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AN OPTIMIZATION PROBLEM OF PACKING IDENTICAL CIRCLES INTO A MULTIPLY CONNECTED REGION

Part 1. A mathematical model and its characteristics

The paper deals with an optimization problem of packing identical circles into a multiply connected region whose frontier consists of arcs of circles and line segments. The approach that allows to reduce solving the problem to solving a sequence of problems with linear objective functions is suggested. To this end radii of all circles are taken as variables. In order to construct a mathematical model of the problem the concept of Ф-functions is using. Some important characteristics of the mathematical model are considered.

Рассматривается оптимизационная задача упаковки одинаковых кругов в многосвязную область, граница которой состоит из отрезков дуг окружностей и прямых отрезков. Предлагается подход, который позволяет свести решение поставленной задачи к решению последовательности задач с линейной целевой функцией. С этой целью радиусы всех кругов принимаются переменными. При построении математической модели используется метод Ф-функций. Рассматриваются некоторые важные особенности построенной математической модели.

Розглядається оптимізаційна задача пакування однакових кіл у багатозв'язну область, границя якої складається з відрізків дуг окружностей та прямих відрізків. Пропонується підхід, що дозволяє звести розв'язання поставленої задачі до розв'язання послідовності задач із лінійною цільовою функцією. З цією метою радіуси всіх кіл приймаються змінними. При побудові математичної моделі використовується метод Ф-функцій. Розглядаються деякі важливі особливості побудованої математичної моделі.

Introduction

The Euclidean geometry problems of the densest packing of identical circles are important both theoretically and practically. These problems have a wide spectrum of applications. In [1] different industrial applications of these problems are pointed. These problems arise in stocking and transporting of loads [2], in stamping and blanking of round products in metal forming industry, in glass industry and in pulp and paper industry. The problems are of interest in detecting signals in a multisource neighbourhood and measuring solar radiation. Packing of identical circles occurs in the study of protein's structures in biology and the structures of diverse many-body systems in chemistry and physics.

Optimization problems of packing identical circles into a irregular region with prohibited areas also arise in the following real-world applications:

- placement of a maximum number of identical circular containers into a storage building with prohibited zones or into a non-standard storage container;
- cutting off a maximum number of identical circular blanks from return materials in different branch of industry;
- packing identical circular containers with radioactive wastes into storages;
- placement identical circular sensors with maximum density into a given region;

– planning of trees planting with maximum density into a given region.

In addition, as mentioned in [3] the circle packing problems can be applied for two-dimensional image recognition problems and for domain discretization problems such as triangulation and mesh generation techniques [4].

There are many papers devoted to the problem of packing identical circles into regular polygons (such as rectangle, square, triangle) and circular containers.

The packing problem of identical circles into a rectangle (in publications also called as “cylinder packing problem” or “cylinder palletization”) is researched in [5]. In these works the approaches for solving the "pallet loading problem" are mainly heuristic. The paper [2] offers a methodology, based on a nonlinear optimization, for solving problem packing identical circles into a rectangle.

The problem packing identical circles into a square is investigated in many papers, for example, in [6].

The papers [7, 8] are investigations devoted to the solving of the problem of packing equal circles into a larger circle container.

For solving the packing problem of equal circles into a square and into a circle container different approaches are suggested, for example, such as: monotonic basin hopping approach [8], beam search algorithm [7], branch-and-bound algorithms, computer-aided optimality proofs approaches [6], nonlinear programming approaches [9].

At present many papers consider the packing problem of a fixed set of items into a given region to minimize the region sizes. Some authors solve the problem using nonlinear optimization models and techniques. In [10] the extensive review of such researches is performed. Also, in [10] a large variety of doubly differentiable nonlinear programming models for the problem of minimizing the region sizes in 2D (square, rectangle, triangle and circle) and 3D (cuboid and sphere) packing problems are considered.

Since all known us researches do not solve the packing problems of identical circles into a multiply connected region whose frontier consists of arcs of circles and line segments, then the aim of our paper is construction of a mathematical model of the problem, investigation of its characteristics and development of solution methods.

