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The Asymptotic Behavior of Viscous Incompressible Fluid Small Oscillations with Solid Interacting Particles

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The motion of viscous incompressible fluid with a large number of small solid interacting particles is considered. The asymptotic behavior of small oscillations of the system is studied, when the radii of particles, distances between the nearest particles and their interaction power are decreased in the prescribed way. The equations describing the homogenized model of the system are derived.

Key words: homogenization, viscous incompressible fluid, interacting particles.

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1. Formulation of the Problem

Consider a bounded domain Ω in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. This domain is filled with viscous incompressible fluid with a large number $N_\varepsilon = O(\varepsilon^{-3})$ of interacting small ball-shaped solids Q_ε^i. Further we will call them "the particles". We suppose that the radii of particles $r_\varepsilon^i = r^i\varepsilon^{1+\alpha}$, $0 < \alpha < 2$, the distances $r_{ij}^\varepsilon = c_{ij}^r\varepsilon$ between the nearest particles and the interacting forces $f_{ij}^\varepsilon = c_{ij}^f\varepsilon^2$ depend on a small parameter ε. Here $0 < c_1 \leq c_{ij}^r, c_{ij}^f, r_i \leq c_2 < \infty$, where c_1 and c_2 do not depend on ε. We assume that the interacting forces f_{ij}^ε between the particles are central, i.e. their directions coincide with the lines of their centers. Furthermore, we suppose that the particles located in a boundary layer with the thickness ε interact with the boundary, and the system of all particles is in equilibrium when the fluid is at rest. The potential energy due to the interaction between the particles for small displacements $(\underline{u}_\varepsilon^i, \underline{\theta}_\varepsilon^i)$ from the equilibrium state is written in the following form:

$$H_\varepsilon(\underline{u}_\varepsilon) = H_\varepsilon(\underline{0}) + \frac{1}{2}\sum_{i,j=1}^{N_\varepsilon} \langle C_\varepsilon^{ij}[\underline{u}_\varepsilon^i - \underline{u}_\varepsilon^j], \underline{u}_\varepsilon^i - \underline{u}_\varepsilon^j\rangle + h(\underline{u}_\varepsilon), \tag{1.1}$$

where $\langle\,,\rangle$ stands for the dot product in $\mathbb{R}^3$, $\underline{u}_\varepsilon^i$ is the displacement of the particle center Q_ε^i, $\underline{u}_\varepsilon = (\underline{u}_\varepsilon^1, ..., \underline{u}_\varepsilon^{N_\varepsilon})$. We also denote by $h(\underline{u}_\varepsilon)$ the terms of smaller order. Matrix C_ε^{ij} is given by

$$C_\varepsilon^{ij}\underline{u} = k^{ij}\varepsilon^2\langle\frac{u}{|\underline{x}_\varepsilon^i - \underline{x}_\varepsilon^j|}, \underline{e}^{ij}\rangle\underline{e}^{ij}, \tag{1.2}$$

where $0 < k_1 \le k^{ij} \le k_2 < \infty$, the constants k_1 и k_2 do not depend on ε, and $\underline{e}^{ij} = \dfrac{\underline{x}_\varepsilon^i - \underline{x}_\varepsilon^j}{|\underline{x}_\varepsilon^i - \underline{x}_\varepsilon^j|}$. Furthermore, we suppose that only the particles Q_ε^i and Q_ε^j that are close (distance between them is of order ε) interact with each other $O(\varepsilon)$, such that the interaction matrix $C_\varepsilon^{ij} \equiv 0$, if $|\underline{x}_\varepsilon^i - \underline{x}_\varepsilon^j| \ge C_0\varepsilon$, $0 < C_0 < \infty$.

We introduce the following notations:

$\Omega_\varepsilon = \Omega \setminus \bigcup_{i=1}^{N} Q_\varepsilon^i$ is the domain filled with the fluid;

ρ is the specific mass density of the fluid;

μ is the dynamic viscosity of the fluid;

ρ_s is the specific mass density of solid particles;

$\underline{x}_\varepsilon^i$ is the position of center of particle Q_ε^i which corresponds to the equilibrium;

$\underline{\theta}_\varepsilon^i$ is the rotation vector of particle Q_ε^i;

m_ε^i is the mass of particle Q_ε^i;

$I_\varepsilon^i = \frac{2}{5}m_\varepsilon^i(r_\varepsilon^i)^2$ is the inertia moment of the ball-shaped particle Q_ε^i.

Then a linearized system of equations which describes small nonstationary motions of the fluid with solid particles can be written as follows:

$$\rho\frac{\partial \underline{v}_\varepsilon}{\partial t} - \mu\Delta\underline{v}_\varepsilon = \nabla p_\varepsilon, \quad \mathrm{div}\underline{v}_\varepsilon = 0 \quad \underline{x} \in \Omega_\varepsilon; \tag{1.3}$$

$$\underline{v}_\varepsilon = \underline{\dot{u}}_\varepsilon^i + \underline{\dot{\theta}}_\varepsilon^i \times (\underline{x} - \underline{x}_\varepsilon^i), \quad \underline{x} \in S_\varepsilon^i; \tag{1.4}$$

$$m_\varepsilon^i\underline{\ddot{u}}_\varepsilon^i + \int_{S_\varepsilon^i} \sigma[\underline{v}_\varepsilon]\nu\, ds = -\nabla_{\underline{u}_\varepsilon^i} H_\varepsilon; \tag{1.5}$$

$$I_\varepsilon^i\underline{\ddot{\theta}}_\varepsilon^i + \int_{S_\varepsilon^i} (\underline{x} - \underline{x}_\varepsilon^i) \times \sigma[\underline{v}_\varepsilon]\underline{\nu}\, ds = -\nabla_{\underline{\theta}_\varepsilon^i} H_\varepsilon(\equiv 0). \tag{1.6}$$

Here $\underline{v}_\varepsilon = \underline{v}_\varepsilon(\underline{x}, t)$ is the velocity of the fluid, $p_\varepsilon = p_\varepsilon(\underline{x}, t)$ is the pressure, ν is the unit inner normal vector to the surface $S^i_\varepsilon = \partial Q^i_\varepsilon$, and

$$\sigma[\underline{v}] = \left\{\sigma_{kl}[\underline{v}] = \mu\left(\frac{\partial \underline{v}_k}{\partial x_l} + \frac{\partial \underline{v}_l}{\partial x_k}\right) - p_\varepsilon \delta_{kl}\right\}_{k,l=1}^3$$

is the stress tensor.

The system of equations is supplemented by the initial conditions

$$\underline{v}_\varepsilon(\underline{x}, 0) = \underline{v}_{\varepsilon 0}(\underline{x}), \quad \underline{x} \in \Omega_\varepsilon; \tag{1.7}$$

$$\underline{u}^i_\varepsilon(0) = 0,\ \dot{\underline{u}}^i_\varepsilon(0) = \underline{v}^i_\varepsilon,\ \underline{\theta}^i_\varepsilon(0) = 0,\ \dot{\underline{\theta}}^i_\varepsilon(0) = \underline{\theta}^i_{\varepsilon 1} \tag{1.8}$$

and the boundary condition on $\partial\Omega$

$$\underline{v}_\varepsilon(\underline{x}, t) = 0, \quad \underline{x} \in \partial\Omega. \tag{1.9}$$

Theorem 1. *There exists a unique solution of the problem* (1.3)–(1.9).

We do not give here the proof of the theorem as well as the class the solution is sought in.

The main goal of the paper is to study the asymptotic behaviour of the problem (1.3)–(1.9) solution as $\varepsilon \to 0$. The cases of the particles of critical sizes ($d^\varepsilon_i = d_i\varepsilon^3$ and $d^\varepsilon_i = d_i\varepsilon$) were studied in [1] and [2]. Here we study the case of the particles of intermediate size ($d^\varepsilon_i = d_i\varepsilon^{1+\alpha}$, $0 < \alpha < 2$).

Before formulating the main result we introduce some assumptions and definitions.

2. Additional Assumptions and the Main Result

Denote by R^i_ε the distance from the particle Q^i_ε to other particles and to the boundary $\partial\Omega$, and $r^i_\varepsilon = r^i\varepsilon^{1+\alpha}$ is the radius of this particle.

We suppose that

$$C_1\varepsilon \le R^i_\varepsilon \le C_2\varepsilon, \tag{2.1}$$

where constants C_1 and C_2 do not depend on ε, $0 < C_1 < C_2 < \infty$.

Consider a cube K^y_h with the side length h, $\varepsilon \ll h \ll 1$, centered at $\underline{y} \in \Omega$. We assume that the edges of this cube are parallel to the coordinate axis. Introduce the following class of vector-functions:

$$J_\varepsilon[K^y_h] =$$

$$\{\underline{w}_\varepsilon \in H^1(K^y_h);\ \operatorname{div}\underline{w}_\varepsilon(\underline{x}) = 0,\ \underline{x} \in K^y_h;\ \underline{w}_\varepsilon(\underline{x}) = \underline{w}^i_\varepsilon + \underline{v}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon),\ \underline{x} \in Q^i_\varepsilon \cap K^y_h\},$$

where $\underline{w}^i_\varepsilon$ and $\underline{v}^i_\varepsilon$ are arbitrary vectors, and consider a minimization problem in this class for the following functional:

$$A^\gamma_{\varepsilon h}(\underline{w}_\varepsilon, \underline{y}, \lambda, T) = E_{K^y_h}[\underline{w}_\varepsilon, \underline{w}_\varepsilon] + \frac{1}{\lambda} I^\varepsilon_{K^y_h}[\underline{w}_\varepsilon, \underline{w}_\varepsilon]$$

$$+P^{\varepsilon h \gamma T}_{K^y_h}\Big[\underline{w}_\varepsilon(\underline{x}) - \sum_{n,p=1}^{3} T_{np}\underline{\varphi}^{np}(\underline{x}-\underline{y}), \underline{w}_\varepsilon(\underline{x}) - \sum_{q,r=1}^{3} T_{qr}\underline{\varphi}^{qr}(\underline{x}-\underline{y})\Big], \qquad (2.2)$$

where

$$E_G[\underline{u}_\varepsilon, \underline{v}_\varepsilon] = 2\mu \int_G \sum_{k,l=1}^{3} e_{kl}[\underline{u}_\varepsilon] e_{kl}[\underline{v}_\varepsilon] d\underline{x}, \qquad (2.3)$$

$$I^\varepsilon_G[\underline{u}_\varepsilon, \underline{v}_\varepsilon] = \frac{1}{2} \sum_{i,j \;\; G} \langle C^{ij}_\varepsilon [\underline{u}_\varepsilon(\underline{x}^i_\varepsilon) - \underline{u}_\varepsilon(\underline{x}^j_\varepsilon)], \underline{v}_\varepsilon(\underline{x}^i_\varepsilon) - \underline{v}_\varepsilon(\underline{x}^j_\varepsilon)\rangle, \qquad (2.4)$$

$$P^{\varepsilon h \gamma T}_{K^y_h}[\underline{u}_\varepsilon(\underline{x}), \underline{v}_\varepsilon(\underline{x})] = h^{-2-\gamma}\varepsilon^3 \sum_{i \;\; K^y_h} \langle \underline{u}_\varepsilon(\underline{x}^i_\varepsilon), \underline{v}_\varepsilon(\underline{x}^i_\varepsilon)\rangle + h^{-2-\gamma} \int_{K^y_h} \langle \underline{u}_\varepsilon(\underline{x}), \underline{v}_\varepsilon(\underline{x})\rangle \, dx, \qquad (2.5)$$

$$\underline{\varphi}^{qr}(\underline{x}) = \frac{1}{2}(x_r \underline{e}^q + x_q \underline{e}^r) - \frac{\delta_{qr}}{3} \sum_{n=1}^{3} x_n \underline{e}^n, \qquad (2.6)$$

