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A Wegner Estimate for Multi-Particle Random Hamiltonians

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We prove a Wegner estimate for a large class of multi-particle Anderson Hamiltonians on the lattice. These estimates will allow us to prove Anderson localization for such systems. A detailed proof of localization will be given in a subsequent paper.

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Dedicated to V. Marchenko and L. Pastur

1. Introduction

Wegner estimates originate in the famous paper [10]. There, Wegner proved among other things that the integrated density of states for the Anderson Hamiltonian has a bounded density provided the probability distribution of the random potential itself has a bounded density. This implies in particular an upper bound on the probability that an Anderson Hamiltonian on a finite box has eigenvalues close to a given energy E.

Wegner's estimates play a key role in the multiscale method to prove Anderson localization (see, e.g. [5] or [4]). Only recently Bourgain and Kenig [1] proved Anderson localization for a Bernoulli model without an *a priori* Wegner estimate; they prove a Wegner-type estimate inductively within the multiscale scheme.

Wegner's original work was restricted to lattice models. However, the estimate was also proven for the continuum (see [3] for a recent rather optimal result and [9] for a review on this subject).

In this note we prove a Wegner estimate for a multi-particle Anderson model. In a subsequent paper we will also do multiscale analysis for this model. The first

Wegner estimate for a multi-particle random Hamiltonian was proved by Zenk [11]. Chulaevsky and Suhov [2] develope a multiscale analysis for certain (1-d) two-body Hamiltonians. In this paper the Wegner estimate requires strong conditions on the probability density of the random potential (e.g. analyticity). It was one of the motivations of the present note to avoid these strong assumptions.

The method of proof applied here is close to Wegner's original idea and was developed from the paper [6]. Note that there is a refinement of this method by Stollmann [8] which is likely to work in the multi-particle case as well.

We note that the method presented in this paper will also work for alloy-type models in the continuous case. The necessary changes can be read off from the paper [6]. However, in the continuous case we get the volume factor of the bound with an exponent 2. This suffices to do a multiscale analysis, but it gives no result for the regularity of the integrated density of states.

2. Models and Results

We will deal with a system of N interacting particles on a lattice $\mathbb{Z}^d$. We consider these particles on the full Hilbert space, disregarding Fermionic or Bosonic symmetry. Physically speaking we deal with distinguishable particles. Since the full Hilbert space is a direct sum of the irreducible subspaces with respect to S_N-symmetry (including the totally symmetric and the totally antisymmetric subspaces) the Wegner estimates for Fermions and Bosons follow immediately from the result on the full space.

The one-particle Hilbert space we consider is $\ell^2(\mathbb{Z}^d)$ and the Hilbert space for N (distinguishable) particles is consequently $\ell^2(\mathbb{Z}^{Nd})$. Any (bounded) operator A on these Hilbert spaces is uniquely defined through its matrix elements $A(x,y) = (\delta_x, A\,\delta_y)$, where δ_z is the vector in ℓ^2 with component 1 at lattice site z and 0 otherwise.

We write the lattice site $x \in \mathbb{Z}^{Nd}$ as $x = (x_1, \ldots, x_N)$, where $x_i \in \mathbb{Z}^d$ denotes the coordinates of the i^{th} particle.

Each particle (with coordinates ξ) is the subject to a random potential $v_\omega(\xi)$ which is the same for all particles. The random potential $v_\omega(\xi)$ consists of independent identically distributed random variables. Throughout we assume that the distribution of the $v(\xi)$ has a bounded density $\rho(v)$. We denote the underlying probability measure by $\mathbb{P}$ and the expectation with respect to $\mathbb{P}$ by $\mathbb{E}$.

The kinetic energy operator for one particle is given by

$$h_0\, u(\xi) \;=\; \sum_{|n|=1,\; n\in\mathbb{Z}^d} u(\xi+n), \qquad \xi \in \mathbb{Z}^d, \tag{2.1}$$

the single particle Hamiltonian is consequently

$$h_\omega \;=\; h_0 \;+\; v_\omega. \tag{2.2}$$

If h is a one-particle operator acting in $\ell^2(\mathbb{Z}^d)$, we denote by $h^{(i)}$ the corresponding operator on $\ell^2(\mathbb{Z}^{Nd})$ acting on the i^{th} particle only, more precisely: if h has matrix elements $h(\xi,\eta)$, then

$$h^{(i)}\, u(x_1,\dots,x_n) \;=\; \sum_{\eta\in\mathbb{Z}^d} h(x_i,\eta)\, u(x_1,\dots,x_{i-1},\eta,x_{i+1},\dots,x_N). \tag{2.3}$$