Problem statement

Let there be a family $\{C_1, C_2, \ldots, C_i, \ldots\}$ of circles to be congruent to a circle C of radius r and a bounded multiply connected region $P \subset R^2$ (Fig. 1). We suppose that P is formed as follows

$$P = \mathrm{cl}\left(P_0 \setminus \bigcup_{l=1}^{\sigma} A_l\right),$$

where cl(·) is the closure of (·); P_0 is a simple connected set (Fig. 1, a), whose frontier is formed by a sequence of line segments $[x_i, y_i, x_{i+1}, y_{i+1}]$ and arcs of circles $[x_i, y_i, x_{i+1}, y_{i+1}, \bar{x}_j, \bar{y}_j, \bar{r}_j]$, where (x_i, y_i) and (x_{i+1}, y_{i+1}) are coordinates of the origin and the terminus of a line segment or an arc, $(\bar{x}_j, \bar{y}_j)$ and $\bar{r}_j$ are centre coordinates and a radius of a circle respectively; A_l is a prohibited area (Fig. 1, b) represented as

$$A_l = \left(\bigcup_{g=1}^{\psi} C_{lg}\right) \bigcup \left(\bigcup_{q=1}^{\vartheta} M_{lq}\right), \qquad l \in I_\sigma = \{1,2,\ldots,\sigma\};$$

$C_{lg} = \{(x, y) \in R^2 : (x - x_{lg}^0)^2 + (y - y_{lg}^0)^2 - (\rho_{lg}^0)^2 \le 0\}$, $g \in I_\psi = \{1,2,\ldots,\psi\}$, M_{lq} is a convex polygon given by m_{lq} vertices, $q \in I_\vartheta = \{1,2,\ldots,\vartheta\}$, i. e. each prohibited area is non-convex set which can be presented by a finite union of different circles and convex polygons.

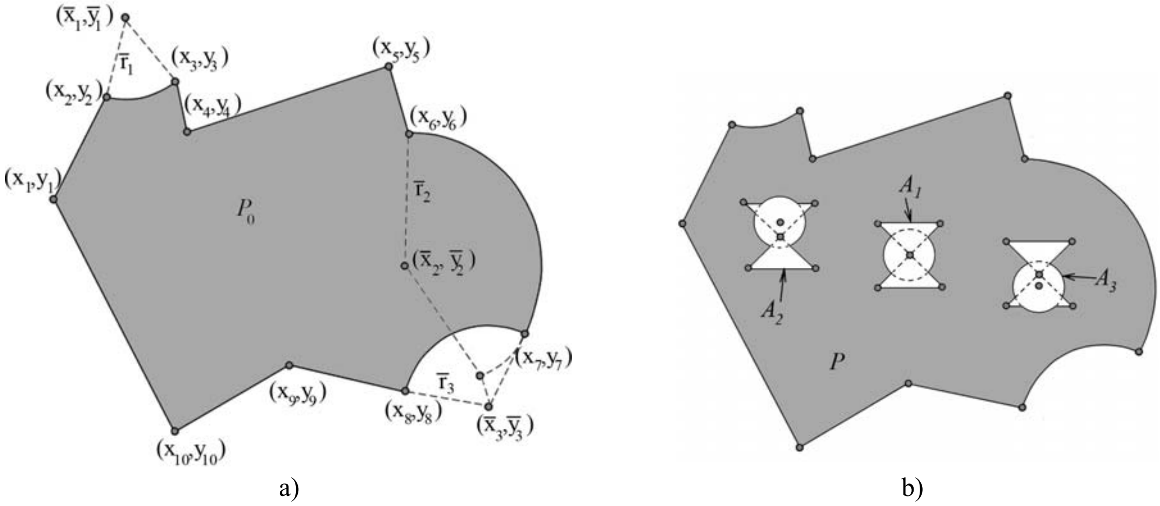


Fig. 1. The region P:
a) – a simple connected set P_0; b) – a multiply connected region P

If $u_i = (x_i, y_i)$ is centre coordinates of C_i, then a location of all C_i, $i \in I_\tau = \{1,2,...,\tau\}$, in R^2 is determined by a vector $u = (u_1, u_2, \dots, u_\tau) \in R^{2\tau}$. In what follows the circle C_i translated by the vector $u_i = (x_i, y_i)$ is denoted by $C_i(u_i)$.

