

For given two distinct points p and q in U, let p_1 and q_1 be the corresponding intersection points of the halflines $p + \mathbb{R}_-(q-p)$ and $p + \mathbb{R}_+(q-p)$ with ∂U (Fig. 1).

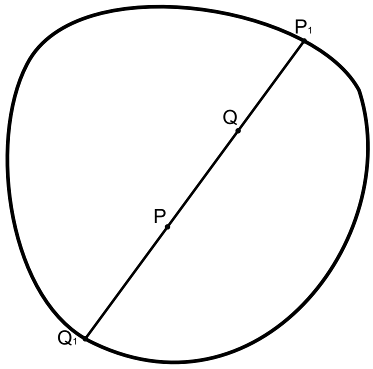

Fig. 1: Hilbert metric

Then consider the following distance function:

$$d_U(p,q) = \frac{1}{2}\ln\frac{\|q-q_1\|}{\|q-p_1\|} \times \frac{\|p-p_1\|}{\|p-q_1\|}, \tag{2}$$
$$d_U(p,p) = 0.$$

The obtained metric space (U, d_U) is called Hilbert geometry and is a complete noncompact geodesic metric space with the $\mathbb{R}^n$-topology and in which the affine open segments joining two points are geodesics [10].

The distance function is naturally associated with the Finsler metric F_U on U. For a point $p \in U$ and a tangent vector $v \in T_pU = \mathbb{R}^n$

$$F_U(p,v) = \frac{1}{2}\|v\| \left(\frac{1}{\|p - p_-\|} + \frac{1}{\|p - p_+\|} \right), \tag{3}$$

where p_- and p_+ are the intersection points of the half-lines $p + \mathbb{R}_- v$ and $p + \mathbb{R}_+ v$ with ∂U.

Then $d_U(p,q) = \inf \int_I F_U(c(t), \dot{c}(t))dt$ when $c(t)$ ranges over all smooth curves joining p and q.

In is known (see for example [16]) that Hilbert metrics are the metrics of constant flag curvature -1.

When $U = B_r^n$, then we obtain the Klein model of the n-dimensional Lobachevsky space $\mathbb{H}^n$, and the Finsler metric has the explicit expression

$$F_{B_r^n}(p,v) = \sqrt{\frac{\|v\|^2}{r - \|p\|^2} + \frac{< v, p >^2}{(r^2 - \|p\|^2)^2}}. \tag{4}$$

In [10] it is proved that the balls of arbitrary radii are convex sets in Hilbert geometry.

The asymptotic properties of Hilbert geometry have been obtained lately. All these properties mean that Hilbert geometry is "almost" Riemannian at infinity. It is proved in [12] that Hilbert metric "tends" to Riemannian metric as follows.

Theorem [12]. *Let $\mathcal{C} \in \mathbb{R}^n$ be a bounded open convex domain whose boundary $\partial\mathcal{C}$ is a hypersurface of class C^3 that is strictly convex. For any $p \in \mathcal{C}$, let $\delta(p) > 0$ be the Euclidean distance from p to $\partial\mathcal{C}$. Then there exists a family $(\vec{l}_p)_{p\in\mathcal{C}}$ of linear transformations in $\mathbb{R}^n$ such that*

$$\lim_{\delta(p)\to 0} \frac{F_C(p,v)}{\|\vec{l}_p(v)\|} = 1$$

uniformly in $v \in \mathbb{R}^n \backslash \{0\}$.

This means that the unit sphere in the tangent space of given Hilbert metric tends to ellipsoid in continuous topology as the tangent point goes to the absolute.

3. Calculating the Volume Growth Entropy of Spheres

In this section we will prove that for an $(n+1)$-dimensional Hilbert geometry

$$\lim_{t\to\infty} \frac{\ln(\mathbf{Vol}(S_t^n))}{t} = n.$$

Consider a bounded open convex domain $U \subset \mathbb{R}^{n+1}$ whose boundary is a C^3 hypersurface with positive normal curvatures in $\mathbb{R}^n$.

Fix a point $o \in U$, we assume this point to be the origin and center of all balls considered. Denote by $\omega(u) : \mathbb{S}^n \to \mathbb{R}_+$ the radial function for ∂U, i.e., the mapping $\omega(u)u$, where $u \in \mathbb{S}^n$ is a parametrization of ∂U. Let $B_r^{n+1}(o)$ be the metric ball of radius r centered at a point o, $S_r^n(o) = \partial B_r^{n+1}(o)$ be the metric sphere.

We will use the following lemma that shows the order of growth of Hilbert distance from the sphere to ∂U in terms of Euclidean distance. We also estimate the deviation of tangent and normal vectors to sphere from those to ∂U.

