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Complete Hypersurfaces in a Real Space Form

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Let M^n be an n-dimensional complete hypersurface with the scalar curvature $n(n-1)R$ and the mean curvature H being linearly related, that is, $n(n-1)R = k'H (k' > 0)$ in a real space form $R^{n+1}(c)$. Assume that the mean curvature is positive and obtains its maximum on M^n. We show that (1) if $c = 1, k' \geq 2n\sqrt{n(n-1)}$, for any i, $\sum_{j\neq i}\lambda_j^2 > n(n-1)$ and $|h|^2 \leq nH^2 + (B_H^+)^2$, then M^n is totally umbilical, or (i) $n \geq 3, M^n$ is locally an $H(r)$-torus with $r^2 < \frac{n-1}{n}$, (ii) $n = 2, M^n$ is locally an $H(r)$-torus with $r^2 \neq \frac{n-1}{n}$; (2) if $c = 0$ and $|h|^2 \leq nH^2 + (\widetilde{B}_H^+)^2$, then M^n is isometric to a standard round sphere, a hyperplane R^n or $S^{n-1}(c_1) \times R^1$; (3) if $c = -1$ and $|h|^2 \leq nH^2 + (\widehat{B}_H^+)^2$, then M^n is totally umbilical or is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$ for some $r > 0$, where $|h|^2$ denotes the squared norm of the second fundamental form of $M^n, B_H^+, \widetilde{B}_H^+$ and $\widehat{B}_H^+$ are denoted by (1.1), (1.2) and (1.3).

Key words: hypersurface, mean curvature, scalar curvature, real space form.

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1. Introduction

Let $R^{n+1}(c)$ be an $(n+1)$-dimensional connected Riemannian manifold with constant sectional curvature c. We also call it a real space form. According to $c > 0$, $c = 0$ or $c < 0$, it is called sphere space, Euclidean space or hyperbolic space, respectively, and it is denoted by $S^{n+1}(c), R^{n+1}$ and $H^{n+1}(c)$. As it is well-known that there are many rigidity results for hypersurfaces with constant mean curvature or with constant scalar curvature in $S^{n+1}(c), R^{n+1}$ and $H^{n+1}(c)$, for example, see [1–3, 5] and [12] etc., but fewer ones are obtained for hypersurfaces

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with the scalar curvature and the mean curvature being linearly related. We know that an $H(r)$-torus in a unit sphere $S^{n+1}(1)$ is the product immersion $S^{n-1}(r) \times S^1(\sqrt{1-r^2}) \hookrightarrow R^n \times R^2$, where $S^{n-1}(r) \subset R^n, S^1(\sqrt{1-r^2}) \subset R^2$, $0 < r < 1$, are standard immersions. In some orientation, $H(r)$-torus has principal curvatures given by $\lambda_1 = \cdots = \lambda_{n-1} = \frac{\sqrt{1-r^2}}{r}$ and $\lambda_n = -\frac{r}{\sqrt{1-r^2}}$.

In [12], the authors obtained the following:

Theorem 1.1. *Let M^n be an n-dimensional complete hypersurface with constant mean curvature H in a unit sphere $S^{n+1}(1)$. (1) If $|h|^2 < D'(n,H)$, then M^n is totally umbilical. (2) If $|h|^2 = D'(n,H)$, then (i) when $H = 0, M^n$ is locally a Clifford torus; (ii) when $H \neq 0, n \geq 3, M^n$ is locally an $H(r)$-torus with $r^2 < \frac{n-1}{n}$; (iii) when $H \neq 0, n = 2, M^n$ is locally an $H(r)$-torus with $r^2 \neq \frac{n-1}{n}$, where*

$$D'(n,H) = n + \frac{n^3H^2}{2(n-1)} - \frac{(n-2)nH}{2(n-1)}[n^2H^2 + 4(n-1)]^{\frac{1}{2}}.$$

In [6], S.Y. Cheng and S.T. Yau obtained the following:

Theorem 1.2. *Let M^n be a complete hypersurface with constant mean curvature in R^{n+1}. If the sectional curvature of M^n is nonnegative, then M^n is isometric to a standard round sphere, a hyperplane R^n or a Riemannian product $S^k(c_1) \times R^{n-k}$, $1 \leq k \leq n-1$.*

In this paper, we study the hypersurfaces in a real space form $R^{n+1}(c)$ with scalar curvature $n(n-1)R$ and the mean curvature H being linearly related. We obtain the following theorems:

Theorem 1.3. *Let M^n be an n-dimensional complete hypersurface with $n(n-1)R = k'H$ in a unit sphere $S^{n+1}(1)$, where $k'(\geq 2n\sqrt{n(n-1)})$ is a positive constant. Assume that the mean curvature H is positive and obtains its maximum on M^n and for any i, $\sum_{j\neq i}\lambda_j^2 > n(n-1)$, where $\lambda_j (j = 1,\ldots,i-1,i+1,\ldots,n)$ are the principal curvatures on M^n. If the squared norm of the second fundamental form $|h|^2$ satisfies*

$$|h|^2 \leq nH^2 + (B_H^+)^2$$

on M^n, then M^n is totally umbilical, or (i) $n \geq 3, M^n$ is locally an $H(r)$-torus with $r^2 < \frac{n-1}{n}$; (ii) $n = 2, M^n$ is locally an $H(r)$-torus with $r^2 \neq \frac{n-1}{n}$, where

$$B_H^+ = -\frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}H + \sqrt{\frac{n^3H^2}{4(n-1)} + n}. \tag{1.1}$$

Theorem 1.4. *Let M^n be an n-dimensional complete hypersurface with $n(n-1)R = k'H$ in a Euclidean space R^{n+1}, where k' is a positive constant. Assume* *that the mean curvature H is positive and obtains its maximum on M^n. If the squared norm of the second fundamental form $|h|^2$ satisfies*

$$|h|^2 \le nH^2 + (\widetilde{B}_H^+)^2$$

on M^n, then M^n is isometric to a standard round sphere, a hyperplane R^n or a Riemannian product $S^{n-1}(c_1) \times R^1$, where

$$\widetilde{B}_H^+ = \sqrt{\frac{n}{n-1}}H. \tag{1.2}$$

Theorem 1.5. *Let M^n be an n-dimensional complete hypersurface with $n(n-1)R = k'H$ in a hyperbolic space $H^{n+1}(-1)$, where k' is a positive constant. Assume that the mean curvature H is positive and obtains its maximum on M^n. If the squared norm of the second fundamental form $|h|^2$ satisfies*

$$|h|^2 \le nH^2 + (\widehat{B}_H^+)^2$$

on M^n, then M^n is totally umbilical or is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$ for some $r > 0$, where

$$\widehat{B}_H^+ = -\frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}H + \sqrt{\frac{n^3H^2}{4(n-1)} - n}, \quad (n^2H^2 \ge 4(n-1)). \tag{1.3}$$

2. Preliminaries

Let M^n be an n-dimensional hypersurface in $R^{n+1}(c)$. For any $p \in M^n$ we choose a local orthonormal frame $e_1, \dots, e_n, e_{n+1}$ in $R^{n+1}(c)$ around p such that $e_1, \dots, e_n$ are tangential to M^n. Take the corresponding dual co-frame $\{\omega_1, \dots, \omega_n, \omega_{n+1}\}$. In this paper we make the following convention on the range of indices,

$$1 \le A, B, C \dots \le n+1; \quad 1 \le i, j, k, \dots \le n.$$

The structure equations of $R^{n+1}(c)$ are

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} = -\omega_{BA},$$
$$d\omega_{AB} = \sum_c \omega_{AC} \wedge \omega_{CB} - c\omega_A \wedge \omega_B.$$

If we denote by the same letters the restrictions of ω_A, ω_{AB} to M^n, we have

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = -\omega_{ji}, \tag{2.1}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.2}$$

where R_{ijkl} is the curvature tensor of the induced metric on M^n.