$e_{kl}[\underline{u}] = \frac{1}{2}(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k})$, $T = \{T_{qr}\}$ is an arbitrary symmetric second rank tensor, and $\sum_{i \;\; G}$ stands for the summation over all particles $Q^i_\varepsilon \subset G$ which are located inside the domain G, $0<\gamma<2$, $\lambda > 0$. It can be proved that there exists the unique vector-function which minimizes the functional (2.2); the minimal value of this functional is a quadratic function of the tensor T:

$$\min_{\underline{w}_\varepsilon \in J_\varepsilon[K^y_h]} A^\gamma_{\varepsilon h}(\underline{w}_\varepsilon, \underline{y}, \lambda, T) = \sum_{n,p,q,r=1}^{3} a^\gamma_{npqr}(\underline{y}, \lambda, \varepsilon, h) T_{np} T_{qr}, \qquad (2.7)$$

where $a^\gamma_{npqr}(\underline{y}, \lambda, \varepsilon, h)$ are the components of the fourth rank tensor, defined as follows

$$a^\gamma_{npqr}(\underline{y}, \lambda, \varepsilon, h) = E_{K^y_h}[\underline{w}^{np}, \underline{w}^{qr}] + \frac{1}{\lambda} I^\varepsilon_{K^y_h}[\underline{w}^{np}, \underline{w}^{qr}]$$

$$+P^{\varepsilon h \gamma T}_{K^y_h}\big[\underline{w}^{np}(\underline{x}) - \underline{\varphi}^{np}(\underline{x}-\underline{y}), \underline{w}^{qr}(\underline{x}) - \underline{\varphi}^{qr}(\underline{x}-\underline{y})\big]. \qquad (2.8)$$

Here $\underline{w}^{np}(\underline{x})$ is the vector-function from $J_\varepsilon[K^y_h]$ that minimizes the functional (2.2) as $T = T^{np} = \frac{1}{2}(\underline{e}^n \otimes \underline{e}^p + \underline{e}^p \otimes \underline{e}^n)$; $\underline{e}^n$, $n = 1, 2, 3$, form an orthonormal basis in $\mathbb{R}^3$.

Starting from the solution $\{\underline{v}_\varepsilon(\underline{x},t),\, \underline{u}^i_\varepsilon,\, \underline{\theta}^i_\varepsilon,\, i=\overline{1,N_\varepsilon}\}$ of the problem (1.3)–(1.9), we construct the vector function

$$\underline{\tilde{v}}_\varepsilon(\underline{x},t) = \chi_\varepsilon(\underline{x})\underline{v}_\varepsilon(\underline{x},t) + \sum_{i=1}^{N_\varepsilon} \chi^i_\varepsilon(\underline{x})[\underline{\dot{u}}^i_\varepsilon + \underline{\dot{\theta}}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon)], \tag{2.9}$$

where $\chi_\varepsilon(\underline{x})$ is the characteristic function of the domain Ω_ε, filled with the fluid, and $\chi^i_\varepsilon(\underline{x})$ is the characteristic function of a particle Q^i_ε.

We assume that the following conditions hold:

2.1) the sequence of initial vector-functions $\underline{\tilde{v}}_{\varepsilon 0}(\underline{x}) = \underline{\tilde{v}}_\varepsilon(\underline{x},0)$ as $\varepsilon \to 0$ converges in $\mathbf{L_2(\Omega)}$ to a continuous vector-function $\underline{v}_0(\underline{x})$;

2.2) for each $\lambda > 0$ and some real number $\gamma > 0$ the following limits exist heterogeneously at $\underline{x} \in \Omega$:

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}} \frac{a^\gamma_{npqr}(\underline{x},\lambda,\varepsilon,h)}{h^3} = \lim_{h\to 0}\underline{\lim_{\varepsilon\to 0}} \frac{a^\gamma_{npqr}(\underline{x},\lambda,\varepsilon,h)}{h^3} = a_{npqr}(\underline{x},\lambda),$$

where $\{a_{npqr}(\underline{x},\lambda)\}$ is a continuous at $\underline{x} \in \Omega$ and $\lambda > 0$ positive definite tensor.

We formulate here the main mathematical result of the paper.

Theorem 2. *Let the conditions 2.1)–2.2) hold. Then the vector-functions* $\underline{\tilde{v}}_\varepsilon(\underline{x},t)$*, defined by* (2.9)*, converge weakly in* $L_2(\Omega \times [0,T])$ *(for any* $T > 0$*) to a vector-function* $\underline{v}(\underline{x},t)$*, which is a solution of the following homogenized problem:*

$$\rho\frac{\partial \underline{v}}{\partial t} - \mu\Delta\underline{v}$$

$$- \sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_p}\{\int_0^t a^1_{npqr}(\underline{x},t-\tau)e_{qr}[\underline{v}(\underline{x},\tau)]\,d\tau\}\underline{e}^n = \nabla p,\ \underline{x}\in\Omega,\ t>0; \tag{2.10}$$

$$div\,\underline{v} = 0 \quad \underline{x}\in\Omega, \quad t>0; \tag{2.11}$$

$$\underline{v}(\underline{x},t) = 0, \quad \underline{x}\in\partial\Omega, \quad t>0; \tag{2.12}$$

$$\underline{v}(\underline{x},0) = \underline{v}_0(\underline{x}), \quad \underline{x}\in\Omega. \tag{2.13}$$

Here $\{a^1_{npqr}(\underline{x},t)\}$ *is a continuous at* $\underline{x}\in\Omega$ *and* $t>0$ *tensor defined by*

$$a_{npqr}(\underline{x},\lambda) - 2\mu I_{npqr} = \int_0^\infty a^1_{npqr}(\underline{x},t)e^{-\lambda t}\,dt, \tag{2.14}$$

where the tensor $\{a_{npqr}(\underline{x},\lambda)\}$ *is defined in condition 2.2) for* $\lambda > 0$, *and the components of the tensor* $\{I_{npqr}\}$ *have the form*

$$I_{npqr} = \frac{1}{2}(\delta_{nq}\delta_{pr} + \delta_{nr}\delta_{pq}) - \frac{1}{3}\delta_{np}\delta_{qr}. \tag{2.15}$$

The problem (2.10)–(2.13) has the unique solution.

We prove this theorem in Sects. 3–5 by using the Laplace transform (Sect. 3) to obtain a time independent analog of the problem (1.3)–(1.9) with the spectral parameter λ. In Section 4 we establish the convergence of this stationary problem solution to the solution of the limiting stationary problem. Then we study the analytical properties of these solutions in the parameter λ and their behaviour as $|\lambda| \to \infty$ and, by taking the inverse Laplace transform, we prove the theorem (Sect. 5).

3. Variational Formulation of the Stationary Problem

Use the Laplace transform of the functions $\underline{v}_\varepsilon(\underline{x},t) \to \underline{v}_\varepsilon(\underline{x},\lambda)$, $p_\varepsilon(\underline{x},t) \to p_\varepsilon(\underline{x},\lambda)$, $\underline{u}^i_\varepsilon(t) \to \underline{u}^i_\varepsilon(\lambda)$, $\underline{\theta}^i_\varepsilon(t) \to \underline{\theta}^i_\varepsilon(\lambda)$. Taking into account the properties of the Laplace transform and (1.1), we rewrite the problem (1.3)–(1.6) in the form

$$-\mu\Delta\underline{v}_\varepsilon + \lambda\rho\underline{v}_\varepsilon - \nabla p_\varepsilon = \rho\underline{v}_{\varepsilon 0}(\underline{x}), \quad \operatorname{div}\underline{v}_\varepsilon = 0, \quad \underline{x} \in \Omega_\varepsilon, \tag{3.1}$$

$$\underline{v}_\varepsilon = \lambda[\underline{u}^i_\varepsilon + \underline{\theta}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon)], \quad \underline{x} \in S^i_\varepsilon, \tag{3.2}$$

$$\lambda^2 m^i_\varepsilon \underline{u}^i_\varepsilon + \int\limits_{S^i_\varepsilon} \sigma[\underline{v}_\varepsilon]\nu\, ds = -\frac{1}{\lambda}\sum_j^i C^{ij}_\varepsilon[\underline{v}_\varepsilon(\underline{x}^i_\varepsilon) - \underline{v}_\varepsilon(\underline{x}^j_\varepsilon)] + m^i_\varepsilon \underline{v}^i_\varepsilon, \tag{3.3}$$

$$\lambda^2 I^i_\varepsilon \underline{\theta}^i_\varepsilon + \int\limits_{S^i_\varepsilon} (\underline{x} - \underline{x}^i_\varepsilon) \times \sigma[\underline{v}_\varepsilon]\nu\, ds = I^i_\varepsilon \underline{\theta}^i_{\varepsilon 1}, \tag{3.4}$$

$$\underline{v}_\varepsilon(\underline{x}) = 0, \quad \underline{x} \in \partial\Omega. \tag{3.5}$$

Here $Re\lambda > 0$, $\sum\limits_j^i$ stands for the summation over all particles Q^j_ε which interact with the particle Q^i_ε. We extend the velocity function $\underline{v}_\varepsilon(\underline{x},\lambda)$ onto the particles Q^i_ε according to (3.2) using the same notations for the extended function. Denote by

$$\rho_\varepsilon(\underline{x}) = \rho\chi_\varepsilon(\underline{x}) + \rho_s \sum_{i=1}^{N_\varepsilon} \chi^i_\varepsilon(\underline{x})$$

the density of suspension of type the fluid-the particles.