Problem. Define a vector $u = (u_1, u_2, \dots, u_\tau) \in R^{2\tau}$ that ensures a placement of the maximal number τ of circles $C_i(u_i)$, $i \in I_\tau$, without mutual overlapping into P.

Mathematical model

One of the most important and complicated problems of computer and mathematical modelling of the problem class is an analytical description of interaction between any circle C_i and the region P. In this paper we follow the concept of Φ-functions, described in [11–14], which has been proved to be very useful when solving similar problems.

In order to formalize the placement conditions of C_i within P by means of Φ-functions we construct the following set

$$G = \mathrm{cl}(R^2 \backslash P_0).$$

The set can be always represented by a finite union of primary objects Q_{ij}, $I = 1, 2, 3, 4$, shown in Fig. 2, i. e.

$$G = \left(\bigcup_{j=1}^{\delta} Q_{1j}\right) \bigcup \left(\bigcup_{j=1}^{\gamma} Q_{2j}\right) \bigcup \left(\bigcup_{j=1}^{\xi} Q_{3j}\right) \bigcup \left(\bigcup_{j=1}^{\zeta} Q_{4j}\right),$$

where

$Q_{1j} = \{(x,y) \in R^2 : \chi_{1j}(x,y) \ge 0\}$, $j \in J_\delta = \{1,2,...,\delta\}$, $\chi_{1j}(x,y) = a_{1j}x + b_{1j}y + c_{1j}$ (Fig. 2, a),

$Q_{2j} = \{(x,y) \in R^2 : \chi_{2jl}(x,y) \ge 0, l = 1,2\}$, $j \in J_\gamma = \{1,2,...,\gamma\}$, $\chi_{2jl}(x,y) = a_{2jl}x + b_{2jl}y + c_{2jl}$ (Fig.2,b),

$Q_{3j} = \{(x,y) \in R^2 : \hat{\omega}_{3j}(x,y) \ge 0\}$, $j \in J_\xi = \{1,2,...,\xi\}$, $\hat{\omega}_{3j}(x,y) = \hat{r}_j^2 - (x - \hat{x}_j)^2 - (y - \hat{y}_j)^2$ (Fig.2,c),

$Q_{4j} = \{(x,y) \in R^2 : \chi_{4jl}(x,y) \ge 0, \breve{\omega}_{4j}(x,y) \ge 0, l = 1,2\}, j \in J_\zeta = \{1,2,...,\zeta\}$,

$\chi_{4jl}(x,y) = a_{4jl}x + b_{4jl}y + c_{4jl}$, $\breve{\omega}_{4j}(x,y) = (x - \breve{x}_j)^2 + (y - \breve{y}_j)^2 - \breve{r}_j^2$, $\rho(A,B) \ge 2r$, $\rho(A,B)$ is the distance between points A and B (Fig.2,d), i. e. Q_{1j} is a half plane, Q_{2j} is a convex cone, Q_{3j} is a circle and Q_{4j} is an intersection of half plane and the complement of a circle to R^2.

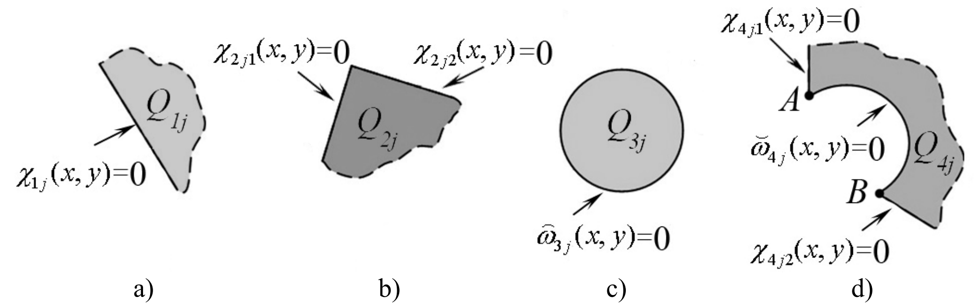


Fig. 2. Primary objects:
a) – Q_{1j}; b) – Q_{2j}; c) – Q_{3j}; d) – Q_{4j}

For example, the set G for the region P_0 depicted in Fig. 1, b can be constructed as follows (Fig. 3)

$$G = \left(\bigcup_{j=1}^{2} Q_{1j}\right) \bigcup \left(\bigcup_{j=1}^{2} Q_{2j}\right) \bigcup \left(\bigcup_{j=1}^{2} Q_{3j}\right) \bigcup \left(\bigcup_{j=1}^{1} Q_{4j}\right).$$

In order to solve the problem we suggest the approach that allows to reduce solving the problem to solving a sequence of problems with linear objective functions. To this end radii r_i of C_i, $i \in I_\tau$, are taken as variables. Thus, the radii form a vector $v^\tau = (r_1, r_2, \dots, r_\tau) \in R^\tau$.