Restricted to $M^n, \omega_{n+1} = 0$, thus

$$0 = d\omega_{n+1} = \sum_i \omega_{n+1i} \wedge \omega_i, \tag{2.3}$$

and by Cartan's lemma we can write

$$\omega_{in+1} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.4}$$

The quadratic form $h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$ is the second fundamental form of M^n. The Gauss equation is

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{2.5}$$

$$n(n-1)R = n(n-1)c + n^2H^2 - |h|^2, \tag{2.6}$$

where R is the normalized scalar curvature, $H = (1/n)\sum_i h_{ii}$ the mean curvature and $|h|^2 = \sum_{i,j} h_{ij}^2$ the squared norm of the second fundamental form of M^n, respectively.

The Codazzi equation and Ricci identity are

$$h_{ijk} = h_{ikj}, \tag{2.7}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}, \tag{2.8}$$

where the first and the second covariant derivatives of the second fundamental form are defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}, \tag{2.9}$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mjk}\omega_{mi} + \sum_m h_{imk}\omega_{mj} + \sum_m h_{ijm}\omega_{mk}. \tag{2.10}$$

In order to represent our theorems, we need some notations, for details see H.B. Lawson [9] and P.J. Ryan [11]. First we give a description of the real hyperbolic space $H^{n+1}(c)$ of constant curvature $c(< 0)$.

For any two vectors x and y in R^{n+2}, we set

$$g(x, y) = x_1y_1 + \ldots + x_{n+1}y_{n+1} - x_{n+2}y_{n+2},$$

(R^{n+2}, g) is the so-called Minkowski space-time. Denote $\rho = \sqrt{-1/c}$. We define

$$H^{n+1}(c) = \{x \in R^{n+2} \mid g(x,x) = -\rho^2, x_{n+2} > 0\}.$$

Then $H^{n+1}(c)$ is a simply-connected hypersurface of R^{n+2}. Hence, we obtain a model of a real hyperbolic space.

We define

$$\begin{aligned} M_1 &= \{x \in H^{n+1}(c) \mid x_1 = 0\}, \\ M_2 &= \{x \in H^{n+1}(c) \mid x_1 = r > 0\}, \\ M_3 &= \{x \in H^{n+1}(c) \mid x_{n+2} = x_{n+1} + \rho\}, \\ M_4 &= \{x \in H^{n+1}(c) \mid x_1^2 + \ldots + x_{n+1}^2 = r^2 > 0\}, \\ M_5 &= \{x \in H^{n+1}(c) \mid x_1^2 + \ldots + x_{k+1}^2 = r^2 > 0, \\ &\qquad x_{k+2}^2 + \ldots + x_{n+1}^2 - x_{n+2}^2 = -\rho^2 - r^2\}. \end{aligned}$$

$M_1, \ldots, M_5$ are often called the standard examples of complete hypersurfaces in $H^{n+1}(c)$ with at most two distinct constant principal curvatures. It is obvious that $M_1, \ldots, M_4$ are totally umbilical. In the sense of Chen [7], they are called the hyperspheres of $H^{n+1}(c)$. M_3 is called the horosphere and M_4 — the geodesic distance sphere of $H^{n+1}(c)$. P.J. Ryan [11] obtained the following:

Lemma 2.1([11]). *Let M^n be a complete hypersurface in $H^{n+1}(c)$. Suppose that, under a suitable choice of a local orthonormal tangent frame field of TM^n, the shape operator over TM^n is expressed as a matrix A. If M^n has at most two distinct constant principal curvatures, then it is congruent to one of the following:*

(1) M_1. In this case, $A = 0$, and M_1 is totally geodesic. Hence M_1 is isometric to $H^n(c)$.