Fix now $\lambda > 0$. Then the problem (3.1)–(3.5) is equivalent to the variational problem

$$\Phi_\varepsilon(\underline{v}_\varepsilon) = \min_{\underline{v}'_\varepsilon \in \overset{\circ}{J}_\varepsilon(\Omega)} \Phi_\varepsilon(\underline{v}'_\varepsilon), \tag{3.6}$$

where $\overset{\circ}{J}_\varepsilon(\Omega)$ is the class of divergence free vector-functions from $\overset{\circ}{H}{}^1(\Omega)$ which are equal to $\underline{a}^i_\varepsilon + \underline{b}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon)$ on the particles Q^i_ε (a^i_ε and b^i_ε are arbitrary vectors), and

$$\Phi_\varepsilon(\underline{v}_\varepsilon) = \int_\Omega \Big\{2\mu \sum_{k,l=1}^{3} e^2_{kl}[\underline{v}_\varepsilon] + \lambda\langle\rho_\varepsilon \underline{v}_\varepsilon, \underline{v}_\varepsilon\rangle - 2\langle\rho_\varepsilon \underline{v}_{\varepsilon 0}, \underline{v}_\varepsilon\rangle\Big\}\, dx \tag{3.7}$$

$$+ \frac{1}{2\lambda} \sum_{i,j=1}^{N_\varepsilon} \langle C^{ij}_\varepsilon[\underline{v}_\varepsilon(\underline{x}^i_\varepsilon) - \underline{v}_\varepsilon(\underline{x}^j_\varepsilon)], \underline{v}_\varepsilon(\underline{x}^i_\varepsilon) - \underline{v}_\varepsilon(\underline{x}^j_\varepsilon)\rangle,$$

where $\lambda > 0$.

Consider the minimization problem

$$\Phi_0(\underline{v}) = \min_{\underline{v}' \in \overset{\circ}{J}(\Omega)} \Phi_0(\underline{v}'), \tag{3.8}$$

where $\overset{\circ}{J}(\Omega)$ is the class of divergence free vector-functions from $\overset{\circ}{H}{}^1(\Omega)$ and

$$\Phi_0(\underline{v}) = \int_\Omega \Big\{ \sum_{n,p,q,r=1}^{3} a_{npqr}(\underline{x}, \lambda) e_{np}[\underline{v}] e_{qr}[\underline{v}] + \lambda\langle\rho\underline{v}, \underline{v}\rangle - 2\langle\rho\underline{v}_0, \underline{v}\rangle\Big\}\, dx. \tag{3.9}$$

The minimizer of this problem is the solution of the following boundary value problem:

$$\lambda\rho\underline{v} - \mu\Delta\underline{v} - \sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_p}\big(a^1_{npqr}(\underline{x}, \lambda) e_{qr}[\underline{v}]\big)\underline{e}^n = \rho\underline{v}_0 + \nabla p, \quad \underline{x} \in \Omega, \tag{3.10}$$

$$\operatorname{div} \underline{v} = 0, \quad \underline{x} \in \Omega, \tag{3.11}$$

$$\underline{v}(\underline{x}, t) = 0, \quad \underline{x} \in \partial\Omega. \tag{3.12}$$

The asymptotic behavior as $\varepsilon \to 0$ of the solution of problem (3.6) is given by the following theorem.

Theorem 3. *Let the conditions 2.1)–2.2) hold. Then the solution $\underline{v}_\varepsilon(\underline{x}, \lambda)$ of the problem (3.6) for any $\lambda > 0$ converges as $\varepsilon \to 0$ to the solution $\underline{v}(\underline{x}, \lambda)$ of the problem (3.8) in the following sense:*

$$\underline{v}_\varepsilon(\underline{x}, \lambda) \xrightarrow[\varepsilon \to 0]{} \underline{v}(\underline{x}, \lambda) \quad \text{strongly in} \quad L_2(\Omega).$$

The proof of this theorem is given in Sect. 4.

4. Convergence Theorem for Variational Problem (3.6)

Let $\underline{v}_\varepsilon(\underline{x},\lambda)$ be the solution of the problem (3.6). Since $0 \in \overset{\circ}{J}_\varepsilon(\Omega)$, we have $\Phi_\varepsilon(\underline{v}_\varepsilon) \le \Phi_\varepsilon(0) = 0$. From this, taking into account (3.7) and nonnegativity of the matrices $C_\varepsilon^{ij}(\underline{x},\underline{y})$, it follows:

$$\int_\Omega \{2\mu \sum_{k,l=1}^{3} e_{kl}^2[\underline{v}_\varepsilon] + \lambda\langle \rho_\varepsilon \underline{v}_\varepsilon, \underline{v}_\varepsilon\rangle\}\,dx \le 2\|\rho_\varepsilon \underline{v}_{\varepsilon 0}\|_{L_2(\Omega)} \cdot \|\underline{v}_\varepsilon\|_{L_2(\Omega)}.$$

Due to the second Korn's inequality

$$\|\underline{v}_\varepsilon\|^2_{H^1(\Omega)} \le c\{\int_\Omega \sum_{n,p=1}^{3} e^2_{np}[\underline{v}_\varepsilon]\,dx + \int_\Omega |\underline{v}_\varepsilon(\underline{x})|^2\,dx\}, \tag{4.1}$$

this implies that

$$\|\underline{v}_\varepsilon\|^2_{H^1(\Omega)} \le C. \tag{4.2}$$

Therefore the set of the vector-functions $\{\underline{v}_\varepsilon(\underline{x},\lambda),\ \varepsilon > 0\}$ is weakly compact in $H^1(\Omega)$. Due to the embedding theorem, this set is compact in $L_2(\Omega)$. Hence, there exists a subsequence of the sequence $\{\underline{v}_\varepsilon(\underline{x},\lambda),\ \varepsilon > 0\}$ which converges to some vector-function $\underline{v}(\underline{x},\lambda)$ (weakly in $H^1(\Omega)$ and strongly in $L_2(\Omega)$). As it is shown below, the limiting vector-function $\underline{v}(\underline{x},\lambda)$ is a solution of the problem (3.8). But since this problem has the unique solution, then the sequence $\{\underline{v}_\varepsilon(\underline{x},\lambda),\ \varepsilon > 0\}$ is also convergent:

$$\underline{v}_\varepsilon \rightharpoonup \underline{v} \text{ weakly in } H^1(\Omega),\ \underline{v}_\varepsilon \to \underline{v} \text{ strongly in } L_2(\Omega). \tag{4.3}$$

Clearly, that $\underline{v}(\underline{x}) \in \overset{\circ}{J}(\Omega)$.

Show that for any vector-function $\underline{w} \in \overset{\circ}{J}(\Omega)$ the following inequality holds:

$$\Phi_0(\underline{v}) \le \Phi_0(\underline{w}). \tag{4.4}$$

1. For any vector-function $\underline{w} \in \overset{\circ}{J}(\Omega) \bigcap C_0^2(\Omega)$ we construct a special vector-function $\underline{w}_{\varepsilon h} \in \overset{\circ}{J}_\varepsilon(\Omega)$, such that

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\,\Phi_\varepsilon(\underline{w}_{\varepsilon h}) \le \Phi_0(\underline{w}). \tag{4.5}$$

Now we describe this construction. Cover the domain Ω with cubes $K_h^{x_\alpha}$ centered at points $x_\alpha \in \Omega$ with the edges of length h, which are parallel to the coordinate axis: $\overline{\Omega} \subset \bigcup_{\alpha\in\Lambda} K_h^{x_\alpha}$. Let the centers $x_\alpha \in \Omega$ of these cubes form a cubic lattice of period $h - h^{1+\frac{\gamma}{2}}$, $0 < \gamma < 2$. Denote by $K_{h'}^{x_\alpha}$ the cubes with the edges of length $h' = h - 2h^{1+\frac{\gamma}{2}}$ which are concentric to $K_h^{x_\alpha}$. It is well known ([2]) that there exists a set of functions $\{\phi_\alpha(\underline{x}) \in C_0^\infty(\Omega)\}_{\alpha\in\Lambda}$ (called *a special partition of unity*) such that:

$$1)\ \phi_\alpha(\underline{x}) = \begin{cases} 1, \underline{x} \in K_{h'}^{x_\alpha} \\ 0, \underline{x} \notin K_h^{x_\alpha} \end{cases}, \quad 2)\ 0 \le \phi_\alpha(\underline{x}) \le 1, \quad 3)\ |\nabla\phi_\alpha(\underline{x})| \le \frac{c}{h^{1+\frac{\gamma}{2}}},$$

$$4)\ \sum_{\alpha\in\Lambda}\phi_\alpha(\underline{x}) \equiv 1,\ \underline{x} \in \overline{\Omega}, \quad 5)\ \phi_\alpha(\underline{x}) = C_\varepsilon^i,\ \underline{x} \in B(Q_\varepsilon^i), \tag{4.6}$$

where C_ε^i are the constants $(0 \le C_\varepsilon^i \le 1)$, and $B(Q_\varepsilon^i)$ are the balls containing the particles Q_ε^i and centered at points $\underline{x}_\varepsilon^i$ and having the radii $\frac{R_\varepsilon^i}{3}$ (see (2.1)).

For any divergence-free vector-function $\underline{w}(\underline{x}) \in C_0^2(\Omega)$ we construct the vector-function $\underline{w}_{\varepsilon h}(\underline{x}) \in \overset{\circ}{J}_\varepsilon(\Omega)$ possessing the following properties. First, it approximates (in $L_2(\Omega)$) a given vector-function $\underline{w}(\underline{x}) \in \overset{\circ}{J}(\Omega)$ for small ε and h. Second, it "almost" minimizes the functional (2.2).