In order to construct a mathematical model of the problem we use the concept of Φ-functions. Values of the Φ-function are some measure both a rate of intersection of two geometric objects and the shortest distance between them depending on their a mutual arrangement in the space.

Making use of the Φ-functions for primary objects [13] and complex 2D objects [14] a mathematical model of the problem sequence can be presented as follows:

$$F_n(X^{n*}) = \max F_n(X^n) = \max \sum_{i=1}^{n} r_i, \quad \text{s. t. } X^n = (u^n, v^n) \in W_n, \; n = 1,2,\dots,\tau+1, \tag{1}$$

$$W_n = \{X^n \in R^{3n} : \Phi_{ij}(u_i, u_j, r_i, r_j) \ge 0, i, j \in I_n = \{1,2,\dots,n\}, i < j,$$
$$\Phi_i(u_i, r_i) \ge 0, i \in I_n, r - r_i \ge 0, \; r_i \ge 0, i \in I_n\}, \tag{2}$$

where $\Phi_{ij}(u_i, u_j, r_i, r_j) = (x_i - x_j)^2 + (y_i - y_j)^2 - (r_i + r_j)^2$, $\Phi_i(u_i, r_i)$ is a Φ-function for C_i and $G_1 = \mathrm{cl}(R^2 \backslash P)$.

The function $\Phi_i(u_i, r_i)$ describes a condition of belonging of a circle C_i to the region P.

The Φ-functions for C_i and $G_1 = \mathrm{cl}(R^2 \backslash P)$ can be represented as follows:

$$\Phi_i(u_i, r_i) =$$
$$= \min\{\Phi_{il}^{CA}(u_i, r_i), \Phi_i^{CG}(u_i, r_i), \; l \in I_\sigma\},$$

where

$$\Phi_{il}^{CA}(u_i, r_i) =$$
$$= \min\{\Phi_{ilg}^{CC}(u_i, r_i), \Phi_{ilq}^{CM}(u_i, r_i), g \in I_\psi, q \in I_\vartheta\}$$