(2) M_2. In this case, $A = \frac{1/\rho^2}{\sqrt{1/\rho^2+1/r^2}} I_n$, where I_n denotes the identity matrix of degree n, and M_2 is isometric to $H^n(-1/(r^2+\rho^2))$.

(3) M_3. In this case, $A = \frac{1}{\rho} I_n$, and M_3 is isometric to a Euclidean space R^n.

(4) M_4. In this case, $A = \sqrt{1/r^2 + 1/\rho^2} I_n, M_4$ is isometric to a round sphere $S^n(r)$ of radius r.

(5) M_5. In this case, $A = \lambda I_k \oplus \mu I_{n-k}$, where $\lambda = \sqrt{1/\rho^2 + 1/r^2}$, and $\mu = \frac{1/\rho^2}{\sqrt{1/r^2+1/\rho^2}}, M_5$ is isometric to $S^k(r) \times H^{n-k}(-1/(r^2+\rho^2))$.

We also need the following algebraic Lemma due to [10] and [1].

Lemma 2.2([10, 1]). *Let $\mu_i, i = 1, \ldots, n$ be real numbers, with $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2 \geq 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{2.11}$$

and equality holds if and only if either $(n-1)$ *of the numbers* μ_i *are equal to* $\beta/\sqrt{n(n-1)}$ *or* $(n-1)$ *of the numbers* μ_i *are equal to* $-\beta/\sqrt{n(n-1)}$.

3. Proof of Theorems

In order to prove our theorems, we introduce an operator $\Box$ due to S.Y. Cheng and S.T. Yau [5] by

$$\Box f = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}, \tag{3.1}$$

where f is a C^2-function on M^n, the gradient and Hessian (f_{ij}) are defined by

$$df = \sum_i f_i\omega_i, \quad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}. \tag{3.2}$$

The Laplacian of f is defined by $\Delta f = \sum_i f_{ii}$.

We choose a local frame field $e_1, \dots, e_n$ at each point of M^n, such that $h_{ij} = \lambda_i\delta_{ij}$. From (3.1) and (2.6), we have

$$\begin{aligned}\Box(nH) &= nH\Delta(nH) - \sum_i \lambda_i(nH)_{ii}\\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i \lambda_i(nH)_{ii}\\ &= \frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta|h|^2 - n^2|\nabla H|^2 - \sum_i \lambda_i(nH)_{ii}.\end{aligned} \tag{3.3}$$

From (2.7) and (2.8), by a standard and direct calculation, we have

$$\frac{1}{2}\Delta|h|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i(nH)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2, \tag{3.4}$$

where $R_{ijij} = c + \lambda_i\lambda_j (i \neq j)$ denotes the sectional curvature of the section spanned by $\{e_i, e_j\}$.

From (3.3) and (3.4), we get

$$\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j}(c + \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2. \tag{3.5}$$

By making use of the similar method in [4], we can prove the following:

Proposition 3.1. *Let* M^n *be an* n*-dimensional hypersurface in a real space form* $R^{n+1}(c)$ *with* $n(n-1)R = k'H$, $k' = constant > 0$. *Assume that the mean curvature* $H > 0$. *Then we have the operator*

$$L = \Box - (k'/2n)\Delta$$

(1) if $c > 0$ and for any i, $\sum_{j \neq i} \lambda_j^2 > n(n-1)c, L$ is elliptic;
(2) if $c \leq 0, L$ is elliptic.