Lemma 1. *Let $\omega(u)u : \mathbb{S}^n \to \mathbb{R}_+$ be the parametrization of ∂U, $\rho_t(u) : \mathbb{S}^n \to \mathbb{R}_+$ be the parametrization of the sphere of radius t.*

Then, as $t \to \infty$:

1. $\omega(u) - \rho_t(u) = \Delta(u)e^{-2t} + \bar{o}(e^{-2t})$,

$$\Delta(u) = \omega(u)\left(\frac{\omega(u)}{\omega(-u)} + 1\right);$$

2. $\omega_i'(u) - \rho_{t,i}'(u) = \Delta_i(u)e^{-2t} + \bar{o}(e^{-2t})$,

$$\Delta_i(u) = \left[\omega_i'(u)\left(2\frac{\omega(u)}{\omega(-u)} + 1\right) + \left(\frac{\omega(u)}{\omega(-u)}\right)^2 \omega_i'(-u)\right];$$

3. $\omega_{ij}''(u) - \rho_{t,ij}''(u) = \Delta_{ij}(u)e^{-2t} + \bar{o}(e^{-2t})$,

$$\begin{aligned}\omega(u)^3\Delta_{ij}(u) &= \omega(u)^2[2\omega_i'(-u)\omega_j'(-u) - \omega(-u)\omega_{ij}''(-u)]\\ &+\omega(-u)^2[2\omega_i'(u)\omega_j'(u) + \omega(-u)\omega_{ij}''(u)]\\ &+2\omega(-u)\omega(u)[\omega_j'(-u)\omega_i'(u) + \omega_i'(-u)\omega_j'(u)].\end{aligned}$$

P r o o f o f L e m m a 1. We are going to obtain the explicit expression for $\rho_t(u)$. Let $q = 0$ be the center of the sphere, p be a point on the sphere. Using formula (3), we obtain the equation on the function $\rho_t(u)$

$$\frac{1}{2}\ln\left[\frac{\omega(u)}{\omega(-u)} \times \frac{\omega(-u) + \rho_t(u)}{\omega(u) - \rho_t(u)}\right] = t.$$

By direct computation we have

$$\rho_t(u) = \frac{\omega(-u)\omega(u)(e^{2t} - 1)}{\omega(u) + \omega(-u)e^{2t}}.$$

1. Consider the difference

$$\omega(u) - \rho_t(u) = \omega(u) - \frac{\omega(-u)\omega(u)(e^{2t}-1)}{\omega(u)+\omega(-u)e^{2t}}$$

$$= \frac{\omega^2(u)+\omega(-u)\omega(u)}{\omega(u)+\omega(-u)e^{2t}} = \omega(u)\left(\frac{\omega(u)}{\omega(-u)}+1\right)e^{-2t} + \bar{o}(e^{-2t}),\ t\to\infty.$$

2. Analogously, we obtain

$$\omega'(u) - \rho'_{t,i}(u)$$

$$= \frac{\omega'_i(u)\omega(-u)^2e^{2t} + 2e^{2t}\omega(u)\omega(-u)\omega'_i(u) + \omega(u)^2(\omega'_i(u)+\omega'_i(-u)(e^{2t}-1))}{(\omega(u)+\omega(-u)e^{2t})^2}$$

$$= \left[\omega'_i(u)\left(2\frac{\omega(u)}{\omega(-u)}+1\right) + \left(\frac{\omega(u)}{\omega(-u)}\right)^2\omega'_i(-u)\right]e^{-2t} + \bar{o}(e^{-2t}),\ t\to\infty.$$

3. It can be proved in the same manner. ■

Denote the minimum and maximum Euclidean normal curvatures of ∂U by k and K, respectively.

We also use the notation $\omega_0 = \min_{u\in\mathbb{S}^n}\omega(u)$, $\omega_1 = \max_{u\in\mathbb{S}^n}\omega(u)$.

The following lemma gives the estimates on the angle between the radial and normal directions at points from ∂U.

Lemma 2. *For a given point $m = \omega(u_m)u_m \in \partial U$ denote by $N(m)$ the normal vector at m. Then*

$$\cos\angle(u_m, N(m)) \geqslant \frac{\omega_0}{R}.$$

P r o o f o f L e m m a 2. This lemma follows from a more general theorem.

Theorem [4, 6, 7]. *Let N be a hypersurface in a Riemannian manifold M. Consider N as defined by the equation $t = \rho(\theta)$ of class C^2, where $\rho(\theta)$ is the distance to point o. N can be seen as the 0-level set of the function $F = t - \rho$. For given point $P \in N$ we consider all the vectors to be attached at P. Consider $Y = \frac{grad_N\rho}{\|grad_N\rho\|}$. Let x, that is orthogonal to the radial direction, be the unit vector in the plane spanned on y and on the radial direction. Let φ be the angle between the normal direction and the radial direction at point $P \in N$.*

If $k_{\mathbf{n}}$ is the normal curvature at P in the direction given by Y, $\mu_{\mathbf{n}}$ is the normal curvature in the direction of x of the sphere centered at o of radius ρ, and $\frac{d\varphi}{ds}$ is the derivative of φ with respect to the arc parameter of the integral curve of Y by P, then

$$k_{\mathbf{n}} = \mu_{\mathbf{n}}\cos\varphi + \frac{d\varphi}{ds}.$$

Now we can prove Lem. 2.