,

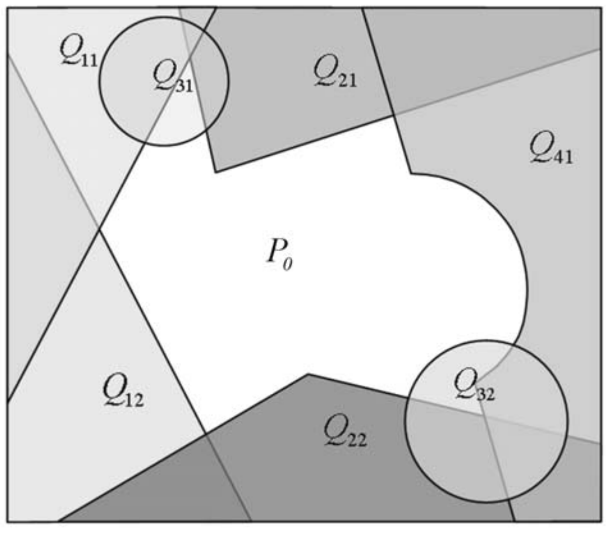
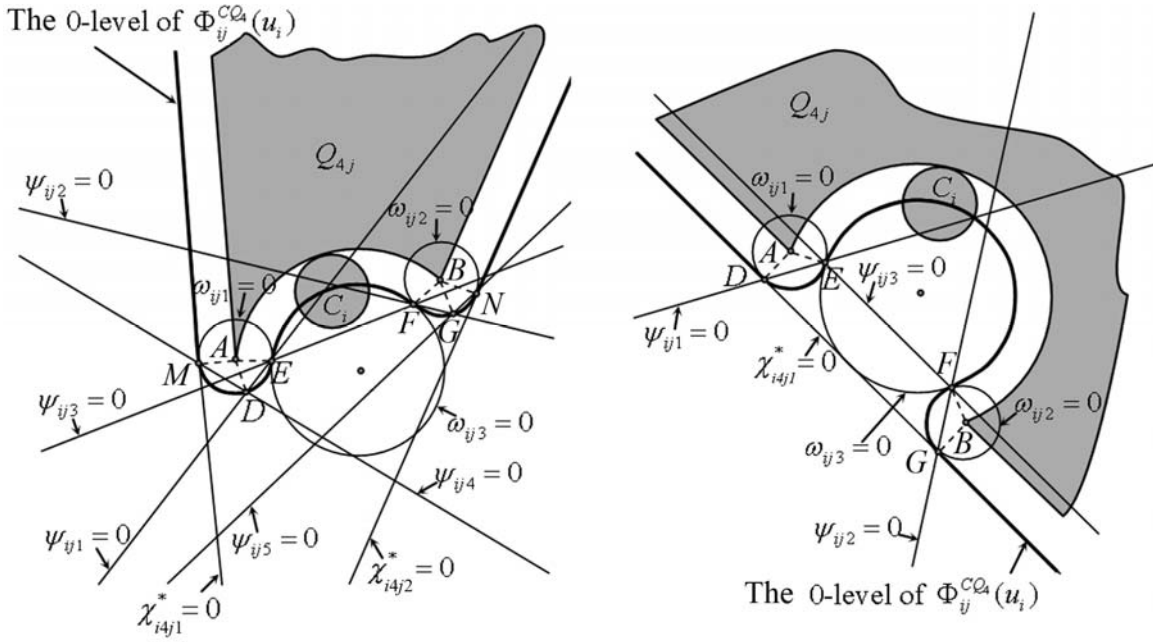


Fig. 3. A representation of the set G



a) b)

Fig.4. The 0-level of $\Phi_{ij}^{CQ_4}(u_i)$ **:**

a) – the frontier of Q_{4j} is formed by two straight line;
b) – the frontier of Q_{4j} is formed by one straight line

$\Phi_{il}^{CA}(u_i, r_i)$ is a Φ-function for C_i and A_l;

$$\Phi_{ilg}^{CC}(u_i, r_i) =$$
$$= (x_i - x_{lg}^0)^2 + (y_i - y_{lg}^0)^2 - (r_i + \rho_{lg}^0)^2,$$

$\Phi_{ilg}^{CC}(u_i, r_i)$ is a Φ-function for C_i and C_{lg};

$$\Phi_{ilq}^{CM}(u_i, r_i) = \max_{k=1,2,\ldots,m_{lq}} \left\{ \max\left\{ \min\{\psi_{ilqk}(u_i, r_i), \omega_{ilqk}(u_i, r_i)\}, \chi^*_{ilqk}(u_i, r_i) \right\}\right\},$$

$\Phi_{ilq}^{CM}(u_i, r_i)$ is a Φ-function for C_i and M_{lg} [13];

$$\Phi_i^{CG}(u_i, r_i) = \min\left\{ \Phi_{ij}^{CQ_1}(u_i, r_i), j \in I_\delta, \Phi_{ij}^{CQ_2}(u_i, r_i), j \in I_\gamma, \Phi_{ij}^{CQ_3}(u_i, r_i), j \in I_\xi, \Phi_{ij}^{CQ_4}(u_i, r_i), j \in J_\zeta \right\},$$

$\Phi_i^{CG}(u_i, r_i)$ is a Φ-function for C_i and G;

$$\Phi_{ij}^{CQ_1}(u_i, r_i) = \chi^*_{i1j}(u_i, r_i) = -\chi_{1j}(u_i) - r_i,$$

$$\Phi_{ij}^{CQ_2}(u_i, r_i) = \max\left\{ \min\{\psi_{ij}(u_i, r_i), \omega_{ij}(u_i, r_i)\}, \chi^*_{i2j1}(u_i, r_i), \chi^*_{i2j2}(u_i, r_i) \right\},$$

$$\Phi_{ij}^{CQ_3}(u_i, r_i) = (x_i - \hat{x}_j)^2 + (y_i - \hat{y}_j)^2 - (r_i + \hat{r}_j)^2,$$

$$\Phi_{ij}^{CQ_4}(u_i, r_i) = \max\left\{ \phi_{ij1}(u_i, r_i), \phi_{ij2}(u_i, r_i), \phi_{ij3}(u_i, r_i), \omega_{ij3}(u_i, r_i), \chi^*_{i4j1}(u_i, r_i), \chi^*_{i4j2}(u_i, r_i) \right\},$$