Note that any vector-function $\underline{w}(\underline{x}) \in C^2(K_h^{x_\alpha})$ can be written in the form

$$\underline{w}(\underline{x}) = \underline{w}(\underline{x}^\alpha) + \sum_{n,p=1}^{3}\big(e_{np}[\underline{w}(\underline{x}^\alpha)]\underline{\varphi}^{np}(\underline{x}-\underline{x}^\alpha)$$

$$+w_{np}[\underline{w}(\underline{x}^\alpha)]\underline{\psi}^{np}(\underline{x}-\underline{x}^\alpha)\big) + \underline{g}_\alpha(\underline{x}), \quad \underline{x} \in K_h^{x_\alpha}, \tag{4.7}$$

where

$$\varepsilon_{np}[\underline{w}(\underline{x}_\alpha)] = \frac{1}{2}\Big(\frac{\partial \underline{w}_n}{\partial x_p}(\underline{x}_\alpha) + \frac{\partial \underline{w}_p}{\partial x_n}(\underline{x}_\alpha)\Big), \quad w_{np}[\underline{w}(\underline{x}_\alpha)] = \frac{1}{2}\Big(\frac{\partial \underline{w}_n}{\partial x_p}(\underline{x}_\alpha) - \frac{\partial \underline{w}_p}{\partial x_n}(\underline{x}_\alpha)\Big),$$

the vector-function $\underline{\varphi}^{np}(\underline{x})$ is defined in (2.6),

$$\underline{\psi}^{np}(\underline{x}) = \frac{1}{2}(x_n\underline{e}^p - x_p\underline{e}^n), \tag{4.8}$$

and $D^k\underline{g}_\alpha(\underline{x}) = \underline{O}(h^{2-k})$, $k = \overline{0,2}$. Define the quasiminimizer $\underline{w}_{\varepsilon h}(\underline{x})$ as follows:

$$\underline{w}_{\varepsilon h}(\underline{x}) = \sum_{\alpha\in\Lambda}\big\{\underline{w}(\underline{x}_\alpha) + \sum_{n,p=1}^{3}\varepsilon_{np}[\underline{w}(\underline{x}_\alpha)]\underline{v}_{\alpha,\varepsilon h}^{np}(\underline{x})$$

$$+\sum_{n,p=1}^{3}w_{np}[\underline{w}(\underline{x}_\alpha)]\underline{\psi}^{np}(\underline{x}-\underline{x}_\alpha)\big\}\cdot\phi_\alpha(\underline{x}) + \underline{\zeta}_{\varepsilon h}(\underline{x}) = \underline{z}_{\varepsilon h}(\underline{x}) + \underline{\zeta}_{\varepsilon h}(\underline{x}); \tag{4.9}$$

here the vector-functions $\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x})$ are the minimizers of the functional (2.2) as $T = T^{np} = \frac{1}{2}(\underline{e}^n \otimes \underline{e}^p + \underline{e}^p \otimes \underline{e}^n)$, and the vector-function $\underline{\zeta}_{\varepsilon h}(\underline{x})$ is constructed according to the following lemma (see [2]).

Lemma 1. *For any function $F_\varepsilon(\underline{x}) \in L_2(\Omega)$ which satisfies the conditions:*

$$1.\ F_\varepsilon(\underline{x}) = 0, \quad x \in \bigcup_i B(Q^i_\varepsilon),$$

$$2.\ \int_\Omega F_\varepsilon(\underline{x})\, dx = 0,$$

there exists a function $\underline{\zeta}_\varepsilon(\underline{x}) \in H^1_0(\Omega)$ such that

$$div\underline{\zeta}_\varepsilon(\underline{x}) = F_\varepsilon(\underline{x}), \quad x \in \Omega;$$

$$\underline{\zeta}_\varepsilon(\underline{x}) = \underline{\zeta}^i_\varepsilon, \quad x \in B(Q^i_\varepsilon); \quad \|\underline{\zeta}_\varepsilon\|_{H^1(\Omega)} \le C\|F_\varepsilon(\underline{x})\|_{L_2(\Omega)},$$

where $\underline{\zeta}^i_\varepsilon$ are constant vectors, and C does not depend on ε.

Due to (4.9), the vector-function $\underline{z}_{\varepsilon h}(\underline{x}) \in H^1(\Omega)$ is equal to zero on the boundary $\partial\Omega$, and since

$$\int_\Omega \text{div } \underline{z}_{\varepsilon h}(\underline{x}) = 0.$$

Moreover, we can show that

$$\text{div } \underline{z}_{\varepsilon h}(\underline{x}) = 0, \quad x \in B(Q^i_\varepsilon).$$

Applying Lemma 1 to the function $F_\varepsilon(\underline{x}) = -\text{div}\, \underline{z}_{\varepsilon h}(\underline{x})$, we construct the divergence-free vector-function $\underline{\zeta}_{\varepsilon h}(\underline{x})$, which is equal to the constant vectors $\underline{\zeta}^i_\varepsilon$ on the balls $B(Q^i_\varepsilon)$ and zero on $\partial\Omega$. Now it is obvious that $\underline{w}_{\varepsilon h}(\underline{x}) \in \overset{\circ}{J}_\varepsilon (\Omega)$.

Let us calculate the functional (3.7) on the vector-function $\underline{w}_{\varepsilon h}(\underline{x})$.

Similarly to [2], we can show that

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} \|\underline{\zeta}_{\varepsilon h}\|_{H^1(\Omega)} = 0; \quad \lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} I_\Omega[\underline{\zeta}_{\varepsilon h}, \underline{\zeta}_{\varepsilon h}] = 0;$$

$$E_\Omega[\underline{w}_{\varepsilon h}, \underline{w}_{\varepsilon h}] = \sum_{\alpha\in\Lambda} \sum_{n,p,q,r=1}^{3} e_{np}[\underline{w}(\underline{x}_\alpha)] e_{qr}[\underline{w}(\underline{x}_\alpha)] E_{K^{x_\alpha}_{h'}}[\underline{v}^{np}_{\alpha,\varepsilon h}, \underline{v}^{qr}_{\alpha,\varepsilon h}] + L_1(\varepsilon, h); \tag{4.10}$$

$$I^\varepsilon_\Omega[\underline{w}_{\varepsilon h}, \underline{w}_{\varepsilon h}] \le \sum_{\alpha\in\Lambda} \sum_{n,p,q,r=1}^{3} e_{np}[\underline{w}(\underline{x}_\alpha)] e_{qr}[\underline{w}(\underline{x}_\alpha)] I^\varepsilon_{K^{x_\alpha}_{h'}}[\underline{v}^{np}_{\alpha,\varepsilon h}, \underline{v}^{qr}_{\alpha,\varepsilon h}]$$

$$+L_2(\varepsilon,h)+c\,I^{\varepsilon}_{\Omega^{\varepsilon_1}_{h'}}[\underline{z}_{\varepsilon h},\underline{z}_{\varepsilon h}]. \tag{4.11}$$

Here $\lim\limits_{h\to 0}\overline{\lim\limits_{\varepsilon\to 0}}\,L_i(\varepsilon,h)=0$, $i=\overline{1,2}$, and $\Omega^{\varepsilon_1}_{h'}=\Omega\setminus\bigcup\limits_{\alpha\in\Lambda}K^{x_\alpha}_{h''}$, where $K^{x_\alpha}_{h''}$ are cubes with the edges of length $h''=h'-\varepsilon_1$ which are concentric to $K^{x_\alpha}_{h'}$; $\varepsilon_1=2C_0\varepsilon$ is a doubled radius of the particles interaction (see (1.2)).

To prove that

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\,I^{\varepsilon}_{\Omega^{\varepsilon_1}_{h'}}[\underline{z}_{\varepsilon h},\underline{z}_{\varepsilon h}]=0, \tag{4.12}$$

we use the following lemma (see [3]).

Lemma 2. *Let $\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x})$ be the minimizer of the functional* (2.2) *as $T=T^{np}=\frac{1}{2}(\underline{e}^n\otimes\underline{e}^p+\underline{e}^p\otimes\underline{e}^n)$. If condition 2.2) holds, then for sufficiently small h and $\varepsilon<\hat{\varepsilon}(h)$ the following estimates can be obtained:*

$$E_{K^{x_\alpha}_h\setminus K^{x_\alpha}_{h'}}[\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}),\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x})]=\overline{o}(h^3);$$

$$I^{\varepsilon}_{K^{x_\alpha}_h\setminus K^{x_\alpha}_{h'}}[\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}),\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x})]=\overline{o}(h^3);$$

$$\varepsilon^3\sum_{i\quad K^{x_\alpha}_h\setminus K^{x_\alpha}_{h'}}|\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{\varphi}^{np}(\underline{x}^i_\varepsilon-\underline{x}_\alpha)|^2=\overline{o}(h^{5+\gamma}).$$

We use these estimates to obtain (4.12). We only have to show that

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\sum_{\alpha\in\Lambda}\sum_{i,j\quad K^{x_\alpha}_h\setminus K^{x_\alpha}_{h'}}\langle C^{ij}_\varepsilon[\underline{z}_{\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{z}_{\varepsilon h}(\underline{x}^j_\varepsilon)],\underline{z}_{\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{z}_{\varepsilon h}(\underline{x}^j_\varepsilon)\rangle=0.$$

For this purpose we write the vector-function $\underline{z}_{\varepsilon h}(\underline{x})$ in the form

$$\underline{z}_{\varepsilon h}(\underline{x})=\sum_{\alpha\in\Lambda}\{\underline{w}(\underline{x})+\sum_{n,p=1}^{3}\varepsilon_{np}[\underline{w}(\underline{x}_\alpha)]\big(\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x})-\underline{\varphi}^{np}(\underline{x}-\underline{x}_\alpha)\big)-\underline{g}_\alpha(\underline{x})\}\cdot\phi_\alpha(\underline{x})$$
$$=\underline{w}(\underline{x})+\sum_{\alpha\in\Lambda}\{\sum_{n,p=1}^{3}\varepsilon_{np}[\underline{w}(\underline{x}_\alpha)]\big(\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x})-\underline{\varphi}^{np}(\underline{x}-\underline{x}_\alpha)\big)-\underline{g}_\alpha(\underline{x})\}\cdot\phi_\alpha(\underline{x}).$$

Hence

$$\underline{z}_{\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{z}_{\varepsilon h}(\underline{x}^j_\varepsilon)=\underline{w}(\underline{x}^i_\varepsilon)-\underline{w}(\underline{x}^j_\varepsilon)+\sum_{\alpha\in\Lambda}\Bigg\{\sum_{n,p=1}^{3}\varepsilon_{np}[\underline{w}(\underline{x}_\alpha)]$$
$$\times\big(\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^j_\varepsilon)-\underline{\varphi}^{np}(\underline{x}^i_\varepsilon-\underline{x}^j_\varepsilon)\big)+\underline{g}_\alpha(\underline{x}^j_\varepsilon)-\underline{g}_\alpha(\underline{x}^i_\varepsilon)\Bigg\}\cdot\phi_\alpha(\underline{x}^i_\varepsilon)$$

$$+\sum_{\alpha\in\Lambda}\Big\{\sum_{n,p=1}^{3}\varepsilon_{np}[\underline{w}(\underline{x}_\alpha)]\big(\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{\varphi}^{np}(\underline{x}^i_\varepsilon-\underline{x}_\alpha)\big)-\underline{g}_\alpha(\underline{x}^i_\varepsilon)\Big\}\cdot\big(\phi_\alpha(\underline{x}^i_\varepsilon)-\phi_\alpha(\underline{x}^j_\varepsilon)\big).$$