P r o o f. We choose an orthonormal frame field $\{e_j\}$ at each point in M^n so that $h_{ij} = \lambda_i \delta_{ij}$. For any i,

$$\begin{aligned}
(nH - \lambda_i - k'/2n) &= \sum_j \lambda_j - \lambda_i - (1/2)[-\sum_j \lambda_j^2 + n^2H^2 + n(n-1)c]/(nH) \\
&= [(\sum_j \lambda_j)^2 - \lambda_i \sum_j \lambda_j - (1/2)\sum_{l \neq j} \lambda_l \lambda_j - (1/2)n(n-1)c](nH)^{-1} \\
&= [\sum_j \lambda_j^2 + (1/2)\sum_{l \neq j} \lambda_l \lambda_j - \lambda_i \sum_j \lambda_j - (1/2)n(n-1)c](nH)^{-1} \\
&= [\sum_{i \neq j} \lambda_j^2 + (1/2)\sum_{\substack{l \neq j \\ l,j \neq i}} \lambda_l \lambda_j - (1/2)n(n-1)c](nH)^{-1} \\
&= (1/2)[\sum_{j \neq i} \lambda_j^2 + (\sum_{j \neq i} \lambda_j)^2 - n(n-1)c](nH)^{-1}. \qquad (3.6)
\end{aligned}$$

(1) If $c > 0$ and for any i, $\sum_{j \neq i} \lambda_j^2 > n(n-1)c$, from (3.6), we have

$$(nH - \lambda_i - k'/2n) \geq (1/2)[\sum_{j \neq i} \lambda_j^2 - n(n-1)c](nH)^{-1} > 0.$$

Therefore, we know that L is an elliptic operator.
(2) If $c \leq 0$, from (3.6) again, we have

$$(nH - \lambda_i - k'/2n) > 0.$$

Therefore, we also know that L is an elliptic operator. This completes the proof of Prop. 3.1.

We can also prove the following:

Proposition 3.2. *Let M^n be an n-dimensional hypersurface in a real space form $R^{n+1}(c)$ with $n(n-1)R = k'H$, $k' =$ constant > 0. Assume that the mean curvature $H > 0$. Then we have:*
(1) if $c > 0$ and $k' \geq 2n\sqrt{n(n-1)c}$, then

$$|\nabla h|^2 \geq n^2 |\nabla H|^2;$$

(2) if $c \leq 0$, for all $k' > 0$, then

$$|\nabla h|^2 \geq n^2 |\nabla H|^2.$$

P r o o f. Since $H > 0$, we have $|h|^2 \neq 0$. In fact, if $|h|^2 = \sum_i \lambda_i^2 = 0$ at a point of M^n, then $\lambda_i = 0$, $i = 1, 2, \ldots, n$, at this point. Therefore $H = 0$ at this point. This is impossible.

From (2.6) and $n(n-1)R = k'H$, we have

$$k'\nabla_i H = 2n^2 H\nabla_i H - 2\sum_{j,k} h_{kj}h_{kji},$$
$$(\tfrac{1}{2}k' - n^2 H)\nabla_i H = -\sum_{j,k} h_{kj}h_{kji},$$
$$(\tfrac{1}{2}k' - n^2 H)^2|\nabla H|^2 = \sum_i(\sum_{j,k} h_{kj}h_{kji})^2 \leq \sum_{i,j} h_{ij}^2 \sum_{i,j,k} h_{ijk}^2 = |h|^2|\nabla h|^2.$$

Therefore, we have

$$\begin{aligned}|\nabla h|^2 - n^2|\nabla H|^2 &\geq [(\frac{k'}{2} - n^2 H)^2 - n^2|h|^2]|\nabla H|^2 \frac{1}{|h|^2}\\ &= [\frac{(k')^2}{4} - n^3(n-1)c]|\nabla H|^2 \frac{1}{|h|^2}.\end{aligned} \tag{3.7}$$

(1)If $c > 0$ and $k' \geq 2n\sqrt{n(n-1)c}$, from (3.7), we have

$$|\nabla h|^2 - n^2|\nabla H|^2 \geq 0.$$

(2) If $c \leq 0$, from (3.7), we also have

$$|\nabla h|^2 - n^2|\nabla H|^2 \geq 0.$$

This completes the proof of Prop. 3.2.