In other words,

$$h^{(i)} \;= \underbrace{\mathbf{1}_{\ell^2(\mathbb{Z}^d)}\otimes\dots\otimes\mathbf{1}_{\ell^2(\mathbb{Z}^d)}}_{i-1 \text{ times}}\otimes\; h\; \otimes \underbrace{\mathbf{1}_{\ell^2(\mathbb{Z}^d)}\otimes\dots\otimes\mathbf{1}_{\ell^2(\mathbb{Z}^d)}}_{n-i-1 \text{ times}}. \tag{2.4}$$

The N-particle Hamiltonian without interaction is defined by

$$H_{\omega,0} \;=\; \sum_{i=1}^{N} h^{(i)}. \tag{2.5}$$

The interaction term U can be a rather general function on $\mathbb{Z}^{Nd}$. We assume it to be bounded for simplicity. We also suppose that U is a deterministic function, it would be sufficient for our purpose to have U independent of v_ω. In most cases U is a pair potential of the form $U(x) = \sum_{i\neq j} u(x_i - x_j)$.

The N-particle Hamiltonian with interaction $U$is then given by

$$H_{\omega,U} \;=\; H_{\omega,0} \;+\; U. \tag{2.6}$$

We will deal with this operator restricted to a bounded (hence finite) domain Λ. The number of elements of Λ will be denoted by $|\,\Lambda\,|$.

We call a subset R of $\mathbb{Z}^d$ a rectangle if

$$R = \{\xi\in\mathbb{Z}^d \mid L_\nu \le \xi_\nu \le M_\nu \;\text{ for } \nu = 1\dots N\,\}. \tag{2.7}$$

A rectangular domain in $\mathbb{Z}^{Nd}$ is a set Λ of the form

$$\Lambda \;=\; \Lambda_1\,\times\,\Lambda_2\times\dots\times\Lambda_N, \tag{2.8}$$

where the Λ_i are rectangles in $\mathbb{Z}^d$. We use the notation $\Pi_i(\Lambda) = \Lambda_i$. We call a rectangular domain Λ *regular* if for all $i,j = 1,\dots,N$ either $\Lambda_i\cap\Lambda_j = \emptyset$ or $\Lambda_i = \Lambda_j$.

For any subset Λ of $\mathbb{Z}^{Nd}$ we define the operator $H^\Lambda = H^\Lambda_{\omega,U}$ by its matrix elements

$$H^\Lambda\,(x,y) \;=\; H_{\omega,U}(x,y) \qquad\qquad \text{for } x,y\in\Lambda. \tag{2.9}$$

The main result of this note is the following Wegner estimate for multi-particle operators:

Theorem 2.1. *If Λ is a regular rectangular domain, then*

$$\mathbb{P}\Big(\operatorname{dist}\left(\sigma\left(H^{\Lambda}\right), E\right) < \kappa\Big) \leq C\, ||\,\rho\,||_{\infty}\, |\,\Lambda\,|\, \kappa. \tag{2.10}$$

The assumption of regularity of the set Λ can be avoided. However, the proof is more transparent with this assumption. The proof of Anderson localization by multiscale analysis, which we will present in a forthcoming paper, will deal with regular domains only.

The proof of Theorem 2.1 implies also that the integrated density of states has a bounded density. This result can also be read off from the explicitly known form of the integrated density of states (see [7]).