Using the inequality

$$\sum_{i,j}\sum_{G}\langle C^{ij}_\varepsilon\underline{u},\underline{v}\rangle\le\Big(\sum_{i,j}\sum_{G}\langle C^{ij}_\varepsilon\underline{u},\underline{u}\rangle\Big)^{\frac12}\Big(\sum_{i,j}\sum_{G}\langle C^{ij}_\varepsilon\underline{v},\underline{v}\rangle\Big)^{\frac12}$$

and the fact that the support of function $\phi_\alpha(\underline{x})$ belongs only to the finite number of cubes $K_h^{x_\gamma}$ containing the slab $K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}$, we obtain the following estimate:

$$\lim_{h\to0}\overline{\lim_{\varepsilon\to0}}\sum_{\alpha\in\Lambda}\sum_{i,j}\sum_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\langle C^{ij}_\varepsilon[\underline{z}_{\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{z}_{\varepsilon h}(\underline{x}^j_\varepsilon)],\underline{z}_{\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{z}_{\varepsilon h}(\underline{x}^j_\varepsilon)\rangle$$

$$\le c\lim_{h\to0}\overline{\lim_{\varepsilon\to0}}\Big(\sum_{\alpha\in\Lambda}\sum_{i,j}\sum_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\langle C^{ij}_\varepsilon[\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^j_\varepsilon)],\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^j_\varepsilon)\rangle$$

$$+\sum_{\alpha\in\Lambda}\sum_{i,j}\sum_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\big\langle C^{ij}_\varepsilon[\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{\varphi}^{np}(\underline{x}^i_\varepsilon-\underline{x}_\alpha)],\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{\varphi}^{np}(\underline{x}^i_\varepsilon-\underline{x}_\alpha)\big\rangle$$

$$\times\big(\phi_\alpha(\underline{x}^i_\varepsilon)-\phi_\alpha(\underline{x}^j_\varepsilon)\big)\Big)+\overline{o}(1), \tag{4.13}$$

where by $\overline{o}(1)$ we denote a contribution of the terms

$$\underline{w}(\underline{x}^i_\varepsilon)-\underline{w}(\underline{x}^j_\varepsilon),\ \underline{\varphi}^{np}(\underline{x}^i_\varepsilon-\underline{x}^j_\varepsilon)\phi_\alpha(\underline{x}^i_\varepsilon),$$

$$\big(\underline{g}_\alpha(\underline{x}^j_\varepsilon)-\underline{g}_\alpha(\underline{x}^i_\varepsilon)\big)\phi_\alpha(\underline{x}^i_\varepsilon),\ \underline{g}_\alpha(\underline{x}^i_\varepsilon)\big(\phi_\alpha(\underline{x}^i_\varepsilon)-\phi_\alpha(\underline{x}^j_\varepsilon)\big).$$

Next, due to the second estimate from Lem. 2 for small ε and h, the term

$$\sum_{\alpha\in\Lambda}\sum_{i,j}\sum_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\langle C^{ij}_\varepsilon[\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^j_\varepsilon)],\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^j_\varepsilon)\rangle$$

is of order $\dfrac{|\Omega|\cdot\overline{o}(h^3)}{h^3}=\overline{o}(1)$.

Evaluate now the second term in (4.13) by using (1.2) and (4.6):

$$\sum_{\alpha\in\Lambda}\sum_{i,j}\sum_{K_h^{x_\alpha}\setminus K_{h'}^{x_\alpha}}\big\langle C^{ij}_\varepsilon[\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{\varphi}^{np}(\underline{x}^i_\varepsilon-\underline{x}_\alpha)],\underline{v}^{np}_{\alpha,\varepsilon h}(\underline{x}^i_\varepsilon)-\underline{\varphi}^{np}(\underline{x}^i_\varepsilon-\underline{x}_\alpha)\big\rangle$$

$$\times\big(\phi_\alpha(\underline{x}^i_\varepsilon)-\phi_\alpha(\underline{x}^j_\varepsilon)\big)^2$$

$$\leq \frac{c\varepsilon^2}{h^{2+\tau}} \cdot \frac{|\Omega|}{h^3} \sum_{i,j} \sum_{K_h^{x_\alpha} \setminus K_{h'}^{x_\alpha}} \frac{\varepsilon^2 k^{ij}}{|\underline{x}_\varepsilon^i - \underline{x}_\varepsilon^j|} \left| \langle \underline{v}_{\alpha,\varepsilon h}^{np}(\underline{x}_\varepsilon^i) - \underline{\varphi}^{np}(\underline{x}_\varepsilon^i - \underline{x}_\alpha), \underline{e}^{ij} \rangle \right|^2$$

$$\leq \frac{c\varepsilon^3}{h^{5+\tau}} \sum_{i} \sum_{K_h^{x_\alpha} \setminus K_{h'}^{x_\alpha}} |\underline{v}_{\alpha,\varepsilon h}^{np}(\underline{x}_\varepsilon^i) - \underline{\varphi}^{np}(\underline{x}_\varepsilon^i - \underline{x}_\alpha)|^2,$$

which vanishes in the limit due to Lem. 2. So, the equality (4.12) is obtained.

From (4.10) and (4.11), taking into account (2.8), we obtain

$$E_\Omega[\underline{w}_{\varepsilon h}, \underline{w}_{\varepsilon h}] + \frac{1}{\lambda} I_\Omega^\varepsilon[\underline{w}_{\varepsilon h}, \underline{w}_{\varepsilon h}]$$

$$\leq \sum_{\alpha \in \Lambda} \sum_{n,p,q,r=1}^{3} a_{npqr}^\gamma(\underline{x}, \lambda, \varepsilon, h) \varepsilon_{np}[\underline{w}(\underline{x}_\alpha)] \varepsilon_{qr}[\underline{w}(\underline{x}_\alpha)] + \overline{o}(1), \quad \varepsilon \ll h \ll 1. \tag{4.14}$$

Now we make use of inequality (4.14) to estimate the functional (3.7):

$$\Phi_\varepsilon(\underline{w}_{\varepsilon h}) \leq \sum_{\alpha \in \Lambda} h^3 \sum_{n,p,q,r=1}^{3} \frac{a_{npqr}^\gamma(\underline{x}_\alpha, \lambda, \varepsilon, h)}{h^3} \varepsilon_{np}[\underline{w}(\underline{x}_\alpha)] \varepsilon_{qr}[\underline{w}(\underline{x}_\alpha)]$$

$$+ \lambda \int_\Omega \langle \rho_\varepsilon \underline{w}_{\varepsilon h}, \underline{w}_{\varepsilon h} \rangle - 2 \int_\Omega \langle \rho_\varepsilon \underline{v}_{\varepsilon 0}, \underline{w}_{\varepsilon h} \rangle \, dx + \Delta(\varepsilon, h), \tag{4.15}$$

where

$$\lim_{h \to 0} \overline{\lim_{\varepsilon \to 0}} \, \Delta(\varepsilon, h) = 0.$$

Taking into account (4.9), we can show that

$$\lim_{h \to 0} \overline{\lim_{\varepsilon \to 0}} \, \|\underline{w}_{\varepsilon h} - \underline{w}\|_{L_2(\Omega)} = 0.$$

Then, passing to the limit in (4.15) as $\varepsilon \to 0$ and $h \to 0$ and taking into consideration 2.1)–2.2) and the fact that $\underline{w}(\underline{x}) \in C^2(\overline{\Omega})$, we obtain

$$\lim_{h \to 0} \overline{\lim_{\varepsilon \to 0}} \, \Phi_\varepsilon(\underline{w}_{\varepsilon h}) \leq \Phi_0(\underline{w}).$$

Thus inequality (4.5) is proved. Next, from (4.5) and an obvious inequality $\Phi_\varepsilon(\underline{v}_\varepsilon) \leq \Phi_\varepsilon(\underline{w}_{\varepsilon h})$ there follows the upper bound:

$$\overline{\lim_{\varepsilon \to 0}} \, \Phi_\varepsilon(\underline{v}_\varepsilon) \leq \Phi_0(\underline{w}), \quad \forall \underline{w} \in \overset{\circ}{J}(\Omega). \tag{4.16}$$

2. Prove now the lower bound

$$\Phi_0(\underline{v}) \leq \underline{\lim}_{\varepsilon \to 0} \, \Phi_\varepsilon(\underline{v}_\varepsilon), \tag{4.17}$$

where the vector-function $\underline{v}(\underline{x})$ is defined in (4.3). We need the following lemma.

Lemma 3. *Let the sequence of vector-functions $\underline{u}_\varepsilon(\underline{x})$ is bounded in $H^1(\Omega)$ uniformly in ε. Denote by B_a and B_d the concentric balls in Ω, such that $B_a \subset B_d \subset \Omega$ ($a<1$). Then the following estimate holds:*

$$| < \underline{u}_\varepsilon >_d - < \underline{u}_\varepsilon >_a | \leq \frac{\|\underline{u}_\varepsilon\|_{H^1(B_d)}}{\sqrt{a}},$$

where by $< \underline{u}_\varepsilon >_a$ and $< \underline{u}_\varepsilon >_d$ we denote the mean values of the vector-functions $\underline{u}_\varepsilon(\underline{x})$ on the balls B_a and B_d respectively.