Proposition 3.3. *Let M^n be an n-dimensional hypersurface in a real space form $R^{n+1}(c)$ with $n(n-1)R = k'H$, $k' = constant > 0$. Then we have*

$$nLH \geq (|\nabla h|^2 - n^2|\nabla H|^2) + |g|^2\{nc + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||g| - |g|^2\},$$

where $|g|^2$ is a nonnegative C^2-function on M^n defined by $|g|^2 = |h|^2 - nH^2$.

P r o o f. From (3.5) we have

$$\begin{aligned}nLH &= n[\Box H - (k'/2n)\Delta H]\\ &= \Box(nH) - (1/2)\Delta[n(n-1)R]\\ &= |\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j}(c + \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2\\ &= |\nabla h|^2 - n^2|\nabla H|^2 + nc|h|^2 - n^2H^2c - |h|^4 + nH\sum_i \lambda_i^3.\end{aligned} \tag{3.8}$$

Let $|g|^2$ be a nonnegative C^2-function on M^n defined by $|g|^2 = |h|^2 - nH^2$. Since $\sum_i (H - \lambda_i) = 0$, $\sum_i (H - \lambda_i)^2 = |h|^2 - nH^2 = |g|^2$, by Lem. 2.2 we get

$$\begin{aligned} nH\sum_i \lambda_i^3 &= 3nH^2|h|^2 - 2n^2H^4 - nH\sum_i (H-\lambda_i)^3 \\ &\geq 3nH^2|g|^2 + n^2H^4 - n|H|\frac{n-2}{\sqrt{n(n-1)}}|g|^3. \end{aligned} \tag{3.9}$$

Therefore, from (3.8) and (3.9), we have

$$nLH \geq |\nabla h|^2 - n^2|\nabla H|^2 + |g|^2\{nc + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||g| - |g|^2\}.$$

This completes the proof of Prop. 3.3.

P r o o f o f T h e o r e m 1.3. From the assumption of Th. 1.3, Prop. 3.2 and Prop. 3.3 for $c = 1$, we have

$$nLH \geq |g|^2\{n + nH^2 - \tfrac{n(n-2)}{\sqrt{n(n-1)}}H|g| - |g|^2\} = |g|^2 P_H(|g|), \tag{3.10}$$

where

$$P_H(|g|) = n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|g| - |g|^2.$$

$P_H(|g|)$ has two real roots B_H^- and B_H^+ given by

$$B_H^{\pm} = -\frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}H \pm \sqrt{\frac{n^3H^2}{4(n-1)} + n}.$$

Therefore, we know that

$$P_H(|g|) = (|g| - B_H^-)(-|g| + B_H^+).$$

Clearly, we know that $|g| - B_H^- > 0$. From the assumption of Th. 1.3, we infer that $P_H(|g|) \geq 0$ on M^n. This implies that the right-hand side of (3.10) is nonnegative. Since, from Prop. 3.1, the operator L is elliptic, and H obtains its maximum on M^n, from (3.10) we know that $H = const$ on M^n. Therefore, we know that M^n is an n-dimensional complete hypersurface with constant mean curvature $H(> 0)$ in a unit sphere $S^{n+1}(1)$. By the assumption of Th. 1.3 and the result of Th. 1.1, we can check directly that $|h|^2 \leq nH^2 + (B_H^+)^2 = n + \frac{n^3H^2}{2(n-1)} - \frac{(n-2)nH}{2(n-1)}[n^2H^2 + 4(n-1)]^{\frac{1}{2}} = D'(n, H)$. Therefore we have either M^n is totally umbilical, or (i) $n \geq 3, M^n$ is locally an $H(r)$-torus with $r^2 < \frac{n-1}{n}$; (ii) $n = 2, M^n$ is locally an $H(r)$-torus with $r^2 \neq \frac{n-1}{n}$. This completes the proof of Th. 1.3.