3. Proof

We prove Theorem 2.1. Let $\Lambda = \Lambda_1 \times \Lambda_2 \cdots \times \Lambda_N$. We may assume that

$$\Lambda_1 = \Lambda_2 = \cdots = \Lambda_K \tag{3.1}$$

$$\text{and} \qquad \Lambda_1 \cap \Lambda_i = \emptyset \quad \text{for all } i > K. \tag{3.2}$$

We denote the eigenvalues of H^{Λ} by $E_n = E_n(H^{\Lambda})$. We order them so that $E_1 \leq E_2 \leq \ldots$ and repeat any eigenvalue according to its multiplicity. The eigenvalue counting function is denoted by

$$N(H^{\Lambda}, E) = \#\{E_n(H^{\Lambda}) \leq E\}. \tag{3.3}$$

We will need the following Lemma:

Lemma 3.1. *Suppose (3.1) and (3.2) hold. Denote by $v(\xi)$ the value of the random potential v_ω evaluated at the lattice site $\xi \in \mathbb{Z}^d$. Then*

$$\sum_{\xi \in \Lambda_1} \frac{\partial E_n(H^{\Lambda})}{\partial v(\xi)} = K. \tag{3.4}$$

P r o o f. Set $V(x) = \sum_{i=1}^{N} v(x_i)$ Then for $\xi \in \Lambda_1$:

$$\frac{\partial V}{\partial v(\xi)}(x_1, \ldots, x_N) = \sum_{i=1}^{K} \delta_{\xi\, x_i}. \tag{3.5}$$

Hence for each $(x_1, \ldots, x_N) \in \Lambda$ we have

$$\sum_{\xi \in \Lambda_1} \frac{\partial V}{\partial v(\xi)}(x_1, \ldots, x_N) = \sum_{\xi \in \Lambda_1} \sum_{i=1}^{K} \delta_{\xi\, x_i} = K. \tag{3.6}$$

Let us denote by ψ_n the normalized eigenfunction of H^Λ for the eigenvalue $E_n = E_n(H^\Lambda)$. The Feynman–Hellman theorem tells us that

$$\frac{\partial E_n}{\partial v(\xi)} \;=\; \langle\, \psi_n,\, \frac{\partial V}{\partial v(\xi)}\, \psi_n\, \rangle \;=\; \sum_{x\in\Lambda} |\, \psi_n(x)\,|^2\, \frac{\partial V}{\partial v(\xi)}. \tag{3.7}$$

Thus from (3.6) we obtain

$$\sum_{\xi\in\Lambda_1} \frac{\partial E_n}{\partial v(\xi)} \;=\; \sum_{x\in\Lambda} |\, \psi_n(x)\,|^2\, \big(\sum_{\xi\in\Lambda_1} \frac{\partial V(x)}{\partial v(\xi)}\, \big) \tag{3.8}$$

$$= \; K \sum_{x\in\Lambda} |\, \psi_n(x)\,|^2 \; = K \tag{3.9}$$

since ψ_n is normalized. ■

Let φ be an increasing C^∞−function on $\mathbb{R}$, $0 \le \varphi \le 1$ with $\varphi = 1$ on (κ, ∞) and $\varphi = 0$ on $(-\infty, -\kappa)$.

Then:

$$\mathbb{P}\Big(\, \mathrm{dist}\,\big(\,\sigma\,\big(H^\Lambda\big)\,,\,E\,\big)\; < \kappa\,\Big) \tag{3.10}$$

$$\le\; \mathbb{E}\Big(\; N(H^\Lambda, E+\kappa) \;-\; N(H^\Lambda, E-\kappa)\Big) \tag{3.11}$$

$$=\; \mathbb{E}\Big(\;\mathrm{tr}\;\big(\chi_{(E-\kappa,E+\kappa]}(H^\Lambda)\,\big)\Big) \tag{3.12}$$

$$\le\; \mathbb{E}\Bigg(\;\mathrm{tr}\;\Big(\varphi(H^\Lambda\; - E\, + 2\kappa) \;-\varphi\,(H^\Lambda\; - E\; - 2\kappa\Big)\Bigg) \tag{3.13}$$

$$\le\; \mathbb{E}\Big(\;\int_{-2\kappa}^{2\kappa}\,\mathrm{tr}\;\Big(\,\varphi'(H^\Lambda\; - E\; + t)\Big)\, dt\Big) \tag{3.14}$$

by Lemma 3.1:

$$\le\; \frac{1}{K}\sum_n \int_{-2\kappa}^{2\kappa} \mathbb{E}\Big(\;\varphi'\Big(E_n(H^\Lambda)\; - E\; + t\Big) \sum_{\xi\in\Lambda_1} \frac{\partial E_n(H^\Lambda)}{\partial v(\xi)}\Big)\, dt \tag{3.15}$$

$$\le\; \sum_n \int_{-2\kappa}^{2\kappa} \sum_{\xi\in\Lambda_1} \mathbb{E}\Big(\; \frac{\partial\,\varphi\Big(E_n(H^\Lambda)\; - E\; + t\Big)}{\partial v(\xi)}\;\Big)\, dt. \tag{3.16}$$

Since $\mathbb{E}$ is a product measure, we can split it into an integration over $v(\xi)$ which we write as $\int\,\cdot\;\rho(v)\,dv$ and the expectation with respect to the other random variables, which expectation we denote as $\mathbb{E}^-_{v(\xi)}$.