$\Phi_{ij}^{CQ_1}(u_i, r_i)$, $\Phi_{ij}^{CQ_2}(u_i, r_i)$, $\Phi_{ij}^{CQ_2}(u_i, r_i)$, $\Phi_{ij}^{CQ_4}(u_i, r_i)$ are the Φ-functions for C_i and Q_{1j}; C_i and Q_{2j} [13]; C_i and Q_{3j}; C_i and Q_{4j} respectively.

The 0-level of the function $\Phi_{ij}^{CQ_4}(u_i, r_i)$ is depicted in fig. 4. Components of $\Phi_{ij}^{CQ_4}(u_i, r_i)$ have the kind:

$$\phi_{ij1}(u_i,r_i)=\min\{\omega_{ij1}(u_i,r_i),\omega_{ij2}(u_i,r_i),\psi_{ij1}(u_i,r_i),\psi_{ij2}(u_i,r_i),\psi_{ij3}(u_i,r_i)\},$$

$$\phi_{ij2}(u_i,r_i)=\min\{\omega_{ij1}(u_i,r_i),\psi_{ij4}(u_i,r_i)\},\phi_{ij3}(u_i,r_i)=\min\{\omega_{ij2}(u_i,r_i),\psi_{ij5}(u_i,r_i)\},$$

$$\omega_{ij1}(u_i,r_i)=(x_i-x_{j1})^2+(y_i-y_{j1})^2-r_i^2,\qquad \omega_{ij2}(u_i,r_i)=(x_i-x_{j2})^2+(y_i-y_{j2})^2-r_i^2,$$

$$\omega_{ij3}(u_i,r_i)=(\breve{r}_j-r_i)^2-(x_i-\breve{x}_j)^2-(y_i-\breve{y}_j)^2,$$

$$\psi_{ij1}(u_i,r_i)=a_{j1}x_i+b_{j1}y_i+c_{j1},\quad a_{1j}=y_{j4}-y_{j3},\quad b_{j1}=-(x_{j4}-x_{j3}),\quad c_{1j}=-(a_{1j}x_{j3}+b_{1j}y_{j3}),$$

$$\psi_{ij2}(u_i,r_i)=a_{j2}x_i+b_{j2}y_i+c_{j2},\quad a_{j2}=y_{j6}-y_{j5},\quad b_{j2}=-(x_{j6}-x_{j5})\quad ,c_{j2}=-(a_{j2}x_{j5}+b_{j2}y_{j5}),$$

$$\psi_{ij3}(u_i,r_i)=a_{j3}x_i+b_{j3}y_i+c_{j3},\quad a_{j3}=y_{j5}-y_{j4},\quad b_{j3}=-(x_{j5}-x_{j4}),\quad c_{j3}=-(a_{j3}x_5+b_{j3}y_5),$$

$$\psi_{ij4}(u_i,r_i)=a_{j4}x_i+b_{j4}y_i+c_{j4},\quad a_{j4}=y_{j3}-y_{j7},\quad b_{j4}=-(x_{j3}-x_{j7}),\quad c_{j4}=-(a_{j4}x_{j3}+b_{j4}y_{j3}),$$

$$\psi_{ij5}(u_i,r_i)=a_{j5}x_i+b_{j5}y_i+c_{j5},\quad a_{j5}=y_{j8}-y_{j6},\quad b_{j5}=-(x_{j8}-x_{j6}),\quad c_{j5}=-(a_{j5}x_{j6}+b_{j5}y_{j6}),$$

$$\chi^*_{i4j1}(u_i,r_i)=-\chi_{4j1}(u_i)-r_i,\quad \chi^*_{i4j2}(u_i,r_i)=-\chi_{4j2}(u_i)-r_i,$$

(x_{j1}, y_{j1}), (x_{j2}, y_{j2}), (x_{j3}, y_{j3}), (x_{j4}, y_{j4}), (x_{j5}, y_{j5}), (x_{j6}, y_{j6}), (x_{j7}, y_{j7}) and (x_{j8}, y_{j8}) are coordinates of points A, B, D, E, F, G, M and N respectively (Fig. 4, a).