Consider any integral curve γ of $\frac{y}{\|y\|}$. Since the angle φ takes its value in the interval $[0, \pi/2]$, then there is a supremum φ_0 of it. If at some point $\gamma(s_0)$ the value φ_0 is achieved, then at this point we have $\varphi' = 0$ and

$$\cos\varphi = \frac{k_{\mathbf{n}}}{\mu_{\mathbf{n}}}.$$

The minimum possible value of $k_{\mathbf{n}}$ is equal to $k = \frac{1}{R}$, and the maximum possible value of $\mu_{\mathbf{n}}$ is equal to $\frac{1}{\omega_0}$. Hence we have

$$\cos\varphi = \frac{k_{\mathbf{n}}}{\mu_{\mathbf{n}}} \geqslant \frac{\omega_0}{R}.$$

And Lemma 2 follows. ■

P r o o f o f T h e o r e m 1. Now we are going to estimate the volume of a sphere S_t^n in Hilbert geometry. The idea of proof is to obtain the Hausdorff measure of this sphere. It follows from the reversibility of Hilbert metrics that the Hausdorff measure coincides with the Finslerian Busemann–Hausdorff volume [9].

Fix a point p on the sphere S_r^n. Since the spheres are convex, we can choose the vector $u \in \mathbb{S}^n$ such that $p = \rho_t(u)$. More generally, for a given origin $o \in \mathbb{R}^{n+1}$ denote the corresponding radius vector by u_p and consider the function $\iota : \mathbb{R}^{n+1} \to \mathbb{S}^n$ such that $\iota(p) = \frac{u_p}{||u_p||}$. Then we can write that $p = \rho_t(\iota(u))$.

Denote the point $\omega(\iota(p))\iota(p) \in \partial U$ by m. Consider the vector v_m which is tangent to ∂U at m, the vector n_m which is orthogonal to v_m with respect to the Euclidean inner product such that the point o belongs to the plane $\mathcal{P}$ spanned on v_m and n_m. Let k_m be the curvature of the section of ∂U by $\mathcal{P}$ at m. Consider a special coordinate system in the plane $\mathcal{P}$ such that the axis z is directed as n_m and the axis x is directed as v_m. Then, in this special coordinate system the section of ∂U can be locally expressed as

$$z(x) = \frac{1}{2}k_m x^2 + \bar{o}(x^2), x \to 0.$$

Later on we will work with this section.

Draw the secant of the sphere that is parallel to the tangent vector at p (Fig. 2).

Put $d = ||a_1 - a_2||$, $\delta(p) = ||m - p||$, $\delta_1 = ||b_1 - f||$, $\delta_2 = ||f - b_2||$, $h = ||p - f||$.

Let us estimate the function $\delta(p)$. From Lemma 1 we have

$$\delta(p) \leqslant \omega(u)\left(\frac{\omega(u)}{\omega(-u)} + 1\right)e^{-2t} + \bar{o}(e^{-2t}), t \to \infty. \tag{5}$$

From the triangle pm_1m we have that $\delta(p) \approx \cos\angle(u_m, v_m)(\omega(u) - \rho_t(u))$. Finally, using Lem. 2, we obtain $\delta(p) \geqslant \frac{\omega_0}{R}(\omega(u) - \rho_t(u))$.

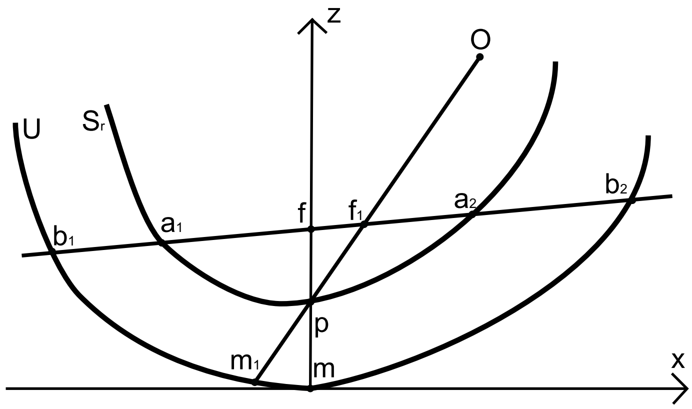

Fig. 2: Proof of Theorem 1

Consequently,

$$\delta(p) \geqslant \frac{\omega_0}{R}\omega(u)\left(\frac{\omega(u)}{\omega(-u)}+1\right)e^{-2t}+\bar{o}(e^{-2t}), t \to \infty. \tag{6}$$

Then we estimate δ_1 and δ_2. Let

$$z = a(p)x + \delta(p) + h$$

be the equation of the secant in a special coordinates system. We suppose h to decrease faster than $\delta(p)$. Thus in further computations h will be neglected.