P r o o f. Write an obvious equality:

$$\underline{u}_\varepsilon(\rho,\varphi) - \underline{u}_\varepsilon(r,\varphi) = \int\limits_r^\rho \frac{\partial \underline{u}_\varepsilon}{\partial R}\, dR.$$

Multiply this equality on $r^2\rho^2$ and then integrate it over the segments $0 \leq r \leq a$, $0 \leq \rho \leq d$ and over a surface of unit ball S_1. We have

$$\frac{a^3}{3}\int\limits_{S_1}\int\limits_0^d \underline{u}_\varepsilon(\rho,\varphi)\rho^2\, d\rho\, dS - \frac{d^3}{3}\int\limits_{S_1}\int\limits_0^a \underline{u}_\varepsilon(r,\varphi) r^2\, dr\, dS$$

$$= \int\limits_{S_1}\int\limits_0^a\int\limits_0^d\int\limits_r^\rho \frac{\partial \underline{u}_\varepsilon}{\partial R}\, dR\, r^2\rho^2\, dr\, d\rho\, dS.$$

It is easy to see that

$$< \underline{u}_\varepsilon >_d - < \underline{u}_\varepsilon >_a = \frac{9}{4\pi a^3 d^3}\int\limits_{S_1}\int\limits_0^a\int\limits_0^d\int\limits_r^\rho \frac{\partial \underline{u}_\varepsilon}{\partial R}\, dR\, r^2\rho^2\, dr\, d\rho\, dS.$$

From this equality we obtain

$$| < \underline{u}_\varepsilon >_d - < \underline{u}_\varepsilon >_a |$$

$$\leq \frac{9}{4\pi a^3 d^3}\int\limits_0^a\int\limits_0^d\int\limits_{S_1}\Big|\int\limits_r^\rho \big(\frac{\partial \underline{u}_\varepsilon}{\partial R}R\big)^2\, dR\Big|^{\frac{1}{2}} dS \Big|\int\limits_r^\rho \frac{dR}{R^2}\Big|^{\frac{1}{2}} r^2\rho^2\, dr\, d\rho$$

$$\leq \frac{9}{4\pi a^3 d^3}\int\limits_0^a\int\limits_0^d\int\limits_{S_1}\Big(\int\limits_0^d |\nabla \underline{u}_\varepsilon|^2 R^2\, dR\Big)^{\frac{1}{2}} dS\sqrt{\Big|\frac{1}{r} - \frac{1}{\rho}\Big|}\, r^2\rho^2\, dr\, d\rho$$

$$\leq \frac{9}{4\pi a^3 d^3} \int\limits_0^a \int\limits_0^d \Big(\int\limits_{S_1} \int\limits_0^d |\nabla \underline{u}_\varepsilon|^2 R^2 \, dR \, dS \Big)^{\frac{1}{2}} |S_1|^{\frac{1}{2}} \sqrt{|\rho - r|}\, r^{\frac{3}{2}} \rho^{\frac{3}{2}} \, dr \, d\rho$$

$$= \frac{9 \|\nabla \underline{u}_\varepsilon\|_{L_2(B_d)}}{2\sqrt{\pi} a^3 d^3} \int\limits_0^a \int\limits_0^d \sqrt{|\rho - r|}\, r^{\frac{3}{2}} \rho^{\frac{3}{2}} \, dr \, d\rho$$

$$\leq \frac{9 \|\underline{u}_\varepsilon\|_{H^1(B_d)}}{2\sqrt{\pi} a^3 d^3} \Big(\int\limits_0^a \int\limits_0^a a^{\frac{1}{2}} r^{\frac{3}{2}} \rho^{\frac{3}{2}} \, dr \, d\rho + \int\limits_0^a \int\limits_a^d \sqrt{\rho}\, r^{\frac{3}{2}} \rho^{\frac{3}{2}} \, dr \, d\rho \Big)$$

$$\leq \frac{9 \|\underline{u}_\varepsilon\|_{H^1(B_d)}}{2\sqrt{\pi} a^3 d^3} \Big(\frac{4}{25} a^{\frac{11}{2}} + \frac{2}{15} a^{\frac{5}{2}} d^3 \Big) \leq \frac{\|\underline{u}_\varepsilon\|_{H^1(B_d)}}{\sqrt{a}}.$$

Thus Lemma 3 is proved.

Prove the inequality (4.17) assuming that the limiting vector-function is smooth enough: $\underline{v}(\underline{x}) \in \overset{\circ}{J}(\Omega) \bigcap C_0^2(\Omega)$.

Consider a partition of the domain Ω by the nonintersecting cubes $K_h^{x_\alpha}$, aligned along the coordinate axes. In each cube the vector-function $\underline{v}(\underline{x})$ can be written in the form

$$\underline{v}(\underline{x}) = \underline{v}(\underline{x}^\alpha) + \sum_{n,p=1}^{3} \big(e_{np}[\underline{v}(\underline{x}^\alpha)] \underline{\varphi}^{np}(\underline{x} - \underline{x}^\alpha)$$

$$+ w_{np}[\underline{v}(\underline{x}^\alpha)] \underline{\psi}^{np}(\underline{x} - \underline{x}^\alpha) \big) + O(h^2), \quad \underline{x} \in K_h^{x_\alpha}. \tag{4.18}$$

Then in every internal cube $K_h^{x_\alpha}$ (which does not intersect the boundary $\partial\Omega$) with respect to the domain Ω, consider a vector-function

$$\underline{u}_\varepsilon^\alpha(\underline{x}) = \underline{v}_\varepsilon(\underline{x}) - \underline{v}(\underline{x}^\alpha) - \sum_{n,p=1}^{3} w_{np}[\underline{v}(\underline{x}^\alpha)] \underline{\psi}^{np}(\underline{x} - \underline{x}^\alpha). \tag{4.19}$$

It is clear that $\underline{u}_\varepsilon^\alpha(\underline{x}) \in J_\varepsilon[K_h^{x_\alpha}]$, $e_{np}[\underline{u}_\varepsilon^\alpha] = e_{np}[\underline{v}_\varepsilon]$ in $K_h^{x_\alpha}$ and $I^\varepsilon_{K_h^{x_\alpha}}[\underline{u}_\varepsilon^\alpha, \underline{u}_\varepsilon^\alpha]$ $= I^\varepsilon_{K_h^{x_\alpha}}[\underline{v}_\varepsilon, \underline{v}_\varepsilon]$. Therefore, from (2.2) and (2.7) for $T_{np} = e_{np}[\underline{v}(\underline{x}_\alpha)]$ we obtain

$$E_{K_h^{x_\alpha}}[\underline{v}_\varepsilon, \underline{v}_\varepsilon] + P^{\varepsilon h \gamma T}_{K_h^y}[\underline{u}_\varepsilon^\alpha(\underline{x})$$

$$- \sum_{n,p=1}^{3} e_{np}[\underline{v}(\underline{x}_\alpha)] \underline{\varphi}^{np}(\underline{x} - \underline{x}_\alpha), \underline{u}_\varepsilon^\alpha(\underline{x}) - \sum_{n,p=1}^{3} e_{np}[\underline{v}(\underline{x}_\alpha)] \underline{\varphi}^{np}(\underline{x} - \underline{x}_\alpha)]$$

$$+ \frac{1}{\lambda} I^\varepsilon_{K_h^{x_\alpha}}[\underline{v}_\varepsilon, \underline{v}_\varepsilon] \geq \sum_{n,p,q,r=1}^{3} a^\gamma_{npqr}(\underline{x}_\alpha, \lambda, \varepsilon, h) \varepsilon_{np}[\underline{v}(\underline{x}_\alpha)] \cdot \varepsilon_{qr}[\underline{v}(\underline{x}_\alpha)]. \tag{4.20}$$

Estimate now the second term in the LHS of inequality (4.20). Taking into account (2.5), (4.18), (4.19), we have

$$\int\limits_{K_h^{x\alpha}} \Big| \underline{u}_\varepsilon^\alpha(\underline{x}) - \sum_{n,p=1}^{3} e_{np}[\underline{v}(\underline{x}_\alpha)]\underline{\varphi}^{np}(\underline{x}-\underline{x}_\alpha) \Big|^2 dx = O(h^7),$$

$$\varepsilon^3 \sum_{i \ K_h^{x\alpha}} |\underline{u}_\varepsilon^\alpha(\underline{x}_\varepsilon^i) - \sum_{n,p=1}^{3} e_{np}[\underline{v}(\underline{x}_\alpha)]\underline{\varphi}^{np}(\underline{x}_\varepsilon^i - \underline{x}_\alpha)|^2$$

$$\le \varepsilon^3 \sum_{i \ K_h^{x\alpha}} |\underline{v}_\varepsilon(\underline{x}_\varepsilon^i) - \underline{v}(\underline{x}_\varepsilon^i)|^2 + \varepsilon^3 \sum_{i \ K_h^{x\alpha}} |\underline{v}(\underline{x}_\varepsilon^i) - \underline{v}(\underline{x}^\alpha)$$

$$- \sum_{n,p=1}^{3} w_{np}[\underline{v}(\underline{x}^\alpha)]\underline{\psi}^{np}(\underline{x}_\varepsilon^i - \underline{x}^\alpha) - \sum_{n,p=1}^{3} e_{np}[\underline{v}(\underline{x}_\alpha)]\underline{\varphi}^{np}(\underline{x}_\varepsilon^i - \underline{x}_\alpha)|^2. \qquad (4.21)$$

The last term in (4.21), due to (4.18), is of order $O(h^7)$. Next, since the vector-function $\underline{v}_\varepsilon(\underline{x})$ satisfies the rigid displacement condition (3.2) and $\underline{v}(\underline{x}) \in C_0^2(\Omega)$, we obtain

$$\varepsilon^3 \sum_{i \ K_h^{x\alpha}} |\underline{v}_\varepsilon(\underline{x}_\varepsilon^i) - \underline{v}(\underline{x}_\varepsilon^i)|^2 \le c\varepsilon^3 \sum_{i \ K_h^{x\alpha}} O(\varepsilon^2)$$

$$+\varepsilon^3 \sum_{i \ K_h^{x\alpha}} | < \underline{v}_\varepsilon >_{r_\varepsilon^i} - < \underline{v}_\varepsilon >_{\frac{R_\varepsilon^i}{2}} |^2 + \varepsilon^3 \sum_{i \ K_h^{x\alpha}} | < \underline{v}_\varepsilon >_{\frac{R_\varepsilon^i}{2}} - < \underline{v} >_{\frac{R_\varepsilon^i}{2}} |^2, \quad (4.22)$$

where the values r_ε^i and R_ε^i are defined at the beginning of Sect. 2.