With this notation (3.16) equals

$$\sum_n \int_{-2\kappa}^{2\kappa} dt \sum_{\xi\in\Lambda_1} \mathbb{E}^-_{v(\xi)} \Big(\int \frac{\partial\varphi(E_m(H^\Lambda - E + t)}{\partial v(\xi)}\, \rho\big(v(\xi)\big)\, dv(\xi) \Big) \tag{3.17}$$

$$\leq \; ||\,\rho\,|| \sum_n \int_{-2\kappa}^{2\kappa} dt \sum_{\xi\in\Lambda_1} \mathbb{E}^-_{v(\xi)} \Big(\int \frac{\partial\varphi(E_m(H^\Lambda - E + t)}{\partial v(\xi)}\, dv(\xi) \Big). \tag{3.18}$$

By the fundamental theorem of calculus we have

$$\int \frac{\partial\varphi\big(E_n(H^\Lambda) - E + t\big)}{\partial v(\xi)}\, dv(\xi) \tag{3.19}$$

$$= \; \varphi\big(E_n(H^\Lambda_{v(\xi)=\max}) - E + t\big) \; - \; \varphi\big(E_n(H^\Lambda_{v(\xi)=\min}) - E + t\big), \tag{3.20}$$

where $H^\Lambda_{v(\xi)=\max}$ (resp. $H^\Lambda_{v(\xi)=\min}$) denotes the operator H^Λ with the potential $v(\xi)$ set to its maximal (resp. minimal) value, i.e. with $v(\xi) = \sup(\mathrm{supp}(\rho))$ or $v(\xi) = \inf(\mathrm{supp}(\rho))$. Note that we include the cases $\sup(\mathrm{supp}(\rho)) = \infty$ and $\inf(\mathrm{supp}(\rho)) = -\infty$.

Changing $v(\xi)$ from its minimal to its maximal value is a (positive) perturbation of rank at most $M = K\frac{|\Lambda|}{|\Lambda_1|}$. Thus

$$E_n(H^\Lambda_{v(\xi)=min}) \;\leq\; E_n(H^\Lambda_{v(\xi)=max}) \;\leq\; E_{n+M}(H^\Lambda_{v(\xi)=min}). \tag{3.21}$$

To estimate (3.18) we use the following simple Lemma:

Lemma 3.2. *Let φ be a nondecreasing function on $\mathbb{R}$ with $0 \leq \varphi \leq 1$. If a_n and b_n are nondecreasing sequences satisfying $a_n \leq b_n \leq a_{n+M}$ for all n, then*

$$\sum_n \big(\, \varphi(b_n) - \varphi(a_n)\,\big) \;\leq\; M. \tag{3.22}$$

Combining the above estimates we get

$$\mathbb{P}\Big(\,\mathrm{dist}\,\big(\sigma\,\big(H^\Lambda\big)\,,\,E\big)\;<\kappa\Big) \tag{3.23}$$

$$\leq \; ||\,\rho\,|| \sum_n \int_{-2\kappa}^{2\kappa} dt \sum_{\xi\in\Lambda_1} \mathbb{E}^-_{v(\xi)} \Big(\int \frac{\partial\varphi(E_m(H^\Lambda - E + t)}{\partial v(\xi)}\, dv(\xi) \Big) \tag{3.24}$$

$$\leq \; ||\,\rho\,|| \int_{-2\kappa}^{2\kappa} dt \sum_{\xi\in\Lambda_1} \mathbb{E}^-_{v(\xi)} \sum_n \Big(\varphi(E_n(H^\Lambda_{v(\xi)=max}) - \varphi(E_n(H^\Lambda_{v(\xi)=min})\Big)$$

$$\leq \; ||\,\rho\,||\, 4\,\kappa\,|\Lambda|. \tag{3.25}$$

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