We find the intersection points of this line with the boundary ∂U. From the expression for the boundary we have

$$a(p)x + \delta(p) = \frac{1}{2}k_m x^2.$$

Thus

$$x_{1,2} = \frac{a(p) \pm \sqrt{a(p)^2 + 2k_m\delta(p)}}{k_m}.$$

It follows from Lem. 1 that $a(p) = a_0(u)e^{-2t}, t \to \infty$, for some function $a_0(u)$ and, consequently, $a(p) = \underline{O}(\delta(p)), \delta(p) \to 0$. Therefore we have

$$x_{1,2} = \pm\sqrt{\frac{2\delta(p)}{k_m}} + \bar{o}(\sqrt{\delta(p)}), \delta(p) \to 0,$$

$$z_{1,2} = \frac{1}{2}k_m x^2 + \bar{o}(x^2)|_{x=x_{1,2}} = \frac{1}{2}\delta(p) + \bar{o}(\delta(p)), \delta(p) \to 0$$

and

$$\delta_1 = \sqrt{x_1^2 + (z_1 - \delta(p))^2} = \sqrt{\frac{2\delta(p)}{k_m}} + \bar{o}(\delta(p)) = \delta_2, \delta(p) \to 0.$$

Therefore, the turning of the tangent, as the point goes to ∂U, does not influence the asymptotic behavior of δ_i.

Compute the Hilbert length of segment a_1a_2. Denote it by d_U. Then, as $h \to 0$,

$$d_U \approx \frac{1}{2}\ln\left[\frac{d+\delta_1}{\delta_2} \times \frac{d+\delta_2}{\delta_1}\right] = \frac{1}{2}\ln\left[\left(\frac{d}{\delta_2} + \frac{\delta_1}{\delta_2}\right) \times \left(\frac{d}{\delta_1} + \frac{\delta_2}{\delta_1}\right)\right]$$

$$\approx \frac{1}{2}\left(\frac{d}{\delta_1} + \frac{d}{\delta_2}\right) \approx \frac{d\sqrt{k_m}}{\sqrt{2(\delta(p)+h)}} + \bar{o}(\sqrt{1/\delta(p)}), \delta(p) \to 0.$$

We are showing now that the limit of ratio of d_U to the Finslerian length $\tilde{d}_U$ of the geodesic arc a_1a_2 is equal to 1 as the arc is subtended to a point. Specialize the coordinate system on $\mathbb{R}^{n+1}$ so as $a_1 = 0$. Let $w(t) : [0,T] \to U$ be a parametrization of the arc. Then the segment from point $a_1 = 0$ to point $a_2 = w(t)$ can be parameterized by $v(s) = \frac{s}{t}w(t) : [0,t] \to U$. Calculate the lengths of v and w

$$\tilde{d}_U = \int_0^t F_U(w(s), \dot{w}(s))ds,$$

$$d_U = \int_0^t F_U(v(s), \dot{v}(s))ds = \int_0^t F_U\left(\frac{s}{t}w(t), \frac{1}{t}w(t)\right)ds.$$

From the intermediate-value theorem for integrals we have

$$\tilde{d}_U = \int_0^t F_U(w(s), \dot{w}(s))ds = tF_U(w(s_0), \dot{w}(s_0)), s_0 \in [0,t],$$

$$d_U = \int_0^t F_U\left(\frac{s}{t}w(t), \frac{1}{t}w(t)\right)ds = tF_U\left(\frac{s_1}{t}w(t), \frac{1}{t}w(t)\right), s_1 \in [0,t].$$

Now we subtend the arc to a point, i.e., let $t \to 0$. Then $s_0, s_1 \to 0$, and we obtain

$$\frac{\tilde{d}_U}{d_U} = \frac{tF_U(w(s_0), \dot{w}(s_0))}{tF_U\left(\frac{s_1}{t}w(t), \frac{1}{t}w(t)\right)} \longrightarrow \frac{F_U(0, \dot{w}(0))}{F_U(0, \dot{w}(0))} = 1.$$

And the statement is proved.

Now our goal is to calculate the Hausdorff measure of sphere S^n_r. Denote by $\delta_0(r)$ the Hausdorff distance from the points of sphere to the absolute ∂U. Consider a covering $\{B_i\}$ of the sphere S^n_r by balls with diameters $\tilde{d}_i$ centered at points $p_i \in S^n_r$. Denote by k_i the normal curvature of ∂U that corresponds to the i-th sphere from the covering (as above). As we saw, we can replace $\tilde{d}_i$ by the lengths of corresponding chords d_i of the sphere S^n_r.

Then the Hausdorff measure and, consequently, the Finslerian Busemann–Hausdorff measure are given by

$$\mathbf{Vol}(S^n_t) = \mathbf{Vol}_E(\mathbb{B}^n) \inf_{d_{B_i}} \sum_i \left(\frac{d_i \sqrt{k_i}}{\sqrt{2(\delta(p_i)+h)}} \right)^n + \bar{o}(\sqrt{1/\delta_0(t)^n}), \delta_0(t) \to 0,$$

where infimum is calculated over all coverings of the sphere S^n_r.