Sum up the inequality (4.22) over all cubes of our partition

$$\varepsilon^3 \sum_{i=1}^{N_\varepsilon} |\underline{v}_\varepsilon(\underline{x}_\varepsilon^i) - \underline{v}(\underline{x}_\varepsilon^i)|^2 \le c\varepsilon^3 \sum_{i=1}^{N_\varepsilon} O(\varepsilon^2)$$

$$+\varepsilon^3 \sum_{i=1}^{N_\varepsilon} | < \underline{v}_\varepsilon >_{r_\varepsilon^i} - < \underline{v}_\varepsilon >_{\frac{R_\varepsilon^i}{2}} |^2 + \varepsilon^3 \sum_{i=1}^{N_\varepsilon} | < \underline{v}_\varepsilon >_{\frac{R_\varepsilon^i}{2}} - < \underline{v} >_{\frac{R_\varepsilon^i}{2}} |^2$$

$$\le c\, O(\varepsilon^2) + \varepsilon^{2-\alpha} \|\underline{v}_\varepsilon\|^2_{H^1(\Omega)} + \int\limits_\Omega |\underline{v}_\varepsilon(\underline{x}) - \underline{v}(\underline{x})|^2 dx. \qquad (4.23)$$

We use Lem. 3 to estimate the second term in (4.23) and the Cauchy–Schwartz inequality to estimate the third term. From (4.20)–(4.23) it follows that

$$\Phi_\varepsilon(\underline{v}_\varepsilon) \ge \sum_{\alpha \in \Lambda} h^3 \sum_{n,p,q,r=1}^{3} \frac{a_{npqr}^\gamma(\underline{x}_\alpha, \lambda, \varepsilon, h)}{h^3} \varepsilon_{np}[\underline{v}(\underline{x}_\alpha)] \varepsilon_{qr}[\underline{v}(\underline{x}_\alpha)]$$

$$+\lambda\int\limits_{\Omega}\langle\rho_\varepsilon\underline{v}_\varepsilon,\underline{v}_\varepsilon\rangle-2\int\limits_{\Omega}\langle\rho_\varepsilon\underline{v}_{\varepsilon 0},\underline{v}_\varepsilon\rangle\,dx+O(h^{2-\gamma})+\overline{o}(1),\quad \varepsilon\ll h\ll 1. \qquad (4.24)$$

Then, passing to the limit as $\varepsilon\to 0$ and $h\to 0$ in (4.24) $\varepsilon\to 0$, and taking into account 2.1)–2.2), the fact that $\underline{v}(\underline{x})\in C^2(\Omega)$ and $\gamma<2$, we obtain

$$\underline{\lim}_{\varepsilon\to 0}\Phi_\varepsilon(\underline{v}_\varepsilon)\geq\int\limits_{\Omega}\Big\{\sum_{n,p,q,r=1}^{3}a_{npqr}(\underline{x})\varepsilon_{np}[\underline{v}(\underline{x})]\cdot\varepsilon_{qr}[\underline{v}(\underline{x})]\,dx$$

$$+\lambda\langle\rho\underline{v},\underline{v}\rangle-2\langle\rho\underline{v}_0,\underline{v}\rangle\Big\}\,dx=\Phi_0(\underline{v}).$$

Thus, the required inequality (4.17) is obtained under the assumption that the limiting vector-function $\underline{v}(\underline{x})$ is smooth. The proof for a nonsmooth case ($\underline{v}(\underline{x})\in\overset{\circ}{J}(\Omega)$) is more technical, though its scheme is the same: it is necessary to construct the smooth approximations $\underline{v}_\sigma(\underline{x})$ of the limiting vector-functions, then to obtain the inequality for these approximations, which is analogous to that of (4.17), and to pass to the limit as $\sigma\to 0$. The details of this construction are presented in [1].

The inequality (4.4) follows from (4.16) and (4.17). Theorem 3 is proved.

5. Proof of Theorem 2

Note, that the convergence in Th. 3 was proved for $\lambda>0$ only. Besides, the coefficients $a_{npqr}(\underline{x},\lambda)$ were defined for $\lambda>0$ only. The following lemma enables us to extend these functions analytically into the complex plane. Moreover, the behavior of the extended functions as $\lambda\to\infty$ is established.

Lemma 4. *The function $a_{npqr}(\underline{x},\lambda)$ defined for $\lambda>0$ can be analytically extended into the complex plane with the section along the line $\lambda\leq 0$. The extended function can be written in the form*

$$a_{npqr}(\underline{x},\lambda)=2\mu I_{npqr}+a^1_{npqr}(\underline{x},\lambda), \qquad (5.1)$$

and for any $\delta>0$ the following estimate holds in the domain $\Phi_\delta=\{\lambda\in\mathbb{C}: |arg\,\lambda-\pi|\geq\delta>0\}$:

$$|a^1_{npqr}(\underline{x},\lambda)|<C\Big(\frac{1}{|\lambda|^{\frac{1}{2}}}\Big),\quad \lambda\to\infty, \qquad (5.2)$$

where $C>0$ does not depend on λ; the tensor $\{I_{npqr}\}$ is defined by equality (2.15).

P r o o f. We write the minimizer $\underline{w}_\varepsilon(\underline{x})$ of the functional (2.2) in the form

$$\underline{w}^T_\varepsilon(\underline{x},\lambda)=\underline{\phi}^T_\varepsilon(\underline{x})+\underline{v}^T_\varepsilon(\underline{x},\lambda), \qquad (5.3)$$

where $\underline{\phi}_\varepsilon^T(\underline{x})$ is the divergence-free vector-function, which is equal to the constant vectors $\underline{\phi}_\varepsilon^i$ on the balls G_ε^i containing the particles and having the radius $(1+\beta)r_\varepsilon^i = O(\varepsilon^{1+\alpha})$, $\beta > 0$. Moreover, this vector-function coincides with the vector-function $\underline{\phi}^T(\underline{x}) \equiv \sum_{q,r=1}^{3} T_{qr}\underline{\varphi}^{qr}(\underline{x}-\underline{y})$ outside the balls with radius $(1+2\beta)r_\varepsilon^i$, which are concentric to G_ε^i. The following estimates hold

$$\|\underline{\phi}_\varepsilon^T - \underline{\phi}^T\|_{L_2(\Omega)} \le c\,\max_i\{r_\varepsilon^i\}, \quad |\underline{\phi}_\varepsilon^T(\underline{x}_\varepsilon^i) - \underline{\phi}^T(\underline{x}_\varepsilon^i)| \le c\,r_\varepsilon^i,$$

$$\|\underline{\phi}_\varepsilon^T\|_{H^1(G)} \le c\,\|\underline{\phi}^T\|_{H^1(G)}, \quad |\underline{\phi}_\varepsilon^i - \underline{\phi}_\varepsilon^j| \le c\,\mathrm{dist}(Q_\varepsilon^i, Q_\varepsilon^j),$$

where G is any subdomain of the domain Ω. The existence of such a vector-function is established in [2].

Then, substituting (5.3) into (2.8), we obtain

$$a_{npqr}(\underline{y},\lambda,\varepsilon,h) = a_{npqr}^0(\underline{y},\varepsilon,h) + a_{npqr}^1(\underline{y},\lambda,\varepsilon,h), \tag{5.4}$$

where

$$a_{npqr}^0(\underline{y},\varepsilon,h) = E_{K_h^y}[\underline{\phi}_\varepsilon^{np}, \underline{\phi}_\varepsilon^{qr}]$$
$$+h^{-2-\gamma}\varepsilon^3 \sum_{i\ K_h^y} \langle \underline{\phi}_\varepsilon^{np}(\underline{x}_\varepsilon^i) - \underline{\varphi}^{np}(\underline{x}_\varepsilon^i - \underline{y}), \underline{\phi}_\varepsilon^{qr}(\underline{x}_\varepsilon^i) - \underline{\varphi}^{qr}(\underline{x}_\varepsilon^i - \underline{y})\rangle, \tag{5.5}$$

$$a_{npqr}^1(\underline{y},\lambda,\varepsilon,h) = E_{K_h^y}[\underline{v}_\varepsilon^{np}, \underline{v}_\varepsilon^{qr}] + h^{-2-\gamma}\varepsilon^3 \sum_{i\ K_h^y} \langle \underline{v}_\varepsilon^{np}(\underline{x}_\varepsilon^i), \underline{v}_\varepsilon^{qr}(\underline{x}_\varepsilon^i)\rangle$$
$$+E_{K_h^y}[\underline{\phi}_\varepsilon^{np}, \underline{v}_\varepsilon^{qr}] + h^{-2-\gamma}\varepsilon^3 \sum_{i\ K_h^y} \langle \underline{\phi}_\varepsilon^{np}(\underline{x}_\varepsilon^i) - \underline{\varphi}^{np}(\underline{x}_\varepsilon^i - \underline{y}), \underline{v}_\varepsilon^{qr}(\underline{x}_\varepsilon^i)\rangle$$
$$+E_{K_h^y}[\underline{v}_\varepsilon^{np}, \underline{\phi}_\varepsilon^{qr}] + h^{-2-\gamma}\varepsilon^3 \sum_{i\ K_h^y} \langle \underline{v}_\varepsilon^{np}(\underline{x}_\varepsilon^i), \underline{\phi}_\varepsilon^{qr}(\underline{x}_\varepsilon^i) - \underline{\varphi}^{qr}(\underline{x}_\varepsilon^i - \underline{y})\rangle$$
$$+\frac{1}{\lambda}\Big(I_{K_h^y}^\varepsilon[\underline{\phi}_\varepsilon^{np}, \underline{\phi}_\varepsilon^{qr}] + I_{K_h^y}^\varepsilon[\underline{v}_\varepsilon^{np}, \underline{v}_\varepsilon^{qr}] + I_{K_h^y}^\varepsilon[\underline{\phi}_\varepsilon^{np}, \underline{v}_\varepsilon^{qr}] + I_{K_h^y}^\varepsilon[\underline{v}_\varepsilon^{np}, \underline{\phi}_\varepsilon^{qr}]\Big). \tag{5.6}$$

Taking into account the properties of the vector-function $\underline{\phi}_\varepsilon^T(\underline{x})$, we can easily show that

$$\lim_{h\to 0}\lim_{\varepsilon\to 0} \frac{a_{npqr}^0(\underline{y},\varepsilon,h)}{h^3} = 2\mu I_{npqr}.$$

The analyticity of the functions $a_{npqr}^1(\underline{x},\lambda)$ over λ and the estimate (5.2) can be obtained similarly to [2].

From Lemma 4 it follows that the function $a_{npqr}(\underline{x}, \lambda)$ is the Laplace transform

$$a_{npqr}(\underline{x}, \lambda) = \int_0^\infty e^{-\lambda t} a_{npqr}(\underline{x}, t)\, dt \tag{5.7}$$

of the function

$$a_{npqr}(\underline{x}, t) = 2\mu I_{npqr} \delta(t) + a^1_{npqr}(\underline{x}, t), \tag{5.8}$$

where $\delta(t)$ is the Dirac delta function, and $a^1_{npqr}(\underline{x}, t)$ is a continuous at $\underline{x} \in \Omega$ and $t > 0$ function.