If the frontier of Q_{4j} is formed by one straight line (Fig. 4, b), then $\Phi_{ij}^{CQ_4}(u_i)$ can be written simpler

$$\Phi_{ij}^{CQ_4}(u_i,r_i)=\max\{\phi_{ij1}(u_i,r_i),\omega_{ij3}(u_i,r_i),\chi^*_{i4j1}(u_i,r_i)\}.$$

Characteristics of the mathematical model

We point out some important characteristics of the mathematical model of the following problem

$$F_n(X^{n*})=\max F_n(X^n)=\max\sum_{i=1}^{n} r_i,\quad \text{s. t.}\quad X^n=(u^n,v^n)\in W_n. \tag{3}$$

1. If $F_n(X^{n*}) = nr$ then the point X^{n*} is a global maximum of the problem (3).

2. If $F_{\tau+1}(X^{(\tau+1)*})<(\tau+1)r$ and $F_\tau(X^{\tau*})=F_\tau(u^{\tau*},v^{\tau*})=\tau\cdot r$, where $X^{(\tau+1)*}$ and $X^{\tau*}$ are global maxima of the problem (3), then a solution of the problem stated is reached at the point $u^*=u^{\tau*}$ to which there corresponds τ packed circles.

3. Since $F_n(X^n)$ is linear then local maxima are achieved at extreme points of W_n.

4. W_n is specified by linear and nonlinear inequalities.

5. It is easily seen that $\Phi_i(u_i,r_i)\ge 0$ if at least one of inequality systems of kind

$$\Gamma_i^s(u_i,r_i)=\begin{cases}\Phi_{ij}^{CQ_1}(u_i,r_i)\ge 0, j\in J_\delta,\\ \Phi_{ij}^{CQ_2}(u_i,r_i)\ge 0, j\in J_\gamma,\\ \Phi_{ij}^{CQ_3}(u_i,r_i)\ge 0, j\in J_\xi,\\ \Phi_{ij}^{CQ_4}\zeta_{ij}(u_i,r_i)\ge 0, j\in J_\zeta,\\ \Phi_{ilg}^{CC}(u_i,r_i)\ge 0, l\in I_\sigma, g\in I_\psi,\\ \Phi_{ilq}^{CM}(u_i,r_i)\ge 0, l\in I_\sigma, q\in I_\vartheta,\end{cases}$$

is satisfied, where $\Phi_{ij}^{CQ_2}(u_i,r_i)\ge 0$ is either one of the inequalities $\chi^*_{i2j1}(u_i,r_i)\ge 0$, $\chi^*_{i2j2}(u_i,r_i)\ge 0$ or the inequality system $\begin{cases}\omega_{ij}(u_i,r_i)\ge 0,\\ \psi_{ij}(u_i,r_i)\ge 0;\end{cases}$ $\Phi_{ij}^{CQ_4}(u_i,r_i)\ge 0$ is either one of the inequalities

$\omega_{ij3}(u_i,r_i)\ge 0$, $\chi^*_{i4j1}(u_i,r_i)\ge 0$, $\chi^*_{i4j2}(u_i,r_i)\ge 0$ or one of the inequality systems $\begin{cases}\omega_{ij1}(u_i,r_i)\ge 0,\\ \omega_{ij2}(u_i,r_i)\ge 0,\\ \psi_{ij1}(u_i,r_i)\ge 0,\\ \psi_{ij2}(u_i,r_i)\ge 0,\\ \psi_{ij3}(u_i,r_i)\ge 0,\end{cases}$ $\begin{cases}\omega_{ij1}(u_i,r_i)\ge 0,\\ \psi_{ij4}(u_i,r_i)\ge 0,\end{cases}$ $\begin{cases}\omega_{ij2}(u_i,r_i)\ge 0,\\ \psi_{ij5}(u_i,r_i)\ge 0;\end{cases}$ $\Phi^{CM}_{ilq}(u_i,r_i)\ge 0$ is either the inequality $\chi^*_{ilqk}(u_i,r_i)\ge 0$ or the inequality system $\begin{cases}\omega_{ilqk}(u_i,r_i)\ge 0,\\ \psi_{ilqk}(u_i,r_i)\ge 0.\end{cases}$