Our metric sphere S^n_r is sufficiently smooth, so we can proceed to the integral over S^n_r

$$\mathbf{Vol}(S^n_t) = \mathbf{Vol}_E(\mathbb{B}^n) \inf_{d_{B_i}} \sum_i \left(\frac{d_i \sqrt{k_i}}{\sqrt{2(\delta(p_i)+h)}} \right)^n + \bar{o}(\sqrt{1/\delta_0(t)^n})$$

$$= \inf_{d_{Bi}} \sum_i \left(\frac{\sqrt{k_i}}{\sqrt{2(\delta(p_i)+h)}} \right)^n \mathbf{Vol}_E(B_i) + \bar{o}(\sqrt{1/\delta_0(t)^n}), \delta_0(t) \to 0.$$

Denote by dp the area element of S^n_t. Proceeding to integral and estimating lead to

$$\mathbf{Vol}(S^n_t) \geqslant k^{\frac{n}{2}} \int\limits_{S^n_t} \left(\frac{1}{2\delta(p)} \right)^{\frac{n}{2}} dp + \bar{o}(\sqrt{1/\delta_0(t)^n}), \delta_0(t) \to 0,$$

$$\mathbf{Vol}(S^n_t) \leqslant K^{\frac{n}{2}} \int\limits_{S^n_t} \left(\frac{1}{2\delta(p)} \right)^{\frac{n}{2}} dp + \bar{o}(\sqrt{1/\delta_0(t)^n}), \delta_0(t) \to 0.$$

The using of the explicit estimates (5), (6) for $\delta(p)$ results

$$\mathbf{Vol}(S^n_t) \geqslant k^{\frac{n}{2}} \int\limits_{S^n_t} \left(2\omega(\iota(p)) \left(\frac{\omega(\iota(p))}{\omega(-\iota(p))} + 1 \right) \right)^{-\frac{n}{2}} dp \cdot e^{nt} + \bar{o}(e^{nt}), t \to \infty, \quad (7)$$

$$\mathbf{Vol}(S^n_t) \leqslant K^{\frac{n}{2}} \int\limits_{S^n_t} \left(2\frac{\omega_0}{R}\omega(\iota(p)) \left(\frac{\omega(\iota(p))}{\omega(-\iota(p))} + 1 \right) \right)^{-\frac{n}{2}} dp \cdot e^{nt} + \bar{o}(e^{nt}), t \to \infty. \quad (8)$$

And Theorem 1 follows. ∎

4. Estimation of Ratio of the Volume of the Ball to the Volume of the Sphere

Here we will find the asymptotic behavior of volume of the metric ball B^{n+1}_{ρ} in Hilbert geometry. Further there will be used the method introduced in [13] where we improve some necessary estimates.

The volume of a metric ball is given by the integral

$$\mathbf{Vol}(B^{n+1}_{\rho}) = \int\limits_{B^{n+1}_{\rho}} \sigma(p)dp.$$

Here $\sigma(p)$ is the Busemann–Hausdorff volume form. And the volume estimating problem is reduced to the estimating of the volume form. Recall (1) that

$$\sigma(p) := \sigma_{F_U}(p) = \frac{\mathbf{Vol}_E(\mathbb{B}^n)}{\mathbf{Vol}_E(B^n_{F_U(p)})}.$$

Thus we have to estimate the volume of the unit sphere in the tangent space at point $p \in U$.

We will use the following simple lemma.

Lemma 3. *There exists a value ρ_0 such that for any points $p \in U$ in the neighborhood $d(p, \partial U) \leqslant \rho_0$ there exists a unique point $\pi(p) \in \partial U$: $d(p, \pi(p)) = d(p, \partial U)$*

Put $m = \pi(p) \in \partial U$. The minimum and maximum Euclidean normal curvatures of ∂U denote by k and K, respectively. Then, at any point $m \in \partial U$ the tangent sphere of radius $R := \frac{1}{k}$ contains U, the tangent sphere of radius $r := \frac{1}{K}$ is contained in U [2]. On two tangent spheres of the radii r and R at this point we construct corresponding Klein metrics F_r and F_R. We can give the explicit expressions (4) for them.

Then the following inequalities hold:

$$\mathbf{Vol}_E\left(B^{n+1}_{F_r(p)}\right) \leqslant \mathbf{Vol}_E\left(B^{n+1}_{F_U(p)}\right) \leqslant \mathbf{Vol}_E\left(B^{n+1}_{F_R(p)}\right). \tag{9}$$

As it was shown in [13]

$$\mathbf{Vol}_E\left(B^{n+1}_{F_R(p)}\right) = \mathbf{Vol}_E(\mathbb{B}^{n+1})R^{n+1}\left\{1 - \left(1 - \frac{d(p,m)}{R}\right)^2\right\}^{\frac{n+2}{2}},$$

$$\mathbf{Vol}_E\left(B^{n+1}_{F_r(p)}\right) = \mathbf{Vol}_E(\mathbb{B}^{n+1})r^{n+1}\left\{1 - \left(1 - \frac{d(p,m)}{r}\right)^2\right\}^{\frac{n+2}{2}}.$$

Thus, we have

$$\frac{1}{R^{n+1}\left\{1-\left(1-\frac{d(p,m)}{R}\right)^2\right\}^{\frac{n+2}{2}}} \leqslant \sigma(p) \leqslant \frac{1}{r^{n+1}\left\{1-\left(1-\frac{d(p,m)}{r}\right)^2\right\}^{\frac{n+2}{2}}}. \qquad (10)$$

Consider the mapping

$$\Phi(u,s) = \tanh(s)\omega(u)u : \mathbb{S}^n \times \mathbb{R} \longrightarrow U.$$

In [13] it was shown that the mapping $\Phi(u,s)$ satisfies the following properties:

1. $\Phi(\mathbb{S}^n,[0,\rho-c]) \subseteq B_\rho^{n+1} \subseteq \Phi(\mathbb{S}^n,[0,\rho+1])$, where $c = \sup_{u\in\mathbb{S}^n}\frac{\omega(u)}{\omega(-u)}$.