P r o o f o f T h e o r e m 1.4. From the assumption of Th. 1.4, Prop. 3.2 and Prop. 3.3, for $c = 0$, we have

$$nLH \geq |g|^2\{nH^2 - \tfrac{n(n-2)}{\sqrt{n(n-1)}}H|g| - |g|^2\} = |g|^2 Q_H(|g|), \tag{3.11}$$

where

$$Q_H(|g|) = nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|g| - |g|^2.$$

$Q_H(|g|)$ has two real roots $\widetilde{B}_H^-$ and $\widetilde{B}_H^+$ given by

$$\widetilde{B}_H^- = -(n-1)\sqrt{\frac{n}{n-1}}H, \quad \widetilde{B}_H^+ = \sqrt{\frac{n}{n-1}}H.$$

Therefore, we know that

$$Q_H(|g|) = (|g| - \widetilde{B}_H^-)(-|g| + \widetilde{B}_H^+).$$

Clearly, we know that $|g| - \widetilde{B}_H^- > 0$. From the assumption of Th. 1.4, we infer that $Q_H(|g|) \geq 0$ on M^n. This implies that the right-hand side of (3.11) is nonnegative. From Proposition 3.1, we know that L is elliptic, and H obtains its maximum on M^n. From (3.11), we have $H = const$ on M^n. From (3.11) again, we get $|g|^2 Q_H(|g|) = 0$. We infer that the equality holds in Lem. 2.2. Therefore, we know that $(n-1)$ of the numbers $H - \lambda_i$ are equal to $|g|/\sqrt{n(n-1)}$. This implies that M^n has $(n-1)$ principal curvatures equal and constant. As H is constant, the other principal curvature is constant as well. From an inequality of Chen–Okumura [8], we know that $|h|^2 \leq n^2H^2/(n-1)$ implies that the sectional curvature K of M^n is nonnegative. Therefore, we know that M^n is a complete hypersurface in R^{n+1} with constant mean curvature and nonnegative sectional curvature. From Theorem 1.2, we have either M^n is isometric to a standard round sphere, a hyperplane R^n or a Riemannian product $S^{n-1}(c_1) \times R^1$. This completes the proof of Th. 1.4.

P r o o f o f T h e o r e m 1.5. From the assumption of Th. 1.5, Prop. 3.2 and Prop. 3.3, for $c = -1$, we have

$$nLH \geq |g|^2\{-n + nH^2 - \tfrac{n(n-2)}{\sqrt{n(n-1)}}H|g| - |g|^2\} = |g|^2 R_H(|g|), \tag{3.12}$$

where

$$R_H(|g|) = -n + nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|g| - |g|^2.$$

$R_H(|g|)$ has two real roots $\widehat{B}_H^-$ and $\widehat{B}_H^+$ given by

$$\widehat{B}_H^{\pm} = -\frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}H \pm \sqrt{\frac{n^3H^2}{4(n-1)} - n}, \quad n^2H^2 \geq 4(n-1).$$

Therefore, we know that

$$R_H(|g|) = (|g| - \widehat{B}_H^-)(-|g| + \widehat{B}_H^+).$$

Clearly, we know that $|g| - \widehat{B}_H^- > 0$. From the assumption of Th. 1.5, we infer that $R_H(|g|) \geq 0$ on M^n. This implies that the right-hand side of (3.12) is nonnegative. From Proposition 3.1, we know that L is elliptic. Since H obtains its maximum on M^n, from (3.12), we have $H = const$ on M^n. From (3.12) again, we get $|g|^2 R_H(|g|) = 0$, so $|g|^2 = 0$, and M^n is totally umbilical, or $R_H(|g|) = 0$. In the latter case, we know that $(n-1)$ of the numbers $H - \lambda_i$ are equal to $|g|/\sqrt{n(n-1)}$. This implies that M^n has $(n-1)$ principal curvatures equal and constant. As H is constant, the other principal curvature is constant as well, so M^n is isoparametric. From the result of Lem. 2.1, M^n is isometric to $S^{n-1}(r) \times H^1(-1/(r^2+1))$ for some $r > 0$. This completes the proof of Th. 1.5.

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