6. It follows from an previous item that $\Phi_i(u_i,r_i)$ includes function $\Phi^{CQ_2}_{ij}(u_i,r_i)$, $j = 1, 2, \ldots, \gamma$, $\Phi^{CQ_4}_{ij}(u_i,r_i)$, $j = 1, 2, \ldots, \xi$, and $\Phi^{CM}_{ilq}(u_i,r_i)$, $l = 1, 2, \ldots, \sigma$, $q = 1, 2, \ldots, \vartheta$, each of which is a function of kind

$$\Phi^{CQ_4}_{ij}(u_i,r_i)=\max\left\{\phi_{ij1}(u_i,r_i),\phi_{ij2}(u_i,r_i),\phi_{ij3}(u_i,r_i),\omega_{ij3}(u_i,r_i),\chi^*_{i4j1}(u_i,r_i),\chi^*_{i4j2}(u_i,r_i)\right\}.$$

It means that if at least one of the functions in the braces is positive then appropriate function $\Phi^{CQ_2}_{ij}(u_i,r_i)$, $\Phi^{CQ_4}_{ij}(u_i,r_i)$ or $\Phi^{CM}_{ilq}(u_i,r_i)$ is positive. So $\Phi_i(u_i,r_i)\ge 0$ if at least one of the inequality systems $\Gamma^s_i(u_i,r_i)\ge 0$, $s\in I_\varpi=\{1,2,\ldots,\varpi=3^\gamma\cdot 6^\zeta\cdot\prod_{l=1}^{\sigma}\prod_{q=1}^{\vartheta}2m_{lq}\}$, is satisfied.

7. On the basis of the previous item and the mathematical model (3) any packing of n circles into P is described by the inequality system of form

$$\begin{cases}\Gamma^{s_t}_i(u_i,r_i)\ge 0, \quad i\in I_n,\\ \Phi_{ij}(u_i,u_j,r_i,r_j)\ge 0, \quad i,j\in I_n, \quad i<j,\\ r-r_i\ge 0, \quad i\in I_n,\\ r_i\ge 0, \quad i\in I_n.\end{cases} \tag{4}$$

The system specifies some subregion W_{nt}. Each inequality system (4) consists of no less than $n(4+\sigma(\psi+\vartheta)+\frac{1}{2}(n+3))$ linear and nonlinear inequalities. The left hand sides of the inequalities specifying the systems are indefinitely differentiable functions.

8. By item 5 the number of the subregions is $\eta = \varpi^n$. Thus, the feasible region W_n can be presented as

$$W_n=\bigcup_{t=1}^{\eta}W_{nt}\ .$$

It is necessary to note that among η systems there is a number of inconsistent systems.

9. If a point $X^{nl*}\in W_{ni_0}$ is a local maximum with respect to W_{ni_0} and simultaneously $X^{nl*}\in W_{ni_j}$, $j\in J_l\subset\{1,2,\ldots,\eta_0\}$, then it is necessary to prove that X^{nl*} is a local maximum with respect to W_n.

10. In [15] it was proved that the problem under consideration is *NP*-hard.

11. In general local minima are non-strict.

12. Since the number of local maxima is much more than n! then an exhaustive search of local maxima is impossible.

Conclusions

This work presents a mathematical model of the problem of packing identical circles into a multiply connected region whose frontier consists of arcs of circles and line segments. The mathematical model (1)–(2) is presented as a constrained non linear optimization problem.

Usage of Ф-function technique to construct a mathematical model of the packing problems allows to build mathematical models in which a feasible region can be presented as an union of subsets. Each of the subsets is described by systems of linear and non-linear inequalities. The left hand sides of the inequalities specifying the systems are indefinitely differentiable functions. It allows to use the modern gradient methods of non-linear optimization for solving of the problem.

The mathematical model suggested can be also used for packing non-identical circles into a multiply connected region.

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