   Hence,

$$\mathbf{Vol}(\Phi(\mathbb{S}^n,[0,\rho-c])) \leqslant \mathbf{Vol}(B_\rho^{n+1}) \leqslant \mathbf{Vol}(\Phi(\mathbb{S}^n,[0,\rho+1])).$$

2. $|\mathbf{Jac}(\Phi(u,s))| = \omega(u)^{n+1}\tanh^n(s)(1-\tanh^2(s))$.

We improve the first property.

Fix $d > 0$. Consider the difference

$$\rho_t(u) - \omega(u)\tanh(t+d) = \omega(u)\left(1 - \frac{\omega(u)+\omega(-u)}{e^{2t}\omega(-u)+\omega(u)} - \tanh(t+d)\right)$$

$$= \omega(u)\left(\frac{2}{e^{2(t+d)}+1} - \frac{\omega(u)+\omega(-u)}{e^{2t}\omega(-u)+\omega(u)}\right)$$

$$= \rho_t(u) - \omega(u)\tanh(t+d) = \omega(u)e^{-2t}\left(2e^{-2d} - 1 - \frac{\omega(u)}{\omega(-u)}\right) + \bar{o}(e^{-2t}),\ t\to\infty.$$

Thus $B_\rho^{n+1} \subseteq \Phi(\mathbb{S}^n,[0,\rho+d])$ for sufficiently large ρ if

$$2e^{-2d} - 1 - \frac{\omega(u)}{\omega(-u)} \leqslant 0,$$

$$d \geqslant -\frac{1}{2}\ln\left[\frac{1}{2}\left(1+\frac{1}{c}\right)\right] := d_1,$$

and $B_\rho^{n+1} \supseteq \Phi(\mathbb{S}^n,[0,\rho+d])$ for sufficiently large finite ρ if

$$d \leqslant -\frac{1}{2}\ln\left[\frac{1}{2}(1+c)\right] := d_2.$$

Fix the values d_2, d_1 and choose a sufficiently large ρ_0.

Then

$$\mathbf{Vol}(\Phi(\mathbb{S}^n,[0,\rho+d_2])) \leqslant \mathbf{Vol}(B_\rho^{n+1}) \leqslant \mathbf{Vol}(\Phi(\mathbb{S}^n,[0,\rho+d_1])). \tag{11}$$

Notice that if the domain U is centrally symmetric, then $d_1 = d_2 = 0$. In the worst case, when $c \to \infty$, we have $d_1 \to \ln\sqrt{2} \approx 0.347 < 1$. Inclusion (11) is more precise than it was obtained in [13]. It will be essentially used in the proof of Th. 2.

The volume of the set $\Phi(\mathbb{S}^n,[\rho_0,\rho])$ is given by

$$\mathbf{Vol}(\Phi(\mathbb{S}^n,[\rho_0,\rho])) = \mathbf{Vol}_E(\mathbb{B}^n)\int\limits_{\mathbb{S}^n}\int\limits_{\rho_0}^{\rho}\sigma(\Phi(u,s))|\mathbf{Jac}(\Phi(u,s))|dsdu.$$

It is known [13] that

$$|Jac(\Phi(u,s))| = \omega(u)^{n+1}\tanh^n(s)(1-\tanh^2(s)) = \omega(u)^{n+1}\frac{4e^{2s}\left(\frac{e^{2s}-1}{e^{2s}+1}\right)^{n+1}}{e^{4s}-1}.$$

And, using the estimates (10), we obtain

$$\int\limits_{\mathbb{S}^n}\int\limits_{\rho_0}^{\rho}\frac{4\omega(u)^{n+1}\frac{e^{2s}\left(\frac{e^{2s}-1}{e^{2s}+1}\right)^{n+1}}{e^{4s}-1}}{R^{n+1}\left(1-\left(1-\frac{d(\Phi(u,s),\partial U)}{R}\right)^2\right)^{\frac{n+2}{2}}}dsdu \leqslant \mathbf{Vol}(\Phi(\mathbb{S}^n,[\rho_0,\rho])),$$

$$\mathbf{Vol}(\Phi(\mathbb{S}^n,[\rho_0,\rho])) \leqslant \int\limits_{\mathbb{S}^n}\int\limits_{\rho_0}^{\rho}\frac{4\omega(u)^{n+1}\frac{e^{2s}\left(\frac{e^{2s}-1}{e^{2s}+1}\right)^{n+1}}{e^{4s}-1}}{r^{n+1}\left(1-\left(1-\frac{d(\Phi(u,s),\partial U)}{r}\right)^2\right)^{\frac{n+2}{2}}}dsdu.$$