It may be shown that the family of the solutions $\underline{v}_\varepsilon(\underline{x}, \lambda)$ of the problem (3.1)–(3.5) is analytic in the domain $G_\varepsilon = \{\mathrm{Re}\lambda > 0\} \bigcup \big\{\Phi_\delta \bigcap \{|\lambda| > \lambda_1(\varepsilon)\}\big\}$. Moreover, in this domain the following estimates hold

$$\|\underline{v}_\varepsilon(\underline{x}, \lambda)\|_{L_2(\Omega)} \le \frac{C}{\mathrm{Re}\lambda},\ \mathrm{Re}\lambda > 0, \tag{5.9}$$

$$\|\underline{v}_\varepsilon(\underline{x}, \lambda)\|_{L_2(\Omega)} \le \frac{C_{1\varepsilon}}{|\lambda|}, \tag{5.10}$$

where the constant C does not depend on ε.

The similar statement is also true for the solution of the problem (3.10)–(3.12). Namely, this solution is analytic in the domain $G = \{\mathrm{Re}\lambda > 0\} \bigcup \big\{\Phi_{\frac{\pi}{3}} \bigcap \{|\lambda| > \lambda_2\}\big\}$, and in this domain

$$\|\underline{v}(\underline{x}, \lambda)\|_{L_2(\Omega)} \le \frac{C}{|\lambda|}. \tag{5.11}$$

Now, taking into account the estimate (5.9) which is a uniform in ε, we can use the Vitaly theorem (see [5]) to show that the sequence of the vector-functions $\underline{v}_\varepsilon(\underline{x}, \lambda)$ converges in $L_2(\Omega)$ uniformly to the vector-function $\underline{v}(\underline{x}, \lambda)$ inside the domain $\mathrm{Re}\lambda > 0$.

Due to the estimates (5.10) and (5.11), we can apply the inverse Laplace transform and prove the statement of Th. 3 (see details in [1, 4]).

6. Explicit Formulas for the Elastic Modulas for Periodic Array of Particles

Now we show the existence of limit 2.2) for a particular example of a periodic cubic lattice. We consider a periodic array when particles Q^i_ε are balls with the radius $r^i_\varepsilon = r\varepsilon^{1+\alpha}$, $r < \frac{1}{2}$, and their centers $\underline{x}^i_\varepsilon$ form a cubic lattice where each vertex is connected by a spring to its nearest neighbors NN (the edges of the periodicity cube), to its next nearest neighbors NNN (the diagonals of the faces of the cube) and to the next-to-next neighbors $NNNN$ (the diagonals of the cube). So, each vertex is connected to $3^3 - 1 = 26$ vertexes in the lattice. The elastic constants k^{ij} (see (1.2)) of these springs are k_1, k_2, k_3 respectively (see figure).

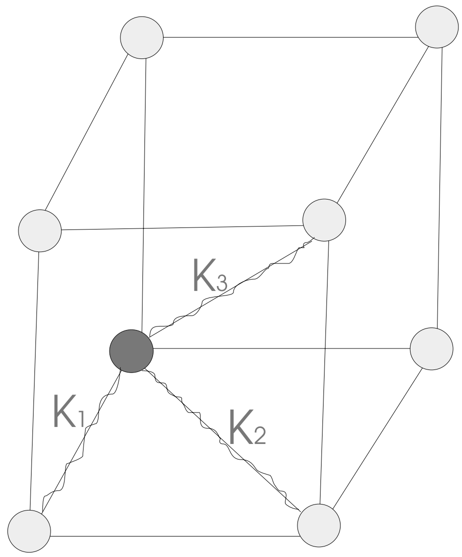

Figure. The basic periodic cell.

On this figure a fixed ball Q^i_ε with the center at the point $\underline{x}^i_\varepsilon$ is shown as a dark ball and all its neighbors Q^j_ε are shown as lighter balls.

We prove the following.

Theorem 4. *For the cubic lattice described above (see also figure) the elastic modulas $a^1_{npqr}(\underline{x}, \lambda)$ in (5.1) are constants with respect to x and given by the following formulas:*

$$a^1_{nnnn}(\lambda) = -2a_{nnpp}(\lambda) = \frac{1}{\lambda}\Big(\frac{2}{3}k_1 + \frac{\sqrt{2}}{3}k_2\Big),$$

$$a^1_{npnp}(\lambda) = \frac{1}{\lambda}\Big(\frac{\sqrt{2}}{2}k_2 + \frac{4\sqrt{3}}{9}k_3\Big),\ n, p = \overline{1, 3},$$

$a^1_{npqr}(\lambda) = 0$ *in all other cases.*

R e m a r k. If we introduce notations $a = a^1_{nnnn}$, $b = a^1_{npnp}$, $c = a^1_{nnpp} = -\frac{1}{2}a$, then the equations (2.10)–(2.13) can be written in terms of displacements $\underline{u}(\underline{x}, t) = \int_0^t \underline{v}(\underline{x}, \tau)\, d\tau$:

$$\begin{cases} \rho(\underline{x})\dfrac{\partial^2 \underline{u}(\underline{x},t)}{\partial t^2} - \mu\Delta\dfrac{\partial \underline{u}(\underline{x},t)}{\partial t} - b\Delta\underline{u}(\underline{x},t) - (a-2b-c)\displaystyle\sum_{r=1}^{3}\dfrac{\partial^2 u_r(\underline{x},t)}{\partial x_r^2}\underline{e}_r \\ = \nabla p(\underline{x},t), \quad \operatorname{div}\,\underline{u}(\underline{x},t) = 0, \quad \underline{x}\in\Omega,\ t>0; \\ \underline{u}(\underline{x},t) = \underline{0}, \quad \underline{x}\in\partial\Omega,\ t\geq 0; \\ \underline{u}(\underline{x},0) = 0,\ \dfrac{\partial \underline{u}(\underline{x},t)}{\partial t}\Bigg|_{t=0} = \underline{v}_0(\underline{x}), \quad \underline{x}\in\Omega. \end{cases}$$

P r o o f. Consider a particle Q_ε^i placed inside a cube K_ε^i of side length ε, so that both the particle and the cube are centered at the point $\underline{x}_\varepsilon^i$. Then $D_\varepsilon^i = K_\varepsilon^i \setminus Q_\varepsilon^i$ is a periodicity cell filled with the fluid. To obtain a standard unit cell, we rescale D_ε^i by the factor ε^{-1} and shift its center to the origin. Then the domain $D_\varepsilon = K \setminus Q_\varepsilon$ is a unit periodicity cell where K is a cube of the side length 1 centered at the origin, and Q_ε is a ball in K with radius $r\varepsilon^\alpha$, $r < \dfrac{1}{4}$.

Let K_h^y be a cube of the side length h, $h \gg \varepsilon$, centered at the point $\underline{y} \in \Omega$. Consider a function

$$\underline{u}_\varepsilon^{np}(\underline{x}) = \operatorname{rot}\big(\phi_\varepsilon(\underline{x})\underline{u}^{np}(\underline{x})\big), \tag{6.1}$$

where

$$\phi_\varepsilon(\underline{x}) = \phi\big(\frac{\underline{x}}{r\varepsilon^\alpha}\big),\ \phi(\underline{x}) = \begin{cases} 1, |\underline{x}| \leq 1 \\ 0, |\underline{x}| > 2 \end{cases},$$

and $\underline{u}^{np}(\underline{x})$ is a smooth vector-function such that

$$\operatorname{rot}\underline{u}^{np}(\underline{x}) = -\underline{\psi}^{np}(\underline{x}), \quad |\underline{u}^{np}(\underline{x})| \leq C|\underline{x}|^2.$$

Since the function $\underline{u}_\varepsilon^{np}(\underline{x})$ is equal to zero on the boundary ∂K, it admits a periodic extension on $\mathbb{R}^3$.

We seek a function $\underline{w}^{np}(\underline{x},\lambda)$ that minimizes the functional (2.2) for $T = T^{np} = \dfrac{1}{2}(\underline{e}^n \otimes \underline{e}^p + \underline{e}^p \otimes \underline{e}^n)$ in the form

$$\underline{w}^{np}(\underline{x},\lambda) = \underline{U}_\varepsilon^{np}(\underline{x}) + \underline{v}_\varepsilon^{np}(\underline{x},\lambda), \tag{6.2}$$

where

$$\underline{U}_\varepsilon^{np}(\underline{x}) = \underline{\psi}^{np}(\underline{x} - \underline{y}_\varepsilon) + \varepsilon\underline{\tilde{u}}_\varepsilon^{np}\big(\frac{\underline{x} - \underline{y}_\varepsilon}{\varepsilon}\big). \tag{6.3}$$

Here $\underline{\tilde{u}}_\varepsilon^{np}(\underline{x})$ is a periodic extension of the function $\underline{u}_\varepsilon^{np}(\underline{x})$ and $\underline{y}_\varepsilon = \underline{x}_\varepsilon^i$ is the nearest to $\underline{y}$ center of particles Q_ε^i. Using the properties of the functions $\underline{\psi}^{np}(\underline{x})$ and $\underline{u}_\varepsilon^{np}(\underline{x})$, we have

$$\underline{U}_\varepsilon^{np}(\underline{x}) = \underline{\psi}^{np}(\underline{x}_\varepsilon^j - \underline{y}_\varepsilon),\ \underline{x} \in Q_\varepsilon^j, \tag{6.4}$$

$$\mathrm{div}\underline{U}_\varepsilon^{np}(\underline{x}) = 0,\ \underline{x} \in K_h^y. \tag{6.5}$$

Next we obtain a variational problem for the corrector $\underline{v}_\varepsilon^{np}(\underline{x}, \lambda)$. Analising the problem and then substituting (6.2)–(6.4) into (2.8) with reference to periodicity of the structure, we get

$$\frac{1}{h^3} a_{npqr}^\gamma(\underline{y}, \lambda, \varepsilon, h) = \frac{1}{h^3} E_{K_h^y}[\underline{\psi}_\varepsilon^{np}, \underline{\psi}_\varepsilon^{qr}] + \frac{1}{h^3\lambda} I_{K_h^y}[\underline{\psi}_\varepsilon^{np}, \underline{\psi}_\varepsilon^{qr}] + \overline{o}(1), \varepsilon \ll h \ll 1.$$

The statement of Th. 4 follows from the above representation.

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