Out next task is to find the asymptotic behavior of the integral

$$\int\limits_0^r\frac{\frac{4e^{2s}\left(\frac{e^{2s}-1}{e^{2s}+1}\right)^{n+1}}{e^{4s}-1}}{(1-(1-Ce^{-2s})^2)^{\frac{n+2}{2}}}ds.$$

After the changing of the variable $y = e^{-2s}$, we obtain the integral

$$\int\limits_{e^{-2r}}^{1}\frac{-8\frac{y^{-2}\left(\frac{y^{-1}-1}{y^{-1}+1}\right)^{n+1}}{y^{-2}-1}}{(1-(1-Cy)^2)^{\frac{n+2}{2}}}dy = \int\limits_{e^{-2r}}^{1}\frac{8(1-y)^{n+1}}{(1+y)^{n+1}(y^2-1)(Cy(2-Cy))^{\frac{n+2}{2}}}dy$$

$$= \int\limits_{e^{-2r}}^{1} \frac{1}{C^{\frac{n+2}{2}} y^{\frac{n+2}{2}} 2^{\frac{n}{2}-2}} \cdot \frac{(1-y)^{-\frac{n}{2}} 2^{\frac{n+2}{2}}}{(1+y)^{n+1}(y^2-1)(2-Cy)^{\frac{n+2}{2}}} dy.$$

Notice that

$$\lim_{y\to 0} \left[\frac{(1-y)^{-\frac{n}{2}} 2^{\frac{n+2}{2}}}{(1+y)^{n+1}(y^2-1)(2-Cy)^{\frac{n+2}{2}}} \right] = -1.$$

Taking into account the above and making the inverse change of variable, we get

$$\int\limits_0^r \frac{\frac{4e^{2s}\left(\frac{e^{2s}-1}{e^{2s}+1}\right)^{n+1}}{e^{4s}-1}}{(1-(1-Ce^{-2s})^2)^{\frac{n+2}{2}}} ds = \frac{1}{nC^{\frac{n+2}{2}} 2^{\frac{n-2}{2}}} e^{nr} + \bar{o}(e^{nr}), \ r \to \infty. \tag{12}$$

The expression for $\mathbf{Vol}_E \left(B^{n+1}_{F_R(p)} \right)$ includes the quantity $d(p,m) = d(p,\partial U)$. Thus we need the estimates of $d(p,m)$ for point $p = \Phi(u,s)$. So,

$$d(\Phi(u,s), \omega(u)u) = \omega(u) - \tanh(s)\omega(u) = \omega(u) - \frac{e^{2s}-1}{e^{2s}+1}\omega(u) = \frac{2\omega(u)}{1+e^{2s}}.$$

Finally,

$$d(\Phi(u,s), \partial U) \leqslant 2\omega(u)e^{-2s} + \bar{o}(e^{-2s}). \tag{13}$$

On the other hand, analogously as formula (6) we get

$$d(\Phi(u,s), \partial U) \geqslant 2\frac{\omega_0}{R}\omega(u)e^{-2s} + \bar{o}(e^{-2s}). \tag{14}$$

Using (12)–(14), one can compute that

$$\frac{1}{n}\mathbf{C}_1 e^{n\rho} + \bar{o}(e^{n\rho}) \leqslant \mathbf{Vol}(\Phi(\mathbb{S}^n, [\rho_0, \rho])) \leqslant \frac{1}{n}\mathbf{C}_2 e^{n\rho} + \bar{o}(e^{n\rho}), \ \rho \to \infty, \tag{15}$$

$$\mathbf{C}_1 = \frac{1}{2^n} \int\limits_{\mathbb{S}^n} \left(\frac{\omega(u)}{R} \right)^{\frac{n}{2}} du,$$

$$\mathbf{C}_2 = \frac{1}{2^n} \frac{R^{\frac{n+2}{2}}}{\omega_0^{\frac{n+2}{2}}} \int\limits_{\mathbb{S}^n} \left(\frac{\omega(u)}{r} \right)^{\frac{n}{2}} du.$$

And, taking into account (11), (15), we have

$$\frac{1}{n}\mathbf{C}_1 e^{nd_2} e^{n\rho} + \bar{o}(e^{n\rho}) \leqslant \mathbf{Vol}(B^{n+1}_\rho) \leqslant \frac{1}{n}\mathbf{C}_2 e^{n\rho} e^{nd_1} + \bar{o}(e^{n\rho}), \ \rho \to \infty. \tag{16}$$

P r o o f o f T h e o r e m 2. It follows from (6), (7), (16) that

$$\lim_{\rho\to\infty}\sup\frac{\mathbf{Vol}(B_\rho^{n+1})}{\mathbf{Vol}(S_\rho^n)} \leqslant \frac{1}{n}\frac{1}{2^{n/2}}e^{nd_1}\frac{R^{\frac{n+2}{2}}}{\omega_0^{\frac{n+2}{2}}}\frac{\int_{\mathbb{S}^n}\left(\frac{\omega(u)}{r}\right)^{\frac{n}{2}}du}{k^{\frac{n}{2}}\int_{\partial U}\left(\omega(\iota(p))\left(\frac{\omega(\iota(p))}{\omega(-\iota(p))}+1\right)\right)^{-\frac{n}{2}}dp}$$

$$\leqslant \frac{1}{n}c^{\frac{n}{2}}\left(\frac{K}{k}\right)^{\frac{n}{2}}\frac{1}{(k\omega_0)^{\frac{n}{2}+1}}\frac{\int_{\mathbb{S}^n}\omega(u)^{\frac{n}{2}}du}{\int_{\partial U}\omega(\iota(p))^{-\frac{n}{2}}dp}$$

$$\leqslant \frac{1}{n}c^{\frac{n}{2}}\left(\frac{K}{k}\right)^{\frac{n}{2}}\frac{\omega_1^n}{(k\omega_0)^{\frac{n}{2}+1}}\frac{\mathbf{Vol}_E(\mathbb{S}^n)}{\mathbf{Vol}_E(\partial U)}.$$

Note that $c \leqslant \frac{\omega_1}{\omega_0}$. Hence

$$\lim_{\rho\to\infty}\sup\frac{\mathbf{Vol}(B_\rho^{n+1})}{\mathbf{Vol}(S_\rho^n)} \leqslant \frac{1}{n}\left(\frac{K}{k}\right)^{\frac{n}{2}}\left(\frac{\omega_1}{\omega_0}\right)^{n+1}\left(\frac{\omega_1}{k}\right)^{\frac{n}{2}}\frac{1}{k\omega_1}\frac{\mathbf{Vol}_E(\mathbb{S}^n)}{\mathbf{Vol}_E(\partial U)},$$

$$\lim_{\rho\to\infty}\inf\frac{\mathbf{Vol}(B_\rho^{n+1})}{\mathbf{Vol}(S_\rho^n)} \geqslant \frac{1}{n}\frac{1}{2^{n/2}}e^{nd_2}\frac{\int_{\mathbb{S}^n}\left(\frac{\omega(u)}{R}\right)^{\frac{n}{2}}du}{K^{\frac{n}{2}}\int_{\partial U}\left(\frac{\omega_0}{R}\omega(\iota(p))\left(\frac{\omega(\iota(p))}{\omega(-\iota(p))}+1\right)\right)^{-\frac{n}{2}}dp}$$

$$\geqslant \frac{1}{n}\frac{1}{c^{\frac{n}{2}}}\left(\frac{k}{K}\right)^{\frac{n}{2}}(k\omega_0)^{\frac{n}{2}}\frac{\int_{\mathbb{S}^n}\omega(u)^{\frac{n}{2}}du}{\int_{\partial U}\omega(\iota(p))^{-\frac{n}{2}}dp}$$

$$\geqslant \frac{1}{n}\frac{1}{c^{\frac{n}{2}}}\left(\frac{k}{K}\right)^{\frac{n}{2}}\omega_0^n(k\omega_0)^{\frac{n}{2}}\frac{\mathbf{Vol}_E(\mathbb{S}^n)}{\mathbf{Vol}_E(\partial U)} \geqslant \frac{1}{n}\left(\frac{k}{K}\right)^{\frac{n}{2}}\left(\frac{\omega_0}{\omega_1}\right)^{\frac{n}{2}}\omega_0^n(k\omega_0)^{\frac{n}{2}}\frac{\mathbf{Vol}_E(\mathbb{S}^n)}{\mathbf{Vol}_E(\partial U)}.$$

And the theorem follows. ■

E x a m p l e 1. Let $U = \mathbb{B}_\rho^{n+1}$. Then we get the Klein model of Lobachevsky space. Applying Theorem 2 to this space, we get

$$\omega(u) = \frac{1}{k} = \frac{1}{K} = r = R = \omega_0 = \rho,$$

$$c = 1,$$

$$\int_{\partial U} du = \rho^n\mathbf{Vol}_E(\mathbb{S}^n).$$

Therefore we have obtained the well-known result

$$\lim_{\rho\to\infty}\frac{\mathbf{Vol}(B_\rho^{n+1})}{\mathbf{Vol}(S_\rho^n)} = \frac{1}{n}.$$

E x a m p l e 2. One should not hope that this result holds for all metrics of negative curvature.

Let U be an open bounded strongly convex domain in $\mathbb{R}^n$, $o = 0 \in \mathbb{R}^n$. Given a point $x \in U$ and a direction $y \in T_xU \backslash \{0\} \simeq U \backslash \{0\}$. The *Funk metric* $F(x,y)$ is a Finsler metric that satisfies the condition

$$x + \frac{y}{F(x,y)} \in \partial U.$$

Then Hilbert metric is a symmetrized Funk metric

$$F_U(x,y) = \frac{1}{2}\left[F(x,y) + F(x,-y)\right].$$

Funk metrics are of constant negative curvature $-1/4$, but for these metrics

$$\lim_{r\to\infty} \frac{\mathbf{Vol}(B_r^{n+1})}{\mathbf{Vol}(S_r^n)} = \infty$$

holds ([5